Kinetic Energy
and the Equivalence Principle

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Abstract

According to the general theory of relativity, kinetic energy contributes to gravitational mass. Surprisingly, the observational evidence for this prediction does not seem to be discussed in the literature. I reanalyze existing experimental data to test the equivalence principle for the kinetic energy of atomic electrons, and show that fairly strong limits on possible violations can be obtained. I discuss the relationship of this result to the occasional claim that “light falls with twice the acceleration of ordinary matter.”

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The principle of equivalence—the exact equality of inertial and gravitational mass—is a cornerstone of general relativity, and experimental tests of the universality of free fall provide a large set of data that must be explained by any theory of gravitation. But the implication that energy contributes to gravitational mass can be rather counterintuitive. Students are often willing to accept the idea that potential energy has weight—after all, potential energy is a rather mysterious quantity to begin with—but many balk at the application to kinetic energy. Can it really be true that a hot brick weighs more than a cold brick?

General relativity offers a definite answer to this question, but the matter is ultimately one for experiment. Surprisingly, while observational evidence for the equivalence principle has been discussed for a variety of potential energies, the literature appears to contain no analysis of kinetic energy. The purpose of this paper is to rectify this omission, by reanalyzing existing experimental data to look for the “weight” of the kinetic energy of electrons in atoms. I will then try to reconcile the results with the occasional (and not completely unreasonable) claim that “objects traveling at the speed of light fall with twice the acceleration of ordinary matter.”

1 The Equivalence Principle and Internal Energies

Modern tests of the principle of equivalence begin with variations of Galileo’s apocryphal free fall experiment, in which the gravitational accelerations of two objects with different compositions are compared. The acceleration due to gravity is proportional to the ratio \( \frac{m_G}{m_I} \) of gravitational to inertial mass, and violations of the principle of equivalence would manifest themselves as differences in acceleration, or “nonuniversality of free fall.” Some experiments\(^1\)–\(^3\) directly measure free fall; others\(^4\)–\(^6\) use torsion pendulums to compare centrifugal acceleration to the acceleration due to gravity. Galileo’s pendulum experiments\(^7\) may have already achieved accuracies of about \( 2 \times 10^{-3} \), and since the pioneering work of Eötvös,\(^8\) modern experiments have pushed uncertainties down to about \( 10^{-12} \).

These experiments directly test the equality of gravitational and inertial mass for a variety of substances. But because different materials have different compositions, the experiments also test the principle of equivalence for various forms of internal energy. For example, the inertial mass of an iron nucleus is about 1% less than the inertial masses of its constituent protons and neutrons, largely because of the (negative) contribution of nuclear binding energy. If this binding energy did not also affect gravitational mass, the ratio \( m_G/m_I \) for iron would be greater than one, and iron would fall more rapidly than, for example, hydrogen.

A convenient measure of potential violations of the equivalence principle is the Eötvös ratio\(^9\)

\[
\eta(A, B) = \frac{m_G(A)}{m_I(A)} - \frac{m_G(B)}{m_I(B)},
\]

where \( m_G \) and \( m_I \) are the gravitational and inertial masses of two materials \( A \) and \( B \). Typical experimental limits, which we shall use later, are

\[
\eta(\text{Be}, \text{Cu}) = (-1.9 \pm 2.5) \times 10^{-12}
\]

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for beryllium and copper\textsuperscript{6} and

\[ \eta(\text{Al}, \text{Pt}) = (-0.3 \pm 0.9) \times 10^{-12} \]  \hspace{1cm} (3)

for aluminum and platinum.\textsuperscript{5} Now, the inertial mass of a material includes a
number of contributions—rest mass, nuclear binding energy, electrostatic energy,
the kinetic energy of constituents, etc. If any of these internal energies \( E_\alpha \)
were to violate the principle of equivalence, one could write, to lowest order in the energies,

\[ m_G = m_I + \sum_\alpha \eta_\alpha \frac{E_\alpha}{c^2}, \]  \hspace{1cm} (4)

where the \( \eta_\alpha \) are dimensionless parameters that measure the strength of the vio-
lation. From (1) and (4), we see that

\[ \eta(A, B) = \sum_\alpha \eta_\alpha \left( \frac{E_\alpha(A)}{m_I(A)c^2} - \frac{E_\alpha(B)}{m_I(B)c^2} \right). \]  \hspace{1cm} (5)

In the absence of fortuitous and fine-tuned cancellations among various types of
energy, observational limits on \( \eta(A, B) \) can thus be used to obtain limits on the
parameters \( \eta_\alpha \).

Expression (4) is, of course, an oversimplification: in practice, we can rarely
separate the internal energy of a material so cleanly into its components. The
binding energy of protons and neutrons in the nucleus, for example, is an important
part of the energy of an atom, typically amounting to a bit less than 1\% of the
mass. But protons and neutrons are made up of quarks, and according to quantum
chromodynamics, nuclear binding energy is merely a remnant of the interaction
energy of quarks and gluons. Should the binding energy in (4) include only the
energy of nucleons, or should we add quark interactions? Should we consider the
kinetic energy of nucleons, or of the quarks and gluons they comprise? What if
quarks themselves have constituents?

Clearly, a complete decomposition of the form (4) would require a complete
understanding of the physics of the materials we are studying. Nevertheless, we
can obtain important, if incomplete, information by isolating a few types of energy
that are relatively well understood, making the assumption that there will be
no precise cancellations between these contributions and those of the energies
we ignore. Without understanding the details of quantum chromodynamics, we
cannot make sweeping statements about the equivalence principle for all binding
energy; but we can draw conclusions about the particular contribution of the
binding energy of nucleons in the atomic nucleus, which is understood empirically
to a rather high accuracy.\textsuperscript{10} Similarly, when we consider kinetic energy below, we
shall not attempt to analyze the kinetic energy of quarks, or even nucleons, but
shall instead focus on the well-understood physics of atomic electrons.

Note that even for well-understood components of internal energy, the energy
of a material is often not directly observable, but must be modeled theoretically.
We cannot, for example, dismantle the Earth to measure its gravitational binding
energy, or catch a single atomic electron to measure its kinetic energy while it is
still in the atom. Often, however, there are useful internal checks on computed
energies. In particular, the virial theorem provides an important consistency check, and can sometimes be used to relate the energies in which we are interested to directly observable quantities. Recall that for a nonrelativistic bound system of \( n \) particles at positions \( \mathbf{r}_i \), having total kinetic energy \( T \) and interacting through a potential \( U(\mathbf{r}) \), the virial theorem states that

\[
\langle T \rangle = \frac{1}{2} \sum_{i=1}^{n} \langle \mathbf{r}_i \cdot \nabla U \rangle ,
\]

where the angle brackets denote a time average. In particular, for an electromagnetically bound system, \( U(\mathbf{r}) \sim \sum q_i/|\mathbf{r} - \mathbf{r}_i| \), and it is not hard to show that

\[
T = -U/2 = -E,
\]

where \( U \) is the electrostatic potential energy and \( E \) is the total (kinetic plus potential) energy. We shall use this relation below to check computed values of kinetic energy for atomic electrons.

Using an analysis based on equation (5), physicists have obtained strong limits on violation of the equivalence principle by strong interaction energy in nuclei, electrostatic and magnetostatic energy in nuclei, the energy of hyperfine interactions, and weak interaction energy.\(^9\) The electrostatic binding energy of electrons in atoms has also been briefly discussed.\(^{11}\) A somewhat different method, which uses Lunar laser ranging to compare the accelerations of the Earth and the Moon, has led to limits on violations of the equivalence principle by gravitational binding energy.\(^{12}\)

### 2 The Case of Kinetic Energy

Our interest in this article is kinetic energy, which has, surprisingly, not yet been analyzed in the literature. Most of the kinetic energy of an atom resides in the nucleus: a typical nucleus of atomic number \( A \) has a radius \( R \sim R_0 A^{1/3} \) with \( R_0 \sim 1.3 \times 10^{-15} \text{m} \),\(^{10}\) and the uncertainty principle yields an estimate \( T_{\text{nuc}}/m c^2 \sim 10^{-2} A^{-2/3} \). This argument only sets a lower bound on the kinetic energy, however, and may not give the actual dependence on \( A \); to use equation (5), we would need a much more sophisticated and model-dependent calculation. For a system described by a simple enough potential, the virial theorem relates the kinetic energy to the (observable) binding energy. In the nucleus, however, several types of potential energy compete, and the virial theorem does not give a unique separation of energies. A decomposition like that of equation (4) is therefore problematic; we do not know enough to distinguish the kinetic energy from other contributions.

For electrons in atoms, on the other hand, these problems largely disappear. Accurate computations of electron kinetic energies are now standard in condensed matter physics, and while exact solutions of the many-body Schrödinger equation are not known, well-understood and well-tested approximations are readily available. Moreover, atomic electrons are bound solely by electromagnetic interactions, and the virial theorem may be used to check computed kinetic energies against observed binding energies. This simplicity comes at a price: electron kinetic energy
is only a small part of total energy, $\frac{T_{\text{elec}}}{m_1c^2} \sim 10^{-7}$, and experiments are thus less sensitive to possible violations of the equivalence principle. Nevertheless, the existing data are accurate enough to provide a good test in this relatively clean system.

The basic physics we wish to explore is fairly simple. The electrons in an atom with high atomic number are, on average, more tightly bound than those in an atom with low atomic number. Their kinetic energy is consequently greater, and constitutes a greater proportion of the total energy of the atom. If, as an extreme example, this kinetic energy had no weight, a high-Z atom would fall measurably more slowly than a low-Z atom.

To obtain quantitative predictions, we need to determine the kinetic energy of atomic electrons. We can begin with the simple Thomas-Fermi model for many-electron atoms,\textsuperscript{13} which treats the electrons statistically as a Fermi gas and uses semiclassical methods to determine their characteristics. In this approximation, a typical electron in an atom of atomic number $Z$ is located roughly $Z^{-1/3}$ Bohr radii from the nucleus. The electrostatic energy of one such electron is proportional to $Z/Z^{-1/3} = Z^{4/3}$; the energy of $Z$ electrons thus goes as $Z^{7/3}$. By the virial theorem (7), the kinetic energy should have the same form, $T \sim cZ^{7/3}$. From the observed ground state energy of the hydrogen atom, we can estimate $c$ to be on the order of 10 eV. The ratios $\frac{T_{\text{elec}}}{m_1c^2}$ in equation (5) thus range from about $10^{-6}$ for platinum to $10^{-8}$ for beryllium, much larger than the limits (2)–(3) for violation of the equivalence principle.

To find more precise results, we could numerically integrate the Thomas-Fermi model to determine the coefficient $c$. But it is almost as easy to employ the much more accurate numerical approaches that are now widely used in condensed matter physics. Consider, for example, the kinetic energies of atomic electrons in beryllium and copper. These can be computed in the local density approximation\textsuperscript{14} (for a review of this method, see reference 15), using standard and widely available computer codes. For isolated atoms, one finds that

$$\frac{T_{\text{elec}}}{m_1c^2} (\text{Be}) = 4.6 \times 10^{-8}$$

$$\frac{T_{\text{elec}}}{m_1c^2} (\text{Cu}) = 7.7 \times 10^{-7}. \quad (8)$$

To check these numbers, we can appeal to the virial theorem (7) for electromagnetically bound systems, which allows us to compare the energies in equation (8) to published values of total energies as computed in the Hartree-Fock approximation\textsuperscript{16} and the local density approximation.\textsuperscript{17} The results agree to within 2%. Better yet, the total energy $E$ can be measured directly—it is the ionization potential, the energy required to totally ionize an atom—and the kinetic energies (8) can be compared to these observations. We again obtain agreement to within 3% for beryllium\textsuperscript{18} and 1% for copper.\textsuperscript{19} One might worry that our computations were performed for isolated atoms, while the experimental test of reference 6 used solid metallic beryllium and copper. But the relevant energy differences, the cohesive energies of the metals, are only a few eV per atom, completely negligible for our purposes.
We can now combine equations (2), (5), and (8), assuming as usual that no precise cancellations occur among different forms of energy. We obtain a limit

\[ |\eta_T| < 6 \times 10^{-6} \]  

for violation of the equivalence principle by the kinetic energy of electrons.

A stronger, although less theoretically certain, limit can be obtained by comparing aluminum and platinum. The local density approximation now gives

\[
\frac{T_{\text{elec}}}{m_e c^2} (\text{Al}) = 2.6 \times 10^{-7} \\
\frac{T_{\text{elec}}}{m_e c^2} (\text{Pt}) = 3.3 \times 10^{-6}.
\]

(10)

For aluminum, these figures are again in good agreement with the theoretical\textsuperscript{16,17} and experimental\textsuperscript{20} results for total energy. For platinum, however, no experimental results appear to be available, and the Hartree-Fock expression for total energy\textsuperscript{16} differs from the local density approximation for kinetic energy by about 15%. The difference may indicate a problem with the computed value; numerical errors in the code used for these calculations are likely to be significant for core electrons in high-Z atoms. In the absence of a better estimate of theoretical uncertainties, let us double this difference, and assume conservatively that the calculation (10) for platinum is accurate to within 30%. We can then combine equations (3), (5), and (10) to obtain

\[ |\eta_T| < 6 \times 10^{-7}. \]  

(11)

The limits (9) and (11) are several orders of magnitude weaker than the corresponding results for nuclear energies. Nevertheless, they are surprisingly strong, and may be counted as good evidence that the equivalence principle holds for kinetic energy.

3 General Covariance and the Weight of Light

The results of the preceding section will come as no surprise to experts in relativity. But perhaps they should. We have another way of “weighing” kinetic energy: we can send a beam of particles past a large mass (the Sun, say) and see how it is deflected. It is well known that the deflection of light is twice that predicted by Newtonian theory; in this sense, at least, light falls with twice the acceleration of ordinary “slow” matter.

Indeed, the general relativistic deflection for a test particle with an arbitrary velocity \( v \) and a large enough impact parameter \( b \) is\textsuperscript{21,22}

\[ \theta = \frac{2GM}{bv^2} \left( 1 + \frac{v^2}{c^2} \right). \]  

(12)

The corresponding angle in Newtonian gravity depends on the ratio of gravitational to inertial mass, and it is easy to check that equation (12) is just the
Newtonian result for a particle with an inertial mass $m_I$ and a gravitational mass $m_G = m_I \left(1 + \frac{v^2}{c^2}\right) = m_I + 2\frac{T}{c^2}$. \hspace{1cm} (13)

For light, the kinetic energy $T$ in this expression should be replaced by the electromagnetic energy $U$, which can be loosely interpreted as the “kinetic energy” of photons.

This argument does not seem to be widely published; instead, many texts rely on a simple “Einstein elevator” analysis that actually gives only half the correct deflection. But it is not uncommon among students, and appears frequently in Internet discussions. In such a simple form, the argument is easily addressed: it is attempting to impose Newtonian categories on general relativity, ignoring in particular the curvature of space. To obtain a value for the deflection of light by the Sun we must at least implicitly compare measurements of direction in widely separated regions of space, and to perform such a comparison correctly, we need to take into account the curvature of space between these two regions.

There is a slightly more sophisticated version of this argument that is harder to dismiss, however. Rather than sending a beam of light past the Sun, let us imagine confining the beam to a mirrored box near the Sun, thus avoiding the problem of comparing distant frames of reference. If the deflection (12) truly reflects the “weight of kinetic energy,” a light beam with energy $U$ should contribute an amount $2U$ to the gravitational mass of the box.

We can analyze this situation in the weak field approximation to general relativity. In this limit, the metric is $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, with

$$h_{00} = 2\phi, \quad h_{ij} = 2\phi \delta_{ij}, \hspace{1cm} (14)$$

where $\phi$ is the Newtonian gravitational potential. The gravitational coupling to a test body with a stress-energy tensor $T^{\mu\nu}$ is thus\(^1\)

$$\frac{1}{2} \int h_{\mu\nu} T^{\mu\nu} d^3x \approx \int \phi \left(T^{00} + \delta_{ij} T^{ij}\right) d^3x. \hspace{1cm} (15)$$

For a slowly moving particle with rest mass $m$ and kinetic energy $T$, we have

$$\int T^{00} d^3x \approx mc^2 + T, \quad \int \delta_{ij} T^{ij} d^3x \approx 2T \hspace{1cm} (16)$$

to lowest order in velocity. For an electromagnetic field with energy $U$, on the other hand—or for a “photon” of energy $U$ moving at light speed—the stress-energy tensor is traceless, so\(^2\)

$$\int T^{00} d^3x \approx U, \quad \int \delta_{ij} T^{ij} d^3x \approx U. \hspace{1cm} (17)$$

These expressions may be checked by considering a gas of particles (or photons) in a volume $V$, for which the diagonal spatial components $T^{ii}$ of the stress-energy\(^3\)

\(^1\)The coupling (15) can be obtained from the action $S[\psi, g]$ for matter in curved spacetime by expanding the metric around its flat Minkowski value and noting that $\delta S = \frac{1}{2} \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}$.\(^2\)\(^3\)
tensor are equal to the pressure \( p \). In that case, (16)–(17) may be recognized as the standard result that

\[
pV = (\gamma - 1)E
\]

where \( E \) is the total (kinetic) energy and the coefficient \( \gamma \) is \( 5/3 \) for nonrelativistic particles, \( 4/3 \) for the extreme relativistic case.

Our “box of light" consists of slowly moving walls with energies of the form (16) and a beam of light with energy of the form (17). For this system—or for any other system consisting of electromagnetic fields and matter—we thus have

\[
\int T^{00} d^3 x \approx mc^2 + T + U,
\]

\[
\int \delta_{ij} T^{ij} d^3 x \approx 2T + U.
\]

If \( \phi \) is nearly constant over the system, equation (15) can thus be interpreted as a coupling of the Newtonian potential to the combination \( mc^2 + 3T + 2U \). For a free beam of light, the first two terms are absent, and this analysis yields a gravitational deflection of twice the Newtonian value, as desired. But comparing section 1, we seem to have found Eötvös parameters \( \eta_T = 2 \) and \( \eta_U = 1 \), in gross violation of the equivalence principle.

Of course, the coupling (19) does not really lead to a disagreement with experiment: we are saved by the virial theorem. Equation (7) was derived for a system of nonrelativistic charged particles coupled by Coulomb interactions. But the relativistic derivation of reference 29 shows that the same relation holds for an arbitrary spatially bounded system of electromagnetically interacting particles and electromagnetic radiation, as long as \( U \) is now understood to be the total electromagnetic energy. For our “box of light,” \( 2T + U \) therefore vanishes, and \( 3T + 2U = T + U = E \). The apparent violation of the equivalence principle has thus rather mysteriously disappeared.

Such an exact cancellation should have a fundamental explanation. In general relativity, it is a consequence of general covariance. Consider a small electromagnetically bound “test body” in a gravitational field, with center of mass \( \bar{x}_i \). Let us start with the expression (15) and perform a coordinate transformation

\[
x^i \rightarrow x^i' = x^i + \phi(\xi(x))(x^i - \bar{x}_i),
\]

where \( \xi \) is any function that is constant inside the test body and falls rapidly to zero outside. In the weak field approximation, the fields \( h_{\mu\nu} \) transform as

\[
h_{\mu\nu} \rightarrow h_{\mu\nu} - \eta_{\mu\rho} \partial_\rho \delta x^\rho - \eta_{\nu\rho} \partial_\rho \delta x^\rho,
\]

and it is easy to check that the coupling (15) becomes

\[
\phi[mc^2 + T' + U' + (1 - \xi(0))(2T' + U')] + \text{higher order terms},
\]

where \( T' \) and \( U' \) are the kinetic and electromagnetic energies in the new coordinate system. The coupling to \( 2T + U \) can thus be altered arbitrarily, and indeed “gauged away,” by a change of coordinates inside the test body. This argument is closely related to our earlier appeal to the virial theorem: the relativistic virial theorem can be derived from conservation of the stress-energy tensor,29 which is in turn a consequence of general covariance in the weak field limit.
Put another way, we have learned that the determination of gravitational mass from constituent energies is not entirely coordinate-independent. But the physical interaction with gravity cannot depend on a choice of coordinates: the coordinate-dependent part of the interaction must vanish. That it does is guaranteed by the virial theorem, and indeed, this argument can be viewed as a derivation of the virial theorem. We have also seen that within the framework of general relativity, coordinates can be chosen to show explicitly that the inertial and gravitational masses of a bound system are equal.

We focused so far on electromagnetic energy. But the coupling (22) illustrates a general ambiguity in determining the parameters $\eta_\alpha$. In any bound system, the virial theorem will require that some linear combination of energies vanish, and that some linear combination of the $\eta_\alpha$ therefore be unmeasurable. For an electromagnetically bound system, for instance, a term in equation (4) of the form $\xi(2T + U)/c^2$ is inherently unobservable.

How can we deal with this ambiguity? One answer is to compare systems with different interactions and different internal energies. For any single system, the virial theorem will give a relationship among energies, and one combination $\sum c_\alpha \eta_\alpha$ will be undetermined. But the appropriate combination will differ from system to system. Moreover, for systems with interactions described by nonpolynomial potentials, it is evident from equation (6) that the coefficients $c_\alpha$ will depend not only on the internal energies, but on the dynamics—the average values of the positions $\mathbf{r}_i$—as well. Observations of nuclei, for example, cannot detect a gravitational coupling to the combination $2T + U_{\text{elec}} + cU_{\text{nuclear}}$, where $c$ depends on mean distances between nucleons.

If the coefficients $\eta_\alpha$ have any universal significance, we can now combine the limits coming from the energy content of nuclei with those coming from atomic electrons to obtain information about $\eta_T$ alone. In particular, the electrostatic Eötvös coefficient $\eta_U$ in nuclei satisfies

$$|\eta_U| < 4 \times 10^{-10},$$

up to an ambiguous coupling that is different from the ambiguous coupling for electrons. It seems safe to assume that there should be no perverse cancellation between the gravitational couplings of, say, electron kinetic energy in beryllium and nuclear binding energy in platinum. Equation (23) can thus be combined with our earlier analysis to remove any “virial theorem ambiguities” in the limits (9) and (11) for kinetic energy.

We can thus tell our students with confidence that kinetic energy has weight, not just as a theoretical expectation, but as an experimental fact.

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