Recursive Construction of Generator for Lagrangian Gauge Symmetries

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Abstract

We obtain, for a subclass of structure functions characterizing a first class Hamiltonian system, recursive relations from which the general form of the local symmetry transformations can be constructed in terms of the independent gauge parameters. We apply this to a non-trivial Hamiltonian system involving two primary constraints, as well as two secondary constraints of the Nambu-Goto type.

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The problem of finding the most general local symmetries of a Lagrangian has been pursued by various authors, using either Lagrangian [1, 2, 3, 4] or Hamiltonian techniques [5, 6, 7, 8].

In a recent paper [9] we had shown that the requirement of commutativity of the time derivative operation with an arbitrary infinitesimal gauge variation generated by the first class constraints was the only input needed for obtaining the restrictions on the gauge parameters entering the most general form of the generator of Lagrangian symmetries. The analysis was performed entirely in the Hamiltonian framework. On the basis of this commutativity requirement, we subsequently derived [10] a simple differential equation for the generator encoding, in particular, the restrictions on the gauge parameters.

In this paper we shall obtain, for a subclass of structure functions characterizing a first class Hamiltonian system, the explicit solution of the above differential equation in the form of simple recursive relations. We then apply this general scheme to a non-trivial model discussed in the literature [12], whose secondary (first class) constraints are identical with the primary constraints of the Nambu-Goto model. Our result for the gauge transformation is found to agree with that quoted in the literature.

We shall consider purely first class systems. The extension to mixed first and second class systems is straightforward. To keep the algebra simple we assume all constraints to be irreducible.

Consider a Hamiltonian system whose dynamics is described by the total Hamiltonian

\[ H_T = H_c + \sum_{a_1} v^{a_1} \Phi_{a_1} . \]  

(1)

where \( H_c \) is the canonical Hamiltonian, \( \{ \Phi_{a_1} \approx 0 \} \) are the (first class) primary constraints, and \( v^{a_1} \) are the associated Lagrange multipliers. We denote the

\[ 4 \text{We follow here the notation of Ref. [9].} \]
complete set of (primary and secondary) constraints by \(\{\Phi_a\} = \{\Phi_{a1}, \Phi_{a2}\}\).

Following the conjecture of Dirac [13], the generator of the gauge transformations \(G\) is given by

\[
G = \sum_a \epsilon^a \Phi_a
\]  

(2)

where the gauge parameters are allowed to depend in general on time, as well as on the phase space variables and Lagrange multipliers. An infinitesimal transformation on the coordinates, generated by \(G\), is then given by

\[
\delta q^\ell = \epsilon^a [q^\ell, \Phi_a],
\]  

(3)

where a summation over repeated indices is always understood.

The Poisson algebra of the constraints with themselves and with the canonical Hamiltonian, is of the form

\[
[H_c, \Phi_a] = V^{b}_{a} \Phi_b
\]  

(4)

\[
[\Phi_a, \Phi_b] = C^{c}_{ab} \Phi_c
\]  

(5)

where \(V^{b}_{a}\) and \(C^{c}_{ab}\) may be functions of the phase-space variables.

As was shown in [6, 9, 10], \(G\) in (2) will generate a local symmetry of the corresponding total Lagrangian, provided the following relations hold:

\[
\delta v^{b_1} = \frac{d}{dt} \epsilon^{a} \left[ V^{b_1}_a + \epsilon^{a_1} C^{b_1}_{a_1 a} \right],
\]  

(6)

\[
0 = \frac{d}{dt} \epsilon^{a} \left[ V^{b_2}_a + \epsilon^{a_1} C^{b_2}_{a_1 a} \right].
\]  

(7)

In the above equations, \(\frac{d}{dt}\) denotes the total time derivative. For obtaining the generator of the symmetries of the original Lagrangian, only eq. (7) is relevant. Eq. (6) is required for consistency on Hamiltonian level.

As was shown in [10], the above equations can be compactly summarized in a simple differential equation for the Generator \(G\) expressing its time independence:

\[
\frac{\partial G}{\partial t} + [G, H_T] = 0.
\]  

(8)

\(^5\)“Secondary” refers to all generations of constraints beyond the primary one.
Equations (6) and (7) describe the restrictions imposed on the Lagrange multipliers and gauge parameters for the most general case where the structure functions depend on coordinates and momenta. We now seek a solution of (7) under two assumptions:

i) The Poisson bracket of any constraint with the primary constraints is a linear combination of only the primary constraints. This implies $C_{a_1}^{b_2} = 0$, and hence the absence of the last term in (7).

ii) The structure functions $V_{ab}^{b_2}$ are constants.

These conditions appear at first sight to be very restrictive. However, many of the physically interesting theories, fall into this class.

The problem of finding the generator of gauge transformations subject to the above assumptions was considered in [14]. Starting from the general set of equations (7), we present here a more compact and transparent approach to the solution, which in fact will include the case of field dependent structure functions $V_{ab}^{b_2}$, if there are no tertiary constraints.

In order to solve the equations (7) it is convenient to organize the constraints into “families”, where the parent of each family “a” is given by a primary constraint $\phi_0^{(a)}$, and the remaining members are recursively derived from [14]

$$[H_c, \phi_0^{(a)}] = \phi_i^{(a)} , \quad i = 1, ..., N_a .$$

With an obvious change in notation, this implies that the corresponding structure functions satisfy

$$V_{ij}^{ab} = \delta^{ab} \delta_{i,j-1} , \quad i = 0, \cdots, N_a - 1 .$$

In order to ensure that the constraints thus obtained are irreducible, we must adopt some systematic procedure. A possibility is to perform this iteration level by level, for all primary constraints, simultaneously. We terminate a family “a”, if at a given level $N_a$, the Poisson bracket of the constraint $\phi_0^{(a)}$ with $H_c$ can be written as a linear combination of the members of all the
families up to this level. However, independent of the specific procedure chosen, the Poisson bracket of the final member of each family with $H_c$ is given by

$$[H_c, \phi^{(a)}_{N_a}] = \sum_{b=1}^{M} \sum_{j=0}^{N_b} V_{N_a j}^{ab} \phi^{(b)}_j ,$$  

(11)

where $M$ is the number of (independent) primary constraints.

In the new notation, equation (7) reads

$$0 = \frac{d\epsilon^{(a)}_i}{dt} - \sum_{b=1}^{M} \sum_{j=0}^{N_b} \epsilon^{(b)}_j V_{ji}^{ba} , \quad i = 1, \cdots, N_a .$$  

(12)

Choosing as our independent parameters the ones associated with the last member in each family,

$$\alpha^a := \epsilon^{(a)}_{N_a}(t) ,$$  

(13)

the equations (12) take the form

$$\frac{d\epsilon^{(a)}_i}{dt} - \epsilon^{(a)}_{i-1} - \sum_{b=1}^{M} \alpha^b V_{N_b i}^{ba} = 0 , \quad i = 1, \cdots, N_a .$$  

(14)

The solution to this set of equations can be constructed iteratively, by starting with the last member of a family:

$$\epsilon^{(a)}_{N_a-1} = \frac{d\alpha^a}{dt} - \sum_{b=1}^{M} \alpha^b V_{N_b N_a}^{ba} .$$  

(15)

Continuing in the same fashion, one easily sees that the general solution can be written in the form

$$\epsilon^{(a)}_i = \sum_{n=0}^{N_a-1} \sum_{b=1}^{M} \frac{d^n \alpha^b}{dt^n} A_{i(n)}^{ba} ,$$  

(16)

with the normalization

$$A_{N_a(0)}^{ab} = \delta^{ab} ,$$  

(17)

following from our choice of parametrization (13). Substituting the above ansatz into (14) and comparing powers in the time derivatives, we obtain the recursion relations

$$A_{i(n-1)}^{ab} = A_{i-1(n)}^{ab} , \quad i = 1, \cdots, N_a ,$$  

(18)
with the “initial conditions”

$$A_{i-1(0)}^{ab} = -V_{Na}^{ab}, \quad i = 1, \ldots, N_a. \tag{19}$$

It is easy to see, that these recursion relations determine the complete solution, from which the generator of the Lagrangian gauge symmetries can be obtained. Using (16) in the generator (2), the infinitesimal gauge transformation (3) takes the form

$$\delta q^\ell = \sum_{b=1}^{M} \sum_{n \geq 0} d^n b \rho_{(n)b}^{\ell}(q, \dot{q}), \tag{20}$$

where

$$\rho_{(n)b}^{\ell}(q, \dot{q}) = \sum_{a=1}^{M} \sum_{j \geq 0} \theta(N_a - n - j) A_{j(n)}^{ba} \frac{\partial \phi^{a}}{\partial p_{\ell}}, \tag{21}$$

with $\theta(0) = 1$, and where it is understood that the canonical momenta are to be replaced by the respective expressions in terms of the Lagrangian variables. Expression (21) is in the form obtained by purely Lagrangian methods [2, 4, 11].

In the case where all the families contain at most two members, the primary constraints will have vanishing Poisson brackets with all of the constraints $^6$, amounting to a vanishing of the last term in (7). In that case we can also relax the above assumption concerning the constancy of the structure functions $V_{ij}^{ab}$, since our iterative scheme already terminates with equation (15) with $N_a = N_b = 1$, and we have for the generator

$$G = \sum_{a=1}^{M} \left[ \left( \frac{d\alpha_a}{dt} - \sum_{b=1}^{M} \alpha_b V_{11}^{ba} \right) \phi^{(a)}_0 + \alpha^a \phi^{(a)}_1 \right]. \tag{22}$$

The following modified version of the Nambu-Goto model has these features.

Consider the Lagrangian [12]

$$L = \int d\sigma \left( \frac{1}{2} x'^2 - \frac{\mu}{\lambda} x'^2 + \frac{1}{2} \frac{\mu^2}{\lambda} x'^2 - \frac{1}{2} \lambda x'^2 \right), \tag{23}$$

$^6$This can be easily verified by noticing that in this case the canonical Hamiltonian can always be written in the form $H_c(q, p, \xi) = H_0(q, p) + \xi^\alpha T_\alpha$ [12], where the Lagrange multipliers $\xi^\alpha$ are the variables conjugate to the primary constraints, and implement the secondary constraints $T_\alpha \approx 0$. 

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where the 4-vector $x^\mu(\tau, \sigma)$ labels the coordinates of a “string” parametrized by $\tau$ and $\sigma$, with the “dot” and “prime” denoting the derivative with respect to $\tau$ and $\sigma$, respectively. There are two primary constraints, $\pi_1 \approx 0$ and $\pi_2 \approx 0$, where $\pi_1$ and $\pi_2$ are the momenta conjugate to the fields $\lambda(\tau, \sigma)$ and $\mu(\tau, \sigma)$, respectively. Hence in our notation

$$\phi_0^{(1)} = \pi_1, \quad \phi_0^{(2)} = \pi_2.$$  \hspace{1cm} (24)

The canonical Hamiltonian reads,

$$H_c = \int d\sigma \left\{ \frac{\lambda}{2}(p^2 + x'^2) + \mu p \cdot x' \right\}, \hspace{1cm} (25)$$

where $p_\mu$ is the four-momentum conjugate to the coordinate $x^\mu$. The conservation in time of the primary constraints leads respectively to secondary constraints, which in our notation read

$$\phi_1^{(1)} = \frac{1}{2}(p^2 + x'^2) \approx 0, \quad \phi_1^{(2)} = p \cdot x' \approx 0.$$  \hspace{1cm} (26)

One readily checks that there are no further constraints.

We see that the secondary constraints are just the primary constraints of the Nambu-Goto string model. They satisfy the familiar Poisson brackets $^7$

$$[\phi_1^{(1)}(\sigma), \phi_1^{(1)}(\sigma')] = \phi_1^{(2)}(\sigma)\partial_\sigma \delta(\sigma - \sigma') - \phi_1^{(2)}(\sigma')\partial_{\sigma'} \delta(\sigma - \sigma')$$  \hspace{1cm} (27)

$$[\phi_1^{(1)}(\sigma), \phi_1^{(2)}(\sigma')] = \phi_1^{(1)}(\sigma)\partial_{\sigma'} \delta(\sigma - \sigma') - \phi_1^{(1)}(\sigma')\partial_{\sigma} \delta(\sigma - \sigma')$$  \hspace{1cm} (28)

$$[\phi_1^{(2)}(\sigma), \phi_1^{(2)}(\sigma')] = \phi_1^{(2)}(\sigma)\partial_{\sigma} \delta(\sigma - \sigma') - \phi_1^{(2)}(\sigma')\partial_{\sigma'} \delta(\sigma - \sigma').$$  \hspace{1cm} (29)

All other Poisson brackets vanish. The constraints are seen to be first class.

In our terminology, we thus have two families, each with two members.

The canonical Hamiltonian is of the form

$$H_c = \int d\sigma (\lambda \pi_1^{(1)}(\sigma) + \mu \phi_1^{(2)}(\sigma)).$$  \hspace{1cm} (30)

$^7$We suppress the $\tau$ variable.
The structure functions $V_{ij}^{ab}$ are read off from the Poisson brackets

$$[H_c, \phi_1^{(1)}] = -\lambda \partial_\sigma \phi_1^{(1)} - 2\lambda' \phi_1^{(2)} - \mu \partial_\sigma \phi_1^{(1)} - 2\mu' \phi_1^{(1)},$$

$$[H_c, \phi_1^{(2)}] = -\lambda \partial_\sigma \phi_1^{(1)} - 2\lambda' \phi_1^{(2)} - \mu \partial_\sigma \phi_1^{(2)} - 2\mu' \phi_1^{(2)},$$

to be

$$V_{11}^{11}(\sigma, \sigma') = V_{11}^{22}(\sigma, \sigma') = -(\mu(\sigma)\partial_\sigma + 2\mu'(\sigma)) \delta(\sigma - \sigma'),$$

$$V_{11}^{12}(\sigma, \sigma') = V_{11}^{21}(\sigma, \sigma') = -(\lambda(\sigma)\partial_\sigma + 2\lambda'(\sigma)) \delta(\sigma - \sigma').$$

Since for the example in question $N_1 = N_2 = 1$, it follows from (15), that our iterative scheme for finding the solution already ends at the first step, with $\epsilon_0^{(a)}$ given by

$$\epsilon_0^{(a)} = \frac{d\alpha^a}{d\tau} - \int d\sigma' \sum_{b=1}^2 \alpha^b(\sigma') V_{11}^{ba}(\sigma', \sigma).$$

We thus obtain

$$\epsilon_0^{(1)} = \frac{d\alpha^1}{d\tau} - \mu \partial_\sigma \alpha^1 + \mu' \alpha^1 - \lambda \partial_\sigma \alpha^2 + \lambda' \alpha^2,$$

$$\epsilon_0^{(2)} = \frac{d\alpha^2}{d\tau} - \mu \partial_\sigma \alpha^2 + \mu' \alpha^2 - \lambda \partial_\sigma \alpha^1 + \lambda' \alpha^1.$$

From (2) and (3) we now compute the corresponding transformation laws for the fields to be

$$\delta x^\mu = \alpha^1 p^\mu + \alpha^2 \partial_\sigma x^\mu,$$

$$\delta \lambda = \epsilon_0^{(1)}, \quad \delta \mu = \epsilon_0^{(2)}.$$

Making use of the expressions for $\epsilon_0^{(a)}$ derived above, we verify that our results (37) agree with that quoted in the literature [12].

To summarize, we have shown that the equations defining the restrictions to be imposed on the gauge parameters in (2) could be solved following a simple iterative scheme, in the case where the structure functions $C_{a1b}^c$ in eq. (7) vanish and $V_a^{b2}$ are constants. We have then applied the general ideas to the case of a non-trivial model with a two-family constraint structure, sharing
some properties with the familiar Nambu-Goto model of string theory. Since each family of constraints consisted only of two members, our general solution was applicable, although the structure functions $V_a^{b_2}$ are functions of the fields. We thereby recovered the local symmetry transformations quoted in the literature.

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References


