DEMONSTRATION OF NO FEASIBILITY OF A CRYSTALLINE BEAM IN A BETATRON MAGNET*

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ABSTRACT
This paper investigates the feasibility of a Crystalline Beam in a weak-focusing Betatron Magnet. The curvature effect due to the bending magnet is also investigated. The case of a circular one-dimensional string of electrically-charged particles is examined. It is found that the motion is unstable due to the dependence of the precession movement with the radial displacement. That is a form of negative-mass instability which can be avoided with an alternating-focussing structure. The calculation of the particle-particle interaction as well as of the forces due to the external magnetic field is done directly in the laboratory frame.

1. INTRODUCTION

One of the most important issue in the study of Crystalline Beams [1-3] is the determination of the best configuration of the storage ring where the formation and detection of the state should occur. Recently, Wei, Li and Sessler [4] have found that a Crystalline Beam cannot be stable in a Betatron. The method used by these Authors is based on a non-obvious transformation of general relativity to the beam rotating frame. We shall deal here with the same issue, and in particular discuss the special case of a Circular String of electrically-charged particles. The method we use applies directly to the laboratory frame and involves the correct calculation of the retarded potentials to describe the particle-particle interaction

We shall consider a circular beam of N ions all with the same electric charge $Q_e$ and mass at rest $m_0$. The ions are moving on the same circular trajectory of radius R, with the same constant angular frequency $\omega_0$, and are equally spaced. The particles are lying on the same plane at $z = 0$. We shall adopt a cylindrical coordinates $(r, \theta, z)$ for our study. We want to prove that the Circular String of particles is indeed an Equilibrium Configuration, compatible with a weak-focussing structure, for which we can subsequently determine the Confinement and Stability Conditions [5]. We shall also prove that such a configuration circulating in a Betatron Magnet is unstable. The instability is caused by the dependence of the precession movement of the particles with respect to each other on their radial displacement. It is the equivalent of the negative-mass instability which can be avoided by choosing a storage ring with alternating-focussing gradients, where the corresponding transition energy is above the energy of the particles. We shall work at the end an example of such a storage ring. It is also found that a large periodicity of the storage ring is required for the beam stability.

2. THE EQUATIONS OF MOTION

The equations of motion of a particle can be written in vectorial form

$$\frac{dp}{dt} = Qe E + Qe (v/c) \times B$$  \hspace{1cm} (1)

where $p = m_0 \gamma v = m v$ is the ion momentum, $v$ its velocity, $\gamma$ the energy relativistic factor, $E$ the total electric field, and $B$ the total magnetic field.

The electric and magnetic fields are the sum of the particle-particle interaction, denoted in the following with the subindex "sc", and of the external fields, denoted with the subindex "ext", that is $E = E_{ext} + E_{sc}$ and $B = B_{ext} + B_{sc}$.

Let us define as the force acting on the particle from the interaction with all other particles

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\[
F_{sc} = Qe \vec{E}_{sc} + Qe (\gamma / c) \times \vec{B}_{sc}
\]  

(2)

We assume there are no external electric fields, that is \( \vec{E}_{ext} = 0 \). The external magnetic field \( \vec{B}_{ext} \) corresponds to the distribution of the field in a typical Betatron Magnet. Performing an expansion and linearization around the reference circular trajectory, the field components are

\[
B_1 = B_0 \, n \, z / R
\]  

(3)

\[
B_2 = -B_0 \, (1 - n \, x / R)
\]  

(4)

where \( B_0 \) is the guiding field on the reference orbit, \( n \) the field index which in a Betatron Magnet ranges between 0 and 1, and \( x = r - R \). The azimuthal component \( B_	heta = 0 \).

In cylindrical coordinates, denoting with \( p_t, p_\theta, p_z \) the components of the momentum \( \vec{p} \) and with \( v_t, v_\theta, v_z \) the components of the velocity \( \vec{v} \), the three components of the vectorial Eq. (1) can be written as

\[
d p_t / d t - p_\theta d \theta / d t = F_t + Qe (\gamma / c) B_z
\]  

(5)

\[
d p_\theta / d t + p_t d \theta / d t = F_\theta + Qe (v_z / c) B_t - Qe (\gamma / c) B_z
\]  

(6)

\[
d p_z / d t = F_z - Qe (\gamma / c) B_t
\]  

(7)

where \( F_t, F_\theta \) and \( F_z \) are the components of the vector \( \vec{F}_{sc} \) given by Eq. (2).

3. THE EQUILIBRIUM CONFIGURATION OF THE CIRCULAR STRING

The Circular String is defined by: \( x = v_t = z = v_z = 0 \) and \( d \theta / d t = \omega_\theta \), constant, at all times. It derives that \( \gamma \) is also a constant, \( v_\theta = \omega_\theta R = v \) is a constant. We have also \( p_t = p_z = 0 \) and that the angular momentum \( p_\theta = p \) is a constant of motion. It is sufficient and necessary that the following conditions are satisfied at the radius \( r = R \) on the \( z = 0 \) plane. From Eq. (5), denoting \( \beta = v/c \),

\[
Qe B_\theta R = pc + R F_t / \beta
\]  

(8)

From Eq. (6), it is required that \( F_\theta = 0 \). Finally, from Eq. (7), we derive the condition \( F_z = Qe \beta B_t \).

Equation (8) is the usual Lorentz condition where the centrifugal force is balanced by the Lorentz force. A repulsive force, due to the particle-particle interaction, appears and has to be taken into account as well. Since the beam has a perfect cylindrical arrangement, from symmetry considerations, it can be determined that \( F_t \) is constant and has the same value for all particles. For typical beam parameters this contribution is small. Using the same symmetry considerations, it is easily seen that \( F_z = 0 \). Thus, we need \( B_z = 0 \) on the \( z = 0 \) plane and at the radius \( r = R \), which is indeed the case as shown by Eq. (3).

We have to deal finally with the condition \( F_\theta = 0 \). We give below a formal prove that this condition is indeed satisfied for the Circular String. We shall calculate explicitly the components of the interaction force in the following section. From symmetry considerations, if there is a non-vanishing azimuthal component of the force due to the particle-particle interaction, this component \( F_\theta \) has to be the same at the same instant to all particles; that is, at the same instant all particles would suffer the same variation of the angular momentum and thus the whole system would increase or decrease its total angular momentum. This violates the principle of conservation of the angular momentum since we are dealing with a conservative isolated system where there is only an external, stationary magnetic field distribution. It derives therefore that \( F_\theta = 0 \) identically.

4. CALCULATION OF THE INTERACTION FORCES AT THE EQUILIBRIUM

The field distribution generated by a point-like charged particle animated of arbitrary motion is given by J. D. Jackson in Classical Electrodynamics [6]. We neglect the "acceleration fields", which depend linearly with \( d \vec{v} / d t \) and that are usually very small, especially for \( \beta < 1 \). Let us denote with \( \vec{E}_{ij} \) and \( \vec{B}_{ij} \) the fields generated by the j-th particle on the i-th particle of the Circular String. Let \( n_{ij} \) be the unit vector directed from the j-th particle to the i-th particle, and \( R_{ij} \) the distance between the two particles. Then according to Jackson,

\[
\vec{B}_{ij} = \{ n_{ij} \times \vec{E}_{ij} \} / R_{ij}
\]  

(9)
\[ E_{ij} = \left( \frac{Qe}{\gamma^2} \right) \left[ \left( \mathbf{n}_{ij} - \mathbf{b}_j \right) / (1 - \beta_j \cdot \mathbf{n}_{ij})^3 \mathbf{r}_{ij}^2 \right]_{\text{ret}} \]  

(10)

where \( \mathbf{b}_j = c \mathbf{v}_j \) is the velocity of the j-th particle. The square brackets with subscript "ret" mean that the quantity in the brackets is to be evaluated at the retarded time \( \tau = t - \mathbf{r}_{ij} / c \).

A more detailed and complete derivation of this result is given by R. Becker in *Theorie der Elektrizitat* [7]. Becker proves that

\[ \left[ \left( \mathbf{n}_{ij} - \mathbf{b}_j \right) \mathbf{r}_{ij} \right]_{\text{ret}} = \mathbf{r}_{ij} \]  

(11)

where \( \mathbf{r}_{ij} \) is the actual distance vector between the two particles at same time \( t \), directed from the j-th toward the i-th particle. It is also seen that

\[ \left[ (1 - \beta_j \cdot \mathbf{n}_{ij}) \mathbf{r}_{ij} \right]_{\text{ret}}^2 = \mathbf{r}_{ij}^2 - (\mathbf{r}_{ij} \times \mathbf{b}_j)^2 = s_{ij}^2 \]  

(12)

where the quantity at the right-hand side is estimated at the same instant \( t \). Thus

\[ E_{ij} = \frac{Qe \mathbf{r}_{ij}}{\gamma^2 s_{ij}^3} \]  

(13)

From symmetry considerations, the field has the same value for all particles. In particular, we can estimate the field acting on the \( i = 0 \) particle when it is located at \( \theta = 0 \) at the time \( t = 0 \). The vector \( \mathbf{r}_{0j} \) has the following components: \( [2 R \sin(\theta_j / 2), -R \sin \theta_j, 0] \) where \( \theta_j \) is the angular separation between the two particles. It is also seen that

\[ (\mathbf{r}_{0j} \times \mathbf{b}_j)^2 = 4 \beta_j^2 R^2 \sin^2(\theta_j / 2) \]  

(14)

and

\[ s_{0j} = \frac{(2 R \sin(\theta_j / 2)) \sqrt{1 - \beta_j^2 \sin^2(\theta_j / 2)}} \]  

(15)

Since for every particle at the angular distance \( \theta_j \) there is another one at \(-\theta_j\), it is immediately seen that the azimuthal component of the electric field \( E_\phi = 0 \) whereas

\[ E_r = \frac{Qe}{2 R^2 \gamma^2} \sum_j \frac{1}{\left[ 1 - \beta_j^2 \sin^2(\theta_j / 2) \right]^{\frac{3}{2}} \sin(\theta_j / 2)} \]  

(16)

where the summation is done only over those particles with \( 0 < \theta_j < \pi \).

It remains to estimate the components of the magnetic field \( \mathbf{B}_{0j} \). According to Becker

\[ \left[ \mathbf{n}_{ij} \times \mathbf{E}_{ij} \right]_{\text{ret}} = \mathbf{b}_j \times \mathbf{E}_{ij} \]  

(17)

where the quantity at the right-hand side is estimated at the instant \( t \). Since the two vectors \( \mathbf{b}_j \) and \( \mathbf{E}_{ij} \) are on the same plane \( (z = 0) \), the magnetic field \( \mathbf{B}_{ij} \) is directed along the z-axis and perpendicular to the \( z = 0 \) plane which contains the Cylindrical String of ions. Since the velocity of any particle is directed along the azimuthal direction \( \theta \), the cross product between \( \mathbf{b}_j \) and \( \mathbf{B}_{ij} \) has only a radial component. In summary, from Eq. (2)

\[ F_r = Qe \left( 1 - \beta_j^2 \right) E_r = F_r \]  

(18)

and \( F_\theta = F_z = 0 \). This completes the prove that the beam made of a Circular String is indeed an Equilibrium Configuration consistent with the field distribution in a Betatron Magnet.

5. DETERMINATION OF STABILITY OF THE CIRCULAR STRING

We shall apply a perturbation method where the motion of a single particle is perturbed by a small amount, whereas the motion of all other particles remains unchanged. The perturbation is described by small displacements \( x, R \), and \( z \) which are smaller than the unperturbed separation of two neighboring particles. Let us take as the perturbed particle the one which would be located at \( \theta = 0 \) at the time \( t = 0 \). Linearization of the expressions of the components of the force resulting from the particle-particle interaction gives
\[ F_r = F_r + k_r x - Q e \beta^2 E_r \left( \frac{d q}{\omega_0 \, dt} \right) \]  \hspace{1cm} (19)

\[ F_\theta = - k_\theta q + Q e \beta^2 E_r \left( \frac{d x}{\nu \, dt} \right) \]  \hspace{1cm} (20)

\[ F_z = k_z z \]  \hspace{1cm} (21)

where the constants \( k_r, k_\theta \) and \( k_z \) have been evaluated in Ref. [8]. It is shown that for nonrelativistic beams

\[ k_r - k_z - \frac{Q^2 e^2 g_0 \, N^3}{4 \pi^3 R^3 \gamma^4} \]  \hspace{1cm} (22)

where \( g_0 = 1.2 \), and \( k_\theta = 2 \pi R k_z \). To first order, the energy relativistic factor \( \gamma \) is constant, whereas

\[ p_\theta \left[ 1 + \frac{d q}{\omega_0 \, dt} + \frac{x}{R} \right] \]  \hspace{1cm} (23)

\[ p_\theta \frac{d \theta}{d t} = p \omega_0 \left[ 1 + \frac{2 d q}{\omega_0 \, dt} + \frac{x}{R} \right] \]  \hspace{1cm} (24)

By taking into account the confinement condition Eq. (8), by inserting Eqs. (3,4) and Eqs.(19-21) into the equations of motion Eqs. (5-7), we derive the following linearized equations

\[ d^2 x / dt^2 + \Omega_{H}^2 x = v \left( \omega_0 x / R + d q / d t \right) \]  \hspace{1cm} (25)

\[ d^2 q / dt^2 + (\omega_0 / R) d x / d t = - \omega_0 k_\theta q / p \]  \hspace{1cm} (26)

\[ d^2 z / dt^2 + \Omega_{V}^2 x = 0 \]  \hspace{1cm} (27)

where

\[ \Omega_{H}^2 = \frac{Q e B_o (1 - n)}{m R} - k_r / m \]  \hspace{1cm} (28)

\[ \Omega_{V}^2 = \frac{Q e B_o n}{m R} - k_z / m \]  \hspace{1cm} (29)

The equations of motion (25-27) are very general. They apply to the motion of a charged particle in a combined function magnet described by the field index \( n \), under the action of other moving particles in a Circular String Configuration. The axial motion can be treated independently from that occurring on the \( z = 0 \) plane. Solving Eq. (27) gives a stable oscillatory solution as long as \( \Omega_{V}^2 > 0 \), that is, from Eq. (29),

\[ n > R^2 k_z / \beta pc \]  \hspace{1cm} (30)

For typical beam parameters, this condition is satisfied for positive and very large values of the field index \( n \).

6. RADIAL-AZIMUTHAL COUPLING. NEGATIVE MASS INSTABILITY.

The first two equations of motion (25 and 26) show coupling between the azimuthal and radial components of motion. They can be solved with the conventional method of eigenvalues by assigning an angular oscillation frequency \( \Omega_0 \) common to the two components of motion. The following equation for \( \Omega_0 \) is derived [8]

\[ \Omega_0^4 - M \Omega_0^2 + \left( \Omega_H^2 - \omega_0^2 \right) \omega_0 k_\theta / p = 0 \]  \hspace{1cm} (31)

where \( M = \Omega_H^2 + \omega_0 k_\theta / p \). Solving Eq. (31) for \( \Omega_0^2 \) gives

\[ 2 \Omega_0^2 = M \pm \sqrt{M^2 - 4 \left( \Omega_H^2 - \omega_0^2 \right) \omega_0 k_\theta / p} \]  \hspace{1cm} (32)

For the motion to be stable \( \Omega_0^2 > 0 \) is required, that is \( \Omega_H^2 > \omega_0^2 \) which yields for the field index

\[ 1 - n > 1 + R^2 k_z / \beta pc \]  \hspace{1cm} (33)
In order to satisfy Eq. (33) the field index $n$ has to take very large negative values. It is not possible that the conditions of Eqs. (30) and (33) can be satisfied simultaneously. It derives that the Equilibrium Configuration of a Circular String is not a Stable Configuration that can be realized in a simple Betatron Magnet. This seems to be in agreement with the results of Wei, Li and Sessler [4].

7. Storage Ring with Alternating Focussing Gradient

Let $r_0 = 1.535 \times 10^{-18}$ m be the classical proton radius and $A$ the mass number of a particle, and

$$n_0 = \frac{R^2 k_z}{\beta pc}$$  \hspace{1cm} (34)

Then the stability condition for the axial component of the motion is, from Eq. (30),

$$n > n_0$$  \hspace{1cm} (35)

whereas the stability condition for the motion on the $z = 0$ plane is, from Eq. (33),

$$n < - n_0$$  \hspace{1cm} (36)

This suggests that a solution can be found with an alternating gradient storage ring where the field index $n$ takes very large values and changes sign periodically.

We shall apply these results to the case of a beam of ions of gold partially stripped ($A = 197, Q = 51$) is injected into a small storage ring from the Tandem-ALPI complex of the Laboratori Nazionali di Legnaro [3] with the magnetic rigidity of $1.2$ T-m ($\beta = 0.1$). We derive $n_0 = 4.9 \times 10^{-18}$ N$^3$. It is seen that already for $N = 10^7$ particles, the field index gets too large value. Not only this may be very difficult to achieve in practice, but also it causes concern about the stability of the storage ring lattice in the limit of a single particle.

We shall assume that the storage ring is made of $n_s$ superperiods, each with a pair of combined function magnets of alternating gradient sign, that is having field index $-n + n$ respectively. Neglecting the presence of drift spaces, the motion of a single particle in the lattice is stable if the following condition is satisfied

$$| \cos \zeta \cosh \zeta | < 1$$  \hspace{1cm} (37)

where $\zeta = \alpha \sqrt{n}$ and $\alpha = \pi / n_p$ is the bending angle per magnet. The stability condition (37) can be satisfied by choosing $\zeta = (2h + 1) \pi / 2$ where $h$ is an integer. This condition is equivalent of setting a betatron phase-advance per superperiod to about 90°.

The relations above combined set a relation between the periodicity of the storage ring and the beam intensity since the critical field index depends with the number $N$ of particles, and one requires $n > n_{cr}$.

$$n_p \sim 4.4 \times 10^{-9} N^{3/2} / (2h + 1)$$  \hspace{1cm} (38)

Small values of $h$ are desirable, since for larger values the stability interval reduces considerably. It is then seen that a large periodicity is needed. For instance for $N = 10^7$ particles $n_p \sim 140 / (2h + 1)$, and a periodicity $n_p = 8$ can be obtained by choosing $h = 8$.

REFERENCES