Chiral fermions on the lattice *

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Chiral fermions resisted being put on the lattice for twenty years. This raised the suspicion that asymptotically free chiral gauge theories were not renormalizable outside perturbation theory and therefore could not be mathematically extended to infinite energies. During the last several years the situation has reversed itself. Today we believe that all the essential ingredients for a full lattice definition of non-anomalous chiral gauge theories are in place within the overlap construction. This construction is based on earlier work by Callan and Harvey, by Kaplan and by Frolov and Slavnov. It can be reinterpreted as coming from the Ginsparg-Wilson relation, but, at the moment, it is a unique construction and therefore might also be the way by which Nature itself regularizes chiral fermions. This is yet another instance in which lattice field theory makes a potentially important contribution to fundamental particle physics.

1. Introduction

My talk is divided into three parts. In the first part the relevance of the chiral fermion issue to fundamental particle physics and to numerical QCD will be explained. The second part is the bulk part of my talk and will present the main ideas and properties of the “overlap”. I shall restrict myself to the period from 1992 to one year ago, summer 1998, the time of the lattice 98 conference, held in Boulder, Colorado. Presumably the next plenary speaker on chiral fermions will focus on the last year, between summer 1998 and summer 1999, so overlap will be avoided. In the last part of my talk I shall try to convince you that these new developments present many opportunities for fresh ideas.

My main partner in the overlap construction was R. Narayanan. I have also collaborated with P. Huet, Y. Kikukawa, A. Yamada and P. Vranas. Important contributions to the overlap development were made by Randjbar-Daemi and J. Strathdee. More recently, the pace of developments has picked up, mainly in the vector-like context, and beautiful work has been done by R. Edwards, U. Heller, J. Kiskis, by Ting-Wai Chiu and by Keh-Fei Liu and his collaborators. New results are coming out almost daily, but this is material for the next lattice conference.

2. Relevance

The minimal standard model (MSM) works well experimentally, is a chiral gauge theory and constitutes a good effective low energy description of the theory of everything (TOE). The relatively varied $U(1)_Y$ charges of the MSM reflect the chiral nature of the MSM by assuring anomaly free couplings to the gauge bosons and gravity. These charges fit snugly into representations of larger groups leading to $SU(5)$ and $SO(10)$ grand unified theories (GUT). All these new gauge theories are also chiral. Supersymmetric extensions of the MSM or of the GUTs also must be chiral. By normal physics standards, the TOE is best described as unknown at present. It seems likely that it is not an ordinary field theory as it contains gravity. It could be the case that in the TOE there is no well defined concept of chirality.

The most basic question about chirality was asked by Holger-Bech Nielsen many years ago[1]. I paraphrase it to: Is a chiral gauge theory completely isolated from ultra-violet effects? In other words is it truly renormalizable? In yet another equivalent form the question is: Is the chiral nature of the theory compatible with an arbitrarily large scale separation between typical scales and new physics scales?

We know very well that the answer to the above question is “yes” in perturbation theory to any order. But, outside perturbation theory, lattice difficulties have raised the suspicion that the answer

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might be “no”. This was the situation from the early 1970s to the early 1990s. The main achievement I am reporting on occurred during the period 1992 to 1997 and amounts to replacing the suspected “no” by an almost compelling “yes”.

Accepting the “yes” from now on, the next question one should ask is: What can we learn about Nature from the lattice difficulties and from their resolution?

A mathematician might answer that we learn nothing because Nature isn’t a lattice. This physicist’s answer is different: If in Nature there were an infinite number of fermions per unit four volume, chirality at low energies can emerge naturally, without fine tuning. The TOE could “know” nothing about chiral gauge theories. New mechanisms in the TOE produce the appropriate set of light degrees of freedom in a natural way. Chiral gauge theories appear because they are the single consistent interaction between these light degrees of freedom in the long distance limit.

So, lattice field theory may have made a contribution to the understanding of one of the most fundamental issues in theoretical particle physics. This would certainly not be the first time, but is worth keeping in mind in the social climate of today’s particle theory.

An important spinoff of the developments on chiral gauge theories is feeding back into our own subfield. For the first time we know, in principle and also in practice (to some degree), that numerical QCD can treat global chiral symmetries exactly, on the lattice. This subfield is rapidly expanding and I am sure more will happen over the next few years.

3. Overlap essentials

3.1. Infinite number of fermions

If the number of fermions is infinite the theory is not precisely defined (yet). This provides an opportunity to “cheat”:

We can write down a theory that looks vectorial, but could equally well be viewed as chiral. Suppose we have a string of right-handed Weyl fermions stretching from $-\infty$ to $+\infty$ and below it another string, now made out of left-handed fermions. We can think about the different fermions being labeled by a new discrete flavor index increasing along the strings. If we pair right-handed with left-handed Weyl particles starting from the middle of the strings outwards, we conclude that we have an infinite number of Dirac fermions and the theory is vector-like. However, suppose we start pairing from both infinite ends inwards. One could easily make a “mistake” and end up with, say, one unpaired Weyl fermion. Now the theory looks chiral.

We are restricting the action to be bilinear in the fermion fields. The action for a multiplet of Weyl fermions would be $\bar{\psi} W \psi$, where $W$ is the Weyl operator in a background gauge field. We shall always assume we are working on a compact Euclidean manifold (it makes no sense to have to worry also about infrared issues, the thermodynamic limit and so forth - we have our plate full already). We know that the gauge fields over a compact manifold fall into distinct “blobs”, each labeled by the topological charge $Q\{U\}$, where $U$ represents the gauge background in lattice notation although at this point we are still in the continuum. Moreover, we know that $W$ is structurally affected by $Q\{U\}$ via the famous Atiyah-Singer index theorem, which, in a loose but quite sensible sense means that \((\text{the number of rows of } W)-(\text{the number of columns of } W)=Q\{U\}\). Clearly, one cannot just write down a finite size matrix $W$ with such a property. But, if the size of $W$ is infinite it is at least conceivable that the Atiyah-Singer index theorem will hold on the lattice since, after all, $\infty - \infty$ can be anything.

It is easy to imagine writing down a formula for $W$ which is gauge covariant. This means that replacing the background gauge field by a gauge transform is equivalent to conjugation by a formally unitary matrix dependent only on the gauge transformation. Thus, $\det W$ would be gauge invariant. But, since $W$ is infinite, $\det W$ isn’t well defined and it is conceivable that gauge invariance does not hold since the manipulations involving the determinant of an infinite matrix, the product of infinite matrices and the only formal unitarity of some of these may end up being incorrect. This creates an opportunity for anomalies to enter, an opportunity we are obligated to create one way or another. It is important to see
that anomalies can show up in a way that is independent of taking the ultraviolet cutoff to infinity or the infrared cutoff to zero.

Thus, postulating an infinite number of fermions creates the right openings, and the problem becomes only how to "cheat" honestly.

3.2. Brief history of chiral issues

In the late sixties Adler, Bell, Jackiw and Bardeen discovered and understood anomalies in the context of particle physics [2]. The importance of this discovery cannot be overstated. Starting from the mid seventies Stora, Zumino [3] and others unveiled the beautiful mathematical structure of anomalies. At the algebraic level the very elegant descent equations were seen to relate anomalies in various dimensions. During the early to mid eighties the understanding of anomalies was enriched by discovering their topological meaning. It is fair to say that during this period the physics of fermions in a classical gauge background was put on firm (and elegant) mathematical grounds [4].

For what follows, a crucial step was taken by Callan and Harvey [5] who provided a physical realization of the relation between anomalies in different dimensions (as algebraically reflected in the descent equations). They connected the convective dimensions in the descent equations by studying physical embeddings in a given manifold of sub-manifolds ("defects") of lower dimension. Prompted by this work, Boyanowski, Dagotto and Fradkin [6] studied similar arrangements in condensed matter. They also proposed that the famous chiral fermion problem of lattice field theory might be solved this way. But, they did not pursue their insight and our community paid no attention.

The situation changed in 1992 when Kaplan [7], again motivated by the work of Callan and Harvey, made a very specific and compelling case that the setup Callan and Harvey used could be put on the lattice and that this was a new way to deal with the lattice fermion problem. During the same year, starting from a completely different point of view, Frolov and Slavnov [8] made a proposal containing an infinite number of auxiliary fields to regulate $SO(10)$ gauge theory with a 16-plet of left-handed Weyl fermions. This is a chiral gauge theory and each irreducible matter multiplet can accommodate one family of the MSM plus one additional $SU(3) \times SU(2) \times U(1)$-neutral right-handed neutrino.

Narayanan and I asked whether the two ways, one by Kaplan, and the other by Frolov and Slavnov had anything in common. Our conclusion was that they did and the heart of the matter in both cases was that although regulators were present, the systems had an infinite number of fermions per unit Euclidean four-volume [9]. This led to the chiral overlap.

We spent several years testing and convincing ourselves that indeed the unbelievable had happened and the lattice fermion problem had been laid to rest [10–12]. Luckily, we were not entirely alone doing this [13]. Obviously, if the chiral gauge theory problem was solved one also had a way to exactly preserve global chiral symmetries in vector-like theories, like QCD [11]. How competitive this was when compared to standard numerical QCD was a question we postponed. Eventually, focusing on QCD, a simplification of the overlap in the case of vector-like gauge theories was found producing a relatively simple formula for the fermion kernel for lattice Dirac fermions with exact global chiral symmetry [14].

Forgotten for many years was a prophetic paper by Ginsparg and Wilson in which the nowadays well known GW relation for the kernel of a vector-like theory was written down [15]. Ginsparg and Wilson showed that if the kernel obeyed their relation one would have both exact global chiral symmetries and the $U(1)$ anomaly, directly on the lattice. They arrived at their relation being motivated by Renormalization Group ideas. In four dimensions a vector-like gauge theory would be classically conformally invariant if matter were massless, which in the fermion case also implies chiral symmetry. Since the Renormalization Group is built to extract the anomalous realization of dilations in the quantum context, it was a natural place to start. The Renormalization Group provided them with a rather formal derivation of their relation. To be sure, the GW relation itself, and its consequences were however well defined and clear. But, they could not find
an explicit kernel satisfying their relation in the presence of a gauge background, and probably because of this the idea was forgotten. It was a pleasant surprise to realize that the overlap Dirac operator relevant to the vector-like case satisfied the GW relation [16,17]. Of course, in the context of the overlap construction it was obvious a priori that one had global chiral symmetries and exactly massless quarks. But, that the GW relation would turn out to be satisfied was not built in as an explicit requirement in the construction.

Very soon after the overlap Dirac operator was shown to obey the GW relation, Narayanan [18] showed that the derivation that took one from the chiral overlap to the vector case by combining a left handed fermion with a right handed one could be reversed. Narayanan factorized the overlap Dirac operator using only that the latter obeyed the GW relation and therefore it became possible to start from the GW relation and get to the chiral case by factorization. No new lattice results have been obtained starting from the GW relation directly, although some derivations can be made to look more familiar. Later on, I shall explain in greater detail the connection between the overlap and the GW relation.

Let me note some important properties of the overlap approach to chiral fermions. It works in any even dimension both for gauge and for gravitational backgrounds. It is independent of the renormalizability of the dynamics of the background. As such, the overlap is not Renormalization Group motivated, although Ginsparg and Wilson were. There is no meaningful GW relation in odd dimensions. However, there are global issues in odd dimensional gauge theories with massless fermions that are intimately related to chiral fermions. These issues are captured in the overlap approach which extends naturally to odd dimensions.

3.3. The basic idea

\( \mathcal{L}_\psi = \bar{\psi} D \psi + \bar{\psi} (P_L \mathcal{M} + P_R \mathcal{M}^\dagger) \psi \) \hspace{1cm} (1)

\( \bar{\psi} \) and \( \psi \) are Dirac and the mass matrix \( \mathcal{M} \) is infinite. It has a single zero mode but its adjoint has no zero modes. This were impossible if the mass matrix were finite. It clearly means we have one massless Weyl fermions whose handedness can be switched by interchanging the chiral projectors. It is very important that as long as \( \mathcal{M} \mathcal{M}^\dagger > 0 \) this setup is stable under small deformations of the mass matrix. This stability comes from the internal supersymmetric quantum mechanics generated by the mass matrix.

Kaplan’s domain wall suggests the following realization:

\( \mathcal{M} = -\partial_s - f(s) \) \hspace{1cm} (2)

where \( s \in (-\infty, \infty) \) and \( f \) is fixed at \( -\Lambda' \) for negative \( s \) and at \( \Lambda \) for positive \( s \). There is no mathematical difficulty associated with the discontinuity at \( s = 0 \). Originally, the \( s \)-variable was discretized, but this is unnecessary.

Before proceeding let me remark that other realizations of \( \mathcal{M} \) ought to be possible, but nothing concrete has been worked out up to now.

The infinite path integral over the fermions is easily “done”: on the positive and negative segments of the real line respectively one has propagation with an \( s \)-independent “Hamiltonian”.

The infinite extent means that at \( s = 0 \) the path integrals produce the overlap (inner product) between the two ground states of the many fermion systems corresponding to each side of the origin in \( s \). The infinite extent also means infinite exponentials linearly proportional to the respective energies - these factors are subtracted. One is left with the overlap formula which expresses the chiral determinant as \( \langle v'\{U\}|v\{U\} \rangle \). The states are in second quantized formalism. By convention, they are normalized, but their phases are left arbitrary. This ambiguity is essential, as we shall see later on. It has no effect in the vector-like case. In first quantized formalism the overlap is:

\( \langle v'\{U\}|v\{U\} \rangle = \det_{k'k} M_{kk} \) \hspace{1cm} (3)

The elements of the matrix \( M \) are the overlaps between single body wave-functions, \( M_{kk} = \psi_{kk}^\dagger \psi_{kk} \). (In the 94 paper[11] \( M \) was denoted by \( \Omega^{RR} \) and the single particle states \( \psi_{kk}', \psi_{kk} \).) The \( \psi'\)'s span the negative energy subspace of \( H' \sim \gamma_5 (D_4 + \Lambda') \) and the \( \psi \)'s span the negative energy subspace of \( H \sim \gamma_5 (D_4 - \Lambda) \). I used continuum like notation to emphasize that the Hamiltonians are arbitrary regularizations of massive four
dimensional Dirac operators with large masses of opposite signs. One may wonder why the different signs can at all matter. A simple way to see the difference is to consider a gauge background consisting of one instanton. While det $H'$ is positive, det $H$ is negative. In a complete, massive, four dimensional theory the mass sign could be traded for a topological $\theta$ parameter and the two cases would correspond to $\theta = 0$ and $\theta = \pi$.

The Hamiltonians only enter as defining the Dirac seas and there is no distinction between the different levels within each sea; all that matters is whether a certain single particle state has negative or positive energy. Thus, all the required information is also contained in the operators $\epsilon = \epsilon(H)$ and $\epsilon' = \epsilon(H')$ where $\epsilon$ is the sign function. Thus the two's are all the $-1$ eigenstates of $\epsilon'$ and the $v$'s are all the $-1$ eigenstates of $\epsilon$. To switch chiralities one only has to switch the sign of the Hamiltonians. This is a result of charge conjugation combined with a particle-hole transformation.

Nowadays the defining equations for the states $v$ and $v'$ are expressed with the help of the projectors $\frac{1+\gamma^5}{2}$, $\frac{1-\gamma^5}{2}$ (they annihilate the states $v$ and $v'$ respectively). But this is only notational novelty.

When $A'$ is taken to infinity in lattice units one is left with $\epsilon' = \gamma_5$. Thus, $\epsilon'$ becomes gauge field independent and so become the associated states. (This simplification was already there in the Boyanowski, Dagotto, Fradkin paper, but has been rediscovered in the domain wall context by Shamir[19].) Physically, $\epsilon'$ can be thought of as a lattice representation of a continuum, positive infinite mass, five dimensional Hamiltonian for Dirac fermions in a static gauge field; one can decouple the fermions from the gauge field entirely. On the other hand, $\epsilon$ always maintains a dependence on the gauge background and its trace gives the gauge field topology. The parameter $A$ is restricted to a finite range in lattice units and cannot be taken to infinity. Physically, one can think of $\epsilon$ as a lattice representation of a continuum, negative infinite mass, five dimensional Hamiltonian for Dirac fermions in a static gauge field; the unavoidable dependence on the gauge field reflects the continuum result that infinitely massive fermions in odd dimensions cannot decouple from the gauge fields for both signs of the mass term.

3.4. The vectorial case

We add the states $w$ corresponding to the chirality opposite to that represented by the states $v$ above. As just said, all this requires is to switch the signs of the $\epsilon$'s. The left handed and right handed fermions do not mix, each being self-coupled by $M^R_{k} = v_k^rv_k$, $M^L_{k} = w_k^iw_k$.

We wish to combine the two systems and get rid of the extra, unused dimensions in each case. This is possible in the vector-like case, but not in the chiral case, because, although the shapes of $M^R$ and $M^L$ change as a function of the gauge background topology, they change in a complementary way: The number of rows is fixed and the number of columns of $M^R$ plus the number of columns of $M^L$ is also fixed, equal to twice the number of rows. Thus $M^R$ and $M^L$ can be packed together into a square matrix of fixed size. To describe $M^R$ or $M^L$ alone one needs a larger space because the shape of these matrices fluctuates. To describe both matrices together however, extra dimensions are not needed.

In the most important case, at zero topology, we are searching for a simplified formula for the product of the determinants of $M^R$ and $M^L$. The two $\epsilon$'s generate a relatively simple algebra; the main new operator in this algebra is the unitary operator $V = \epsilon'\epsilon$. A very basic linear set of elements in that algebra is $O = \alpha \epsilon + \beta \epsilon' + \gamma \epsilon'\epsilon + \delta$. The matrix elements of $O$ between any $v$ or $w$ states are trivially expressible in terms of corresponding overlaps. Picking $\alpha = \beta = 0$, $\gamma = \delta = \frac{1}{2}$ we can kill all $v \sim w$ cross terms and the matrix elements of $O$ are determined by those of $M^R$ and $M^L$:

$$ (w' \quad v')^t \frac{1+\epsilon'\epsilon}{2} (w \quad v) = \begin{pmatrix} M^L & 0 \\ 0 & M^R \end{pmatrix} \tag{4} $$

Both $(w \quad v)$ matrices are unitary since the columns make up orthonormal bases. Using charge conjugation one can assure that the determinants of the two $M$-matrices are complex conjugate of each other. With

$$ D_o = \frac{1+\epsilon'\epsilon}{2} \tag{5} $$
one trivially derives \( \det D_o = |\det M^L|^2 \). When \( \Lambda' = \infty \), \((w' v')\) is the unit matrix. Moving the unitary factor \((w v)\) to the other side of equation (4) we see that \( D_o \) has been “factorized”. The columns \( v \) span the kernel of \( \frac{H^*}{2} \) and the columns \( w \)'s span the orthogonal complement of this subspace.

Another way to decouple \( w \) from \( v \) is to choose in \( \alpha = \beta = \frac{1}{2}, \gamma = \delta = 0 \):

\[
(w' v')^\dagger \frac{\epsilon'}{2} (w v) = \begin{pmatrix} M^L & 0 \\ 0 & -M^R \end{pmatrix}
\]

At \( \Lambda' = \infty \), and after moving the unitary factor \((w v)\) to the other side of equation (6) we obtain the hermitian overlap Dirac operator, \( H_o = c'D_o \) studied at SCR12[21].

It is important that even if we keep \( \Lambda' \) finite in lattice units and \( H' \) has a nontrivial gauge dependence, \( H' \) always has exactly as many negative as positive energy eigenstates and there is an impenetrable (as a function of the gauge background) gap in its spectrum around zero. In other words, \( \epsilon' \) is never sensitive to gauge field topology.

Let me add here that a version of \( D_o \) can be derived starting with a lattice implementation of the see-saw mechanism obeying a Froggatt-Nielsen symmetry. This \( D_o \) is obtained in the limit of infinite see-saw partners [16].

3.5. Topology and fermions

The topological charge \( Q\{U\} \) is the difference between the number of columns and rows of \( M^L \), which is the same quantity for \( M^R \). Since \( tr(\epsilon') = 0 \), \( Q\{U\} = \frac{1}{2} trc. \) When \( Q\{U\} \neq 0 \), \( \det \begin{pmatrix} M^L & 0 \\ 0 & -M^R \end{pmatrix} \equiv 0 \) because either among the first columns or among the last there are too many zeros to maintain linear independence. This implies \( \det D_o = 0 \), and hence exact zero modes for the overlap Dirac operator [20]. Using \( tr\epsilon' = 0 \) the formula \( Q\{U\} = \frac{1}{2} trc \) can be written in many equivalent ways. These days a popular way is \( Q\{U\} = tr\epsilon'D_o \) with the sum over sites contained in the trace made explicit. The summand is a lattice version of the topological density.

The impact of topology on fermion dynamics is easiest to see in second quantized language (our original formulation). We denote second quantized operators by hats: \( H = \hat{a}^\dagger H \hat{a}, \quad H' = \hat{a}^\dagger H' \hat{a}, \quad N = \hat{a}^\dagger N \hat{a}, \quad [H, N] = [H', N] = 0 \). The second quantized states entering the overlap satisfy: \( \hat{N}|v\rangle = \frac{i}{2} N|v\rangle \) and \( \hat{N}|v\rangle = (\frac{1}{2} N + Q\{U\})|v\rangle \).

Here \( N = tr1 \). Clearly, \( Q\{U\} \neq 0 \) forces \( \langle v'|v\rangle = 0 \). This immediately leads to nonvanishing, automatically normalized ‘t Hooft vertices. For example, if \( Q\{U\} = 1 \), \( \langle v'|\hat{a}|v\rangle \neq 0 \).

The consequences of the ‘t Hooft vertices are far reaching. In the vector-like case they provide the solution to the \( U(1)_A \) problem, now in an entirely rigorous setting. In a background that carries topological charge 1 for each flavor we shall have \( \langle v'|\hat{a}|v\rangle \neq 0 \), \( \langle v'|\hat{a}^\dagger|v\rangle \neq 0 \) which gives the two \( \hat{\psi}_R \hat{\psi}_L, \hat{\psi}_L \hat{\psi}_R \) factors per flavor that make up the vector-like ‘t Hooft vertex. In the chiral case one can get explicit fermion number violation. A simple example of this can be found in two dimensions, in an abelian gauge model with fermionic matter consisting of one charge 2 right handed fermion \( \phi \) and four left handed charge \(-1\) fermions \( \chi_{\alpha} \). \( \chi_{\alpha} \) is a \( (1,i) \). The ‘t Hooft vertex gives a nonzero expectation value to the operator \( V = \psi\sigma \cdot \partial \psi \chi_{\alpha} \chi_{\beta} \chi_{\gamma} \chi_{\delta} \). The model is exactly soluble and known to have a massless composite sextet of fermions \( \Phi_{\alpha \beta} = \psi \chi_{\alpha} \chi_{\beta} \). These fermions are actually Majorana-Weyl, although the original fermions were just Weyl. This is needed to match the global \( SU(4) \) anomalies associated with the four \( \chi \) fermion fields at the composite level. The ‘t Hooft vertex \( V \) (note the absence of barred fermion fields) provides the kinetic energy term for these composites.

The confirmation of the above features numerically in the 21111 chiral model represented a major step since it showed that even the subtler details of chiral fermion dynamics were captured by the overlap in an effortless way. More traditional approaches to the chiral fermion problem always had to come up with tenuous explanations for how such effects might be recovered in the continuum.

3.6. Chiral symmetry breaking

We find ourselves on the lattice, with exact global chiral symmetries, with correct anomalies and with exactly obeyed mass inequalities. Based
on the continuum it seems that the day is not far where we shall be able to claim to have a rigorous proof of spontaneous chiral symmetry breaking directly on the lattice.

In the meantime, numerical work on the spectral properties of $H_0$ by SCRi has produced “experimental” evidence for spontaneous chiral symmetry breaking. Related observations were made by the Kentucky group[21,22]. See also [23].

### 3.7. Anomalies

The second quantized states entering the overlap each come from a single body Hamiltonian which is analytic in the gauge link variables. Thus, they carry Berry phases. There is a natural connection (an abelian gauge field) over the space of gauge fields (playing the role of parameters in the familiar Berry setup). Integrating Berry’s connection along a smooth closed loop in gauge field space generates an invariant phase, Berry’s phase. There is a natural choice for the states representing the ground state rays and compute the connection once some representatives of the rays $|\psi(U)\rangle$ are chosen (possibly using patches with overlays). $\mathcal{A}$ is not quite a function over the space of gauge fields; it is a connection, in the sense that it could be defined in patches and in their overlays the several definitions could differ by gauge transformations. Berry’s phase is nontrivial at the “perturbative level” because $\mathcal{A}$ has curvature (abelian field strength) $F = d\mathcal{A}$ which does not vanish. Unlike $\mathcal{A}$, $F$ is a function of the gauge field independent of the ray representatives used to define the connection. In second quantized notation $F = \langle dv(U)|dv(U)\rangle$ (antisymmetrization is implicit). In first quantized language one has $F = \sum_{k} \langle \text{Dirac Sea}|dv\rangle$. Note that the overlaps entering the connection and the curvature only compare variations to the same state. This is why $F$ is a local functional of the gauge background. It does not depend on phase choices, so can be expressed in terms of the sign function alone: $F = -\frac{1}{4}\text{tr} (\varepsilon(H) d\varepsilon(H) d\varepsilon(H))$.

A special case is interesting. Take the gauge group as $SU(2)$ and the fermions in the fundamental representation. There is a simple basis in which $H\{u\}$ is a real matrix with no complex entries. Thus, it is natural to choose all eigenstates real and therefore Berry’s connection and its curvature vanish. Nevertheless, Berry’s phase factors can be nontrivial giving sign flips when states are taken round some loops. The $U(1)$ bundle of states one usually has in the overlap is replaced by a $Z_2$ bundle, and the latter could be twisted. This is how Witten’s global anomalies show up in the overlap [25]. Let me remind you that seeing Witten’s global anomalies was beyond the reach of all approaches to the chiral fermion problem before the overlap.

My main message if that Berry’s phase encodes all anomalies in the theory. Let us see how this works in the ordinary, complex case [24].

In the continuum one defines two currents. (I shall restrict my discussion to a nonabelian semisimple gauge group and to a spacetime topology of a $d$-torus.) The consistent current is the variation of the chiral determinant with respect to the gauge field. As a variation it obeys “curl-free” constraints, also known as the Wess-Zumino conditions. When there are anomalies the consistent current cannot be covariant with respect to gauge transformations. If the determinant were gauge invariant the consistent current would trivially also be covariant. Even when there are anomalies and the consistent current cannot be covariant it can be made so by adding a local, exactly known polynomial in the gauge fields and their derivatives. This quantity is called $\Delta J$. Although both the covariant and the consistent currents are nonlocal functionals (in the continuum) of the gauge background, their difference, $\Delta J$, is local. $\Delta J$, by itself, fixes the anomaly. In short, gauge invariance is restorable if and only if $\Delta J$ vanishes (on account of anomaly cancelation).

Now let us go back to the lattice. We make some smooth phase choice for the states representing the ground state rays and compute the variation of the overlap with respect to the external gauge fields. This should produce a lattice version of the consistent current on the lattice because it is the variation of something. One writes $J^{\text{cons}} = \mathcal{A} - \mathcal{A}' + J^{\text{cov}}$. The Berry phase terms contain the part of $J^{\text{cons}}$ which is not guaranteed to be gauge covariant. The remain-
der, defined as $J^{cov}$ is given (at $N = \infty$) by $rac{(\langle |\psi\rangle, i\psi)\langle i\psi \rangle}{\langle |\psi\rangle^2}$; it is independent of the phase choices and has naive gauge transformation properties. Here, $|x\rangle \equiv |x\rangle - \langle x| \langle x|$. The “curl” of $\Delta J = A - A'$ is $\mathcal{F} - \mathcal{F}'$ and does not vanish in general.

The analogy sketched above has not been yet fully fleshed out, but one result is available. Pick an abelian background in the direction of a $U(1)$ subgroup with charges $q$. In the abelian context, $\mathcal{F}$ is gauge invariant and can be viewed as defined over gauge orbits. If the anomaly does not vanish one can find a two-torus in the space of gauge orbits over which the necessarily quantized integral of $\mathcal{F}$ is nonzero. (In $d$ even dimensions the integral $\int \mathcal{F}$ goes as $\sum_i q_i^{d+1}$.) This implies that no “small” deformation of $H(U)$ can make $\mathcal{F} \equiv 0$ and hence $\Delta J \neq 0$. This leads to two conjectures (for the complex case):

If and only if anomalies cancel it is possible to smoothly deform $H(U)$ and $H'(U)$ such that $\mathcal{F} = \mathcal{F}'$. If $\mathcal{F} = \mathcal{F}'$ one can choose the second quantized states $\langle \psi(U) \rangle$ and $\langle \psi'(U) \rangle > 0$ smoothly such that the action of the gauge group is non-projective: for any $g \in G \ G(g)\langle \psi(U) \rangle > 0 \langle \psi(U) \rangle$ and the same for $\langle \psi' \rangle$. If these conjectures prove true one can preserve exact gauge invariance of the overlap if anomalies cancel, but, to do so, one needs to fine tune the Hamiltonians.

So, we must ask whether this is “natural”. The answer is that fine tuning is not necessary to get full gauge invariance in the continuum limit. Even before fine tuning the gauge breaking of the overlap is of a specific kind because the Hamiltonians are gauge covariant, implying (excluding backgrounds with degenerate fermionic ground states) $G(g)\langle \psi(U) \rangle = e^{i\chi(U)}, g \langle \psi(U) \rangle$. $\chi - \chi'$ is a lattice Wess-Zumino action. By fine tuning we conjectured that one can make $\chi = \chi'$ if anomalies cancel. But even if the lattice Wess-Zumino action is not zero, as long as anomalies cancel, it can be small in the sense that one can expand in it. (The cancelation of anomalies implies that in the continuum limit the lattice Wess-Zumino action will have no contribution from the continuum Wess-Zumino action.) Then the mechanism discovered by Förster, Nielsen and Ninomiya shows that exact gauge invariance will be restored in the continuum limit and the Higgs like degrees of freedom representing gauge transformations decouple [26]. This was checked numerically in the above mentioned abelian two dimensional model already in 1997. Although this was only two dimensions it was not at all trivial.

The next chiral speaker will concentrate on the phase of the overlap [27]. Berry’s connection and curvature will be seen to play a central role. It is important to stress that the problem has reduced to a phase choice only because in the overlap this is the single source of gauge breaking, just as emphasized in the continuum context by Fujikawa: Any fermionic correlation function in a fixed gauge background violates gauge covariance by no more and no less than the determinantal anomaly.

3.8. GW and overlap

We already heard that the chiral overlap produced the vector-like operator $D_o$ and that $D_o$ can be factorized to give back the chiral overlap. Now we focus on the relation between $D_o$ and GW. For related considerations, see [28].

Let us first specify what is meant by GW (I adopt a restricted definition including $\gamma_5$-hermiticity): (1a) One is given a local hermitian positive operator $R$ which commutes with $\gamma_5$. The issue is to find a Dirac operator satisfying $\{\gamma_5, D^{-1} - R\} = 0$ and (1b) $\gamma_5 D = (\gamma_5 D)^\dagger$.

Clearly, the operator $D^{-1} = D^{-1} - R$ anticommutes with $\gamma_5$ and is $\gamma_5$-hermitian. (By standard wisdom, $D_o$ cannot be local.) Define the operator $V = 1 + \frac{1}{1+D^{-1} - R}$. $V$ is seen to be $\gamma_5$-hermitian and unitary. Inverting the relation, we find $D^{-1} = 1 + \frac{1}{1+V}$, leading trivially to $D = (1+V)\frac{1}{1+R-(1-R)V}$ which is the most general solution of (1a+b), in terms of a unitary hermitian operator $\epsilon \equiv \gamma_5 V$. ($D_o$ corresponds to $R = 1$.) Obviously, $\epsilon$ squares to unity. Although this satisfies the GW requirement, it is not enough to produce massless fermions. One also needs that topology be given by $Q\{U\} = \frac{i}{2} \text{tr} \epsilon$. In our realization of the overlap we used $\epsilon = \epsilon(H)$ with a sparse $H\{U\}$ analytic in the link variables and showed that topology and perturbation theory produce the correct chi-
eral answers.

It is trivial that the overlap provides a solution to GW. What is the physical meaning of $\epsilon$?

Physically, $\epsilon$ by itself describes Dirac fermions, but they have infinite mass. Therefore, unlike $D_c$, $\epsilon$ is local (except when ill defined). In the continuum, the infinite mass Hermitian Euclidean Dirac operator would have a spectrum concentrated at $\pm \infty$. $\epsilon$ is a rescaled version, with spectrum at $\pm 1$. Any lattice operator $H$, representing Dirac fermions with order $\frac{1}{2}$ negative mass produces an $\epsilon = \epsilon(H)$. A solution of GW, $\epsilon$, that also satisfies the additional conditions required of massless fermions is an acceptable $H$, and reproduces itself in an overlap construction since $\epsilon = \epsilon(\epsilon)$. The overlap provides more flexibility, allowing the replacement of $\gamma_5$ by $\epsilon'$.

It is unreasonable to view the GW relation as pivotal in Nature because it is just a formula, not the embodiment of a fundamental principle. Moreover, the formula accepts also unphysical solutions. On the other hand, the overlap is a direct reflection of a system consisting of an infinite number of fermions governed by some internal dynamics realizing an internal index; it is easier to accept that this is a natural mechanism, conceivably operative in Nature.

Had events in this decade occurred in reversed chronological order the infinite fermion number “explanation” of the GW relation might have been viewed as an inspired insight.

4. A list of projects

There is plenty to do and you are invited to join the chiral subfield! To make my case, I shall present a list of projects. I don’t suggest that you slavishly execute any one of them. The intention is more to inspire you, so you come up with your own idea. The main project seems difficult to me:

- Find a genuinely non-overlap way to solve the chiral fermion problem. If this is possible we shall conclude that the overlap only solved our problem, not necessarily that faced by Nature.

Let me turn to less ambitious proposals:

4.1. General particle physics

- Examine which aspects of low energy physics would be particularly sensitive to an UV regulator of the overlap type.
- Find a natural way to explain the subtraction of the infinite Dirac sea vacuum energies.
- Prove that one cannot find an acceptable solution to the GW relation which is nearest neighbor even in only one direction. Proceed to argue that unitarity in Minkowski space requires an infinite number of fermions.

4.2. Numerical 4D chiral gauge theories

In order to avoid dealing with a complex measure but still treat a non-trivial chiral model I suggest to solve numerically an $SU(2)$ gauge theory with one Weyl $j = \frac{3}{2}$ multiplet. The chiral determinant is real and there are no Witten anomalies. But, there also is no singlet $\psi - \psi$ bilinear.

- What is the phase structure as a function of $\beta$, the gauge coupling?
- Does the model confine? What is its particle spectrum?
- Are there massless fermion states?

4.3. QCD

- Go to the $F_4$ lattice to disallow terms of the form $\sum_{\mu} p_{\mu}$ which are scalars on a hypercubic lattice but not in nature. (This is analogous to Higgs work, where, strictly speaking, claims about Nature on the basis of lattice work cannot be made using hypercubic lattices without fine tuning away the $\sum_{\mu} p_{\mu}$ term.) Compute, by Monte Carlo simulation, the order $p^4$ coefficients in a chiral effective Lagrangian for pions resulting from massless quarks [29]. It is suggested to do this using finite size soft pion theorems of the type used previously in $F_4$ lattice Higgs work [30]. (Let me take the opportunity to correct a misunderstanding that occurred during the discussion following my talk; contrary to a comment from the audience, this problem has not been solved in a poster presented at this conference [23].)
- Use $D_0$ to define nonperturbative improvement coefficients to standard actions. The improvement intends to hasten the restoration of chirality in the continuum limit. On a gauge configuration typical of a fixed $\beta$ evaluate $c$ and $c'$ by
minimizing \(|c'D_o - (D_W + \sigma \cdot F)|^2\).

- The sign function \(\epsilon(M)\) is well defined even for complex \(M\): investigate QCD at nonzero chemical potential but zero quark mass using the overlap.
- Use the Wilson-Dirac operator \(D_W\) to define the pure gauge action, as well as \(\epsilon\). This should reduce the density of states with low \(H_W^2\) eigenvalues. For example, a pure gauge action could contain the term \(\text{tr}\frac{1}{\pi a^3} H_W^2\), or, alternatively, one could use the determinant of a function of \(H_W\) implemented by auxiliary heavy bosonic fields.

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