STOCHASTIC COOLING REQUIREMENTS FOR A MUON COLLIDER *

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ABSTRACT
The most severe limitation to the muon production for a large-energy muon collider is the short time allowed for cooling the beam to dimensions small enough to provide reasonably high luminosity. The limitation is caused by the short lifetime of the particles. It appears to be desirable to accelerate the beam quickly in very short bunches. This paper describes the requirements of single-pass, fast stochastic cooling for very short bunches. Bandwidth, amplifier gain and Schottky power do not seem to be of major concern. Problems do arise with the ultimate low emittance that can be achieved, the value of which is seriously affected by the front-end thermal noise.

1. INTRODUCTION
In the quest for the Higgs bosons, a muon collider may be perceived as the experimental device more affordable and more feasible than electron-positron or very large hadron colliders [1-3]. Unfortunately, muons do not exist in nature and they have to be produced with the only technique we know these days: impinging an intense beam of protons or electrons on a target. There is also the fact that muons are intrinsically unstable particles with a very short lifetime. Accumulation, acceleration and cooling are then to be executed extremely fast.

The topic of this paper deals with the requirement of betatron stochastic cooling. The situation being described is altogether different from the usual encountered with coasting beams. Now the beam is made of short bunches with a length considerably smaller than the wavelength of the bandwidth of available electronic amplifiers. Thus a different method is to be developed based on the correction of the stochastic signal for all particles at the same time in one single-step. There is a fundamental limitation to the ultimate value of the final emittance that can be achieved, caused by the thermal noise at the front-end of the amplifier. It is found that a luminosity of about $10^{24}$ cm$^{-2}$s$^{-1}$ can be achieved at the very most.

2. THE MUON COLLIDER
A layout of a possible scenario is shown in Fig. 1. A fast-cycling accelerator of either electrons or protons is provided as the source of the primary beam which is made to impinge on a sequence of targets for the production of muon pairs [4-8]. The secondary beam, is collected with a normalized emittance of about 100 $\pi$ mm mmrad. An average intensity of about 50 nA per each component of the pair production is expected. Both types of beam, $\mu^+$ and $\mu^-$, are accelerated in a linear rf structure, operating at 3 GHz, to the final energy which is in the range of 100 to 1000 GeV. At the end of the acceleration, each beam is transferred to a storage ring where fast stochastic cooling is done. Each beam is then taken to a stacking ring of about the same size, where several cooled beam pulses are stacked sidewise in the momentum phase space [7]. Finally, both beams are extracted from their respective stacking ring and transferred to the collider ring proper where they are made to collide.

Fig. 1 - A conceptual layout of the muon collider

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3. THE LUMINOSITY PERFORMANCE

The average luminosity is given by the following expression

\[ L = M F N_0 \beta^* \frac{\gamma}{4 \pi \varepsilon_0} = M F L_0 \]  
(1)

where \( N_0 \approx 100 \) is the initial number of particles per bunch, \( \beta^* \approx 3 \text{ GHz} \) is the beam bunching frequency during acceleration and stochastic cooling, \( \gamma \) is the energy relativistic factor, \( \varepsilon_0 \approx 25 \text{ nm rad} \) is the initial rms normalized emittance, \( \beta^* \) is the focusing amplitude parameter at the interaction point which is 1 cm for the single-pass mode. \( M \) is the number of beam pulses which are stacked in the momentum phase space of the stacking ring. It is to be noticed that the current \( I_{\text{bunch}} = N_0 \varepsilon f_{\text{bunch}} \approx 50 \text{ nA} \) is a constant equal to the average current of each muon beam. Finally \( F \) is the form factor.

\[ F = F_{\text{acc}} F_{\text{sto}} F_{\text{stac}} F_{\text{col}} \]  
(2)

\( F_{\text{acc}} \) is the square of the beam survival fraction after acceleration.

\[ F_{\text{acc}} = (E_{\text{init}} / E_{\text{final}})^{2E_0 / cG \tau_0} \]  
(3)

where \( E_{\text{init}} \approx 1 \text{ GeV} \) is the beam kinetic energy at production, \( E_{\text{final}} \) is the final energy in the collider, \( E_0 = 106 \text{ MeV} \) is the rest energy, \( \tau_0 = 2.2 \mu \text{s} \) is the lifetime of the muon at rest and \( G \) is the accelerating gradient in the linear accelerator. \( F_{\text{sto}} \) reflects the effects of reducing the betatron emittance by stochastic cooling.

\[ F_{\text{sto}} = (\varepsilon_0 / \varepsilon_{\text{w}}) \exp (-2T_{\text{sto}} / \gamma_{\text{final}} \tau_0) \]  
(4)

where \( (\varepsilon_0 / \varepsilon_{\text{w}}) \) is the ratio of the initial to the final betatron emittance and \( T_{\text{sto}} \) is the cooling period. \( F_{\text{stac}} \) is the beam survival fraction after momentum stacking of \( M \) turns during a period \( T_{\text{stac}} \).

\[ F_{\text{stac}} = \left[ 1 - \exp \left( -T_{\text{stac}} / \gamma_{\text{final}} \tau_0 \right) \right] / (T_{\text{stac}} / \gamma_{\text{final}} \tau_0) \]  
(5)

To avoid excessive beam losses, at most \( M \approx 900 \) pulses [9]. Finally, \( F_{\text{col}} \) represents the beam losses during collision and has an expression similar to the Eq. (5), except that \( T_{\text{stac}} \) is replaced by \( 2T_{\text{col}} \), that is twice the period of beam-beam collision.

4. REQUIREMENTS ON STOCHASTIC COOLING

With the values of the parameters given in the previous Sections, we have \( L_0 = 1.0 \times 10^{12} \text{ cm}^2 \text{s}^{-1} \). To achieve a luminosity of about \( 1.0 \times 10^{27} \text{ cm}^2 \text{s}^{-1} \), as required by the high energy physics experimental program, one needs \( M F \approx 1 \times 10^{12} \). Since at most \( M F_{\text{stac}}^2 \approx 1000 \), even assuming \( F_{\text{acc}} F_{\text{col}} \approx 1 \), to meet the requirement one needs \( F_{\text{stac}} \approx 1 \times 10^6 \), which is a very large requirement for the betatron stochastic cooling [9]. The requirement is independent of the beam energy: a normalized emittance of \( 25 \times 10^{15} \text{ rad} \), that is a reduction of the betatron emittance by nine orders of magnitude (!). Thus the fundamental question concerns the ultimate emittance that can be realistically achieved at the end of cooling. Fast cooling can be obtained with a low number of particles per bunch and a large electronic gain. Then a serious problem is associated with the thermal noise at the front-end of the amplifier, which will set a limitation of the final beam transverse dimension.

There are major differences between stochastic cooling for the case of bunched beams we are investigating here and the usual approach for coasting beams encountered, for instance, during production and accumulation of antiproton beams [9-11]. With good approximation, the beam bunches have no longitudinal extension. The beam current signal is therefore highly organized and coherent. The transverse beam position, on the other hand, has a very stochastic behavior. The longitudinal internal motion can be ignored and no mixing occurs between the detection of the beam signal at the pickups and the application of the deflection at the kickers. The lack of mixing causes a serious limitation on the effectiveness of stochastic cooling. Once the initial beam displacement has been corrected, there is no more signal from the beam that can be used. Thus everything is done in a single step with a relatively small reduction of the beam size. Between steps, the signal from the beam has to be regenerated, for instance by rearranging the particle mutual position with the aid of powerful magnetic lenses, like skew quadrupoles, sextupoles and nonlinear lenses. We shall assume below that this is indeed the case.
5. ANALYSIS OF THE COOLING DEVICE

Consider a very narrow bunch made of \( N \) particles all with the same electric charge. The bunch is periodically traversing a sequence of \( n_p \) position pickups made of two parallel striplines each of length \( t \) and separated by a distance \( d \). The striplines are shorted at one end and terminated at the upstream end to their characteristic impedance \( R_p \). The bunch current can be represented as a pulse of zero duration, proportional to the average displacement \( x_0 \). This current leaves a voltage signal which after filtering and amplification is distributed among \( n_k \) kickers of characteristic impedance \( R_k \) and having exactly the same geometrical configuration and size of the pickups.

Using the statistically relation between the average beam displacement \( \langle x_0 \rangle \) and the rms beam size \( \sigma \), that is \( \sigma^2 = N \langle x_0 \rangle^2 \), the total deflection angle each particle receives during the traversal of the kickers is \( \theta = g_0 \sigma / d \) where, with \( E \) the particle total energy and \( \beta c \) the velocity,

\[
g_0 = \frac{\beta E_d}{2 \beta^2 E_d} \sqrt{N e^4 W^2 n_k n_p R_k R_p}
\]

(6)

The finite temperature of the terminating resistors of the loop and of the preamplifiers creates at the input to the preamplifier a signal of power \( P_T = k_B (T_A + T_R) \) W, where \( k_B = 8.6171 \times 10^{-5} \text{ eV/K} \) is the Boltzmann constant, \( T_A \) the equivalent temperature of the amplifier and \( T_R \) of the resistor. We calculate also the total deflection angle due to the thermal noise

\[
\theta_T = \frac{e \beta E_d}{\beta^2 E_d} \sqrt{n_k R_k P_T}
\]

(7)

The effectiveness of stochastic cooling is assessed by the ratio of Schottky to the thermal power

\[
S = \frac{\theta_T^2}{\theta_s^2} = \frac{N e^2 W^2 n_p R_p \sigma^2}{4 d^2 P_T}
\]

(8)

This ratio increases linearly with the bandwidth, differently from the one it was derived for coasting beams [11].

6. THE EQUATION FOR THE EVOLUTION OF THE BEAM EMITTANCE

The beam emittance can be defined as follows

\[
\varepsilon = \Sigma (\gamma x_i^2 + 2\alpha x_i x_i' + \beta x_i'^2) / N
\]

(9)

where \( \alpha, \beta \) and \( \gamma \) are the lattice Twiss parameters, and \( x_i, x_i' \) are respectively the position and angle of the i-th particle. At the kickers each particle receives the same kick, that is \( x_i' \rightarrow x_i' + \theta_s + \theta_T \). Since there is no correlation between particle position and thermal noise, the average change per revolution is [9]

\[
<\Delta \varepsilon> = - (2g - N g^2) \varepsilon + \beta_k \theta_T^2
\]

(10)

having assumed an optimum betatron phase advance between pickups and kickers, and where the dynamical gain

\[
g = g_0 \sqrt{\beta_k \beta_p / N d^2}
\]

(11)

Finally, the evolution of the beam emittance is described by the following equation

\[
\frac{d\varepsilon}{dt} = -\lambda \varepsilon + D
\]

(12)

where, assuming \( n_k \) identical cooling systems in the storage ring, and denoting with \( f_0 \) the revolution frequency,

\[
\lambda = n_k f_0 (2g - N g^2)
\]

(13)

is the cooling rate, and the diffusion coefficient

\[
D = n_k f_0 \beta_k \theta_T^2
\]

(14)
7. OPTIMIZATION OF THE COOLING PERFORMANCE

An optimum cooling rate is obtained by setting \( g = 1 / N \)

\[
\lambda_{\text{opt}} = n_s f_0 / N
\]  

(15)

which corresponds to correcting the instantaneous beam bunch displacement in one single step. At the same time we can also derive the required amplifier gain

\[
A = \frac{3 \beta^2 E d^2}{\sqrt{e^2 n_k n_p R_k R_p \beta_k \beta_p}} \frac{m}{3 \beta^2 E d^2} \quad \text{(16)}
\]

and the equilibrium emittance

\[
\varepsilon_{\infty} = D / \lambda_{\text{opt}} = N \beta_k \Theta_T^2 = \frac{6 d^2 P_T / W}{m n_p \beta_p e f_{\mu} R_p}
\]  

(17)

Both of these expressions show the same dependence with the bunching mode number \( m = 3 W / 2 f_{\text{bunch}} \) and with the average beam current \( I_{\mu} = N e f_{\text{bunch}} \). Noticing that the thermal power \( P_T \) is proportional to \( W \), it is seen that both \( A \) and \( \varepsilon_{\infty} \) do not depend explicitly on how the beam is bunched.

8. AN APPLICATION OF THE OPTIMAL SYSTEM

We take the following values: \( d = 1 \) cm, \( \beta_p = \beta_p = 200 \) m, \( n_p = n_k = 1024 \), \( R_p = R_k = 100 \) ohm. We take also \( m = 3 \), that is a bunching frequency \( f_{\text{bunch}} = 3 \) GHz, and a bandwidth \( W = 6 \) GHz ranging between 6 and 12 GHz. We set the temperature of the amplifier and resistor \( T_A = T_R = 1 \) °K which is very likely an unrealistic value. The summary of the results of our calculations are shown in Table 1 below, where we have taken the optimum gain \( g = 1 / N \).

<table>
<thead>
<tr>
<th>Beam Energy, GeV</th>
<th>100</th>
<th>300</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2 \pi R, m )</td>
<td>700</td>
<td>2100</td>
<td>7000</td>
</tr>
<tr>
<td>( 1 / \lambda, \text{ms} )</td>
<td>0.030</td>
<td>0.030</td>
<td>0.030</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>1 \times 10^9</td>
<td>3 \times 10^9</td>
<td>1 \times 10^{10}</td>
</tr>
<tr>
<td>( \epsilon_{\mu} )</td>
<td>32</td>
<td>96</td>
<td>320</td>
</tr>
<tr>
<td>( f_{\mu}, \text{cm}^{-2} \text{s}^{-1} )</td>
<td>1 \times 10^{18}</td>
<td>3 \times 10^{18}</td>
<td>1 \times 10^{19}</td>
</tr>
<tr>
<td>( L )</td>
<td>1000</td>
<td>300</td>
<td>100</td>
</tr>
<tr>
<td>( \text{cm}^{-2} \text{s}^{-1} )</td>
<td>1 \times 10^{21}</td>
<td>1 \times 10^{21}</td>
<td>1 \times 10^{21}</td>
</tr>
<tr>
<td>( N_{\mu} )</td>
<td>0.0068</td>
<td>0.0023</td>
<td>0.000</td>
</tr>
<tr>
<td>( f_{\mu} )</td>
<td>40</td>
<td>120</td>
<td>400</td>
</tr>
<tr>
<td>( L_{\mu} )</td>
<td>4 \times 10^{22}</td>
<td>1.2 \times 10^{23}</td>
<td>4 \times 10^{23}</td>
</tr>
</tbody>
</table>

The circumference of the storage ring increases with the beam energy, and the number \( n_s \) of cooling systems varies proportionally. As a consequence, the cooling rate \( \lambda \) is constant with energy, whereas the amplifier gain \( A \) and the equilibrium emittance \( \varepsilon_{\infty} \) increase linearly with energy. Even at the very low temperature of 1 °K, thermal noise dominates over the beam signal, and the equilibrium emittance is just about comparable to the initial beam emittance at the energy of 100 GeV. For larger energies, there is actually stochastic heating accompanied by an increase of the beam emittance.

Since for the optimum gain the cooling time is 0.03 ms, which is considerably shorter than the beam lifetime, it is reasonable to lower the amplifier gain. The results are shown in Fig. 2. As the gain \( g \) is lowered, both the amplifier gain and the equilibrium emittance reduce also, but on the other hand, unfortunately, the cooling time increases. If the increase is too large then the particle losses would also be too large.
There is an optimum [9] in correspondence of $N_{g_{\text{max}}} = 1 / \gamma \tau_0 \lambda_{\text{opt}}$ which gives an enhancement of the luminosity by a factor $f_{\text{max}} \sim 0.04 \gamma$. The values of $N_{g_{\text{max}}}$ and of $f_{\text{max}}$ with the corresponding increased luminosity are also shown at the bottom part of Table 1. The luminosity figures are still well below the desired values.

The only other parameter that can be varied is the bunching frequency $f_{\text{bunch}}$. It is found [9] that as a result of the optimization with respect to the dynamical gain $g$, there is no optimum with the bunching mode number $m$.

9. CONCLUSIONS

We have determined that it is indeed feasible that the luminosity of a muon collider scales linearly with the beam energy, as it is required by physics arguments. Unfortunately, even with the stretching of our imagination, it is seen from Table 1 that at the very most only a luminosity of $10^{20} \, \gamma \, \text{cm}^2 \, \text{s}^{-1}$ can be obtained. This is seven orders of magnitude below what it is actually required.

The limitation of thermal noise to the ultimate emittance that can be achieved is to be coupled with the requirement of the cooling rate which is to be large compared to the inverse of the beam lifetime. To achieve very fast cooling, a large amplifier gain is needed, which has also the effect to amplify to a larger level the front-end noise. Moreover large cooling rate can be obtained only with a few number of particles per bunch. Even by postulating the feasibility of momentum stacking, it is rather difficult to accumulate more than $10^7$ particles per bunch.

Our estimates of the performance of stochastic cooling are based on the simple scenario of production, acceleration, cooling and collision we have proposed here. Other scenarios may be possible and we believe that an optimum configuration has still to be searched and is highly desirable. But we also believe that stochastic cooling has to be an integral part of the scheme.

REFERENCES