Asymptotic Dynamics in Quantum Field Theory

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Abstract

A crucial element of scattering theory and the LSZ reduction formula is the assumption that the coupling vanishes at large times. This is known not to hold for the theories of the Standard Model and in general such asymptotic dynamics is not well understood. We give a description of asymptotic dynamics in field theories which incorporates the important features of weak convergence and physical boundary conditions. Applications to theories with three and four point interactions are presented and the results are shown to be completely consistent with the results of perturbation theory.

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1 Introduction

Descriptions of scattering in quantum field theory assume that at large times the particles are widely separated and behave like free particles. This assumption, which underlies the LSZ formalism, is incorrect for many theories. The most obvious example of this is when the incoming or outgoing system includes bound states, but it also fails if the physics is characterised by long range interactions. Since most of the physics of the standard model falls into at least one of these categories (confined quarks, massless gauge bosons) it is very important to have a precise understanding of the dynamics of quantum field theories at large times.

Generally, then, it is assumed that at asymptotic times the Heisenberg fields become free ones

\[ \lim_{t \to \infty} \phi(x) \to Z^{1/2} \phi_{\text{out}}(x), \]

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and similarly for \( t \rightarrow -\infty \). We should note that this behaviour can only be taken to hold as a weak limit, between matrix elements, since otherwise (see Sect. 5-1-2 of [1]) one can show, from the Källen-Lehmann representation, that the fields are free at all times.

The limit in (1) is sometimes discussed in the framework of an ‘adiabatic approximation’, in which the coupling constant is taken to be multiplied by a function which is one during the scattering process and approaches zero for very large (positive or negative) times. This is unsatisfactory since it assumes the desired answer which ought rather to emerge from the theory itself. It can also be wrong, as in the case of Quantum Electrodynamics (QED).

QED, the paradigm for the Standard Model, has long range interactions. The masslessness of the photon means that the potential between static charges falls off only as \( 1/r \). It has been known for a long time [2, 3] that this means that (1) does not hold and that any attempt to impose such a relation generates infra-red (IR) divergences in the wave-function renormalisation constant of (charged) matter fields.

This has been studied [4] in the relativistic theory by Kulish and Faddeev (KF) and their general approach to asymptotic dynamics has been utilised by various authors, see, e.g., [5–13]. We shall now give a brief sketch of the procedure adopted by KF and what their results seem to indicate for QED.

They considered the usual QED interaction Hamiltonian

\[
\mathcal{H}_{\text{int}}(t) = -e \int d^3x \, A_\mu(t, x) J^\mu(t, x),
\]

where \( J^\mu(t, x) = \bar{\psi}(t, x) \gamma^\mu \psi(t, x) \) is the (conserved) matter current. In order to carry out the LSZ reduction of the \( S \)-matrix we must be in the interaction picture so that, although the time evolution of the states is determined by (2), the evolution of the fields themselves is given by the free Hamiltonian. One may then insert the free field expansions in (2). These plane wave expansions are

\[
\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ b(p, s) u^*(p) e^{-ip \cdot x} + d^\dagger(p, s) v^*(p) e^{ip \cdot x} \right\}
\]

where the notation implies a sum over the \( s \) indices. Working in Feynman gauge we have

\[
A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left\{ a_\mu(k) e^{-ik \cdot x} + a_\mu^\dagger(k) e^{ik \cdot x} \right\},
\]

where \( k_0 = E_p = \sqrt{|p|^2 + m^2} \) and \( \omega_k = |k| \). Inserting these into (2) results in eight terms which may be grouped according to the positive and negative frequency components of the fields. Each of these pieces will have a time dependence of the form \( e^{i\psi t} \) where \( \psi \) involves sums and differences of energy terms.

KF claimed that, for \( |t| \rightarrow \infty \), only terms with \( \psi \) tending to zero contribute to the asymptotic dynamics. Since the spatial integration in (2) generates a momentum delta function, only four terms with \( \psi = \pm(E_{p+k} - E_p \pm \omega_k) \) would then have a large \( t \)-limit. This vanishing of \( E_{p+k} - E_p \pm \omega_k \approx 0 \) can only take place in QED because the photon is massless, and it only occurs for soft photons, i.e., for \( \omega_k \approx 0 \). This is in accord with perturbation theory: the breakdown of the \( S \)-matrix occurs for soft photons and giving the photon a small mass acts as a cut-off on these divergences. An asymptotic
approximation to (2) is obtained from the lowest order term of the Taylor expansion, in powers of $k$, of the Hamiltonian. This yields

$$\mathcal{H}^{\text{as}}_{\text{int}}(t) = -e \int d^3x \, A_\mu(t, x) J^\mu_{\text{as}}(t, x)$$

(5)

where

$$J^\mu_{\text{as}}(t, x) = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{E_p} \rho(p) \delta^3\left(x - \frac{p}{E_p} t\right),$$

(6)

and $\rho(p)$ is the charge density

$$\rho(p) = \sum_s \left( b^\dagger(p, s)b(p, s) - d^\dagger(p, s)d(p, s) \right).$$

(7)

According to KF therefore, the asymptotic Hamiltonian is the integral over all momenta of the current associated with a charged particle of velocity $p^\mu/E_p$. This non-vanishing Hamiltonian finds its perturbative expression in the branch cuts, rather than poles, in the on-shell Green’s functions in the matter fields of QED. An attractive aspect of this theory is that it completely dispenses with the adiabatic approximation. Although this discussion seems to pick up the problems in applying the LSZ scheme to QED and, in particular, correctly identifies the problem with long wavelength ($\omega k \approx 0$) photons, it cannot be regarded as the end of the story.

These arguments are used extensively in other theories, such as QCD, where the physics is not well understood, and where greater reliance is put on the mathematics. On the other hand the theory is employed at the level of operators whereas it is more appropriate, in quantum field theories, to work at the level of matrix elements and weak limits. The KF approach also makes no connection between the large time limit and the separation of particles at large distance. As we shall see below, the naive application of this approach to massive $\phi^4$ theory would indicate that the LSZ formalism should also break down, which it very evidently does not.

This paper is concerned with constructing a new approach to asymptotic dynamics within the context of weak convergence, and with appropriate physical boundary conditions corresponding to the separation of particles. We will apply it to a variety of interactions and show that it yields results which are completely consistent with what is known from explicit perturbative calculations.

In Sect. 2 we will show that the KF argument cannot be applied to four point interactions. Sect. 3, which is the heart of this paper, develops the method for massive $\phi^4$ theory and then applies it to both the three and four point interactions of scalar QED. In this way we will see both the well known spin independence of the IR structure in the abelian theory with massive charges and, for the three point vertex, regain the results of [4]. A discussion of the implications of these results for perturbative calculations in the standard model is presented in Sect. 4. Some technical details are given in appendices.

2 Four Point Interactions

As we have indicated in the introduction, there are objections to the KF view of asymptotic dynamics: firstly, that the statements have been framed specifically for the
picture of the $\omega_k \approx 0$ infra-red theory, which is well known to suffer from divergences, and not as a general statement on the behaviour of asymptotic limits; secondly, and more critically, that the prescription does not translate to other quantum field theories. It is the latter objection which is the most serious and the one which we shall now demonstrate.

To be precise, we shall show that the KF argument applied to the case of massive $\phi^4$ theory is not sufficient to show that the asymptotic limit of the coupling term vanishes. However, $\phi^4$ theory is the standard textbook example of an interacting quantum field theory and its perturbation theory is straightforward – the coupling must vanish for well separated particles.

To begin then, let us take the standard free field expansion of the scalar field

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left( a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x} \right),$$  \hspace{1cm} (8)$$

with $E_k = \sqrt{|k|^2 + m^2}$ and $m$ the mass of the particle. The commutator relations are

$$[a(k), a^\dagger(k')] = (2\pi)^3 2E_k \delta^3(k - k')$$ \hspace{1cm} (9)$$

and the interaction part of the Hamiltonian for the theory under consideration is then

$$H_{\text{int}} = \frac{\lambda}{4!} \int d^3x : \phi^4(x, t) :,$$  \hspace{1cm} (10)$$

where the $:\ :$ indicates normal ordering.

When the expansion (8) is inserted into (10) and the resulting expression is simplified, then, after normal ordering, it will be found to consist of twelve terms, each of which has an exponential term where the exponent is made up of sums and differences of the energy eigenvalues. Some of these exponents are obviously non-vanishing. According to the methods of KF which were described in the introduction, the integrals containing these exponentials may be ignored. However, not all of the integrals involved have exponents which are so easily dealt with and one of these, which we shall now consider, is

$$\int \frac{d^3p d^3q d^3k}{(2\pi)^9} \frac{a^\dagger(k)a^\dagger(p+q-k)a(p)a(q)}{2E_p 2E_q 2E_k 2E_{p+q-k}} e^{-it(E_p + E_q - E_k - E_{p+q-k})}.$$  \hspace{1cm} (11)$$

Applying the methods described in the introduction, any non-vanishing asymptotic dynamics will come from those terms and those momenta for which the exponent vanishes. To this end we must determine if the equation

$$E_p + E_q - E_k - E_{p+q-k} = 0$$  \hspace{1cm} (12)$$

has any solutions.

Far from this being a difficult problem, upon reflection it becomes obvious that there are infinitely many solutions! This is most easily seen by noting that the problem is equivalent to that of finding solutions to the simultaneous system of equations

$$\sqrt{|p|^2 + m^2} + \sqrt{|q|^2 + m^2} = \sqrt{|k|^2 + m^2} + \sqrt{|l|^2 + m^2}$$

$$p + q = k + l.$$  \hspace{1cm} (13)$$
The incoming and outgoing momenta must have some connection with a scattering process so that the obvious, trivial solutions, which can be found by taking, say, \( q = l = 0 \), will be ignored.

Referring to Fig. 1, let \( p = \overrightarrow{AB} \), and \( q = \overrightarrow{BC} \). Now imagine pivoting the rigid triangle \( ABC \), with the line \( AC \) as the hinge, to get a new triangle which is congruent to \( ABC \). If the new triangle is \( AB'C \) then take \( k = \overrightarrow{AB'} \) and \( l = \overrightarrow{B'C} \). The vectors \( p, q, k, l \) then automatically satisfy both the conditions of (13).

One can do this with any two non-parallel vectors, for example (with \( m = 1 \)) take \( p = (1, 1, 0), q = (1, -1, 0), k = (1, 1/\sqrt{2}, 1/\sqrt{2}), l = (1, -1/\sqrt{2}, -1/\sqrt{2}) \). Another example is \( p = (-4/\sqrt{3}, 2\sqrt{2}/\sqrt{3}, 0), q = 0, k = (-2/\sqrt{3}, \sqrt{2}/\sqrt{3}, 1) \), and \( l = p + q - k \).

No physical meaning could be given to such an arbitrary set of momenta. Following the arguments of KF however, we conclude that the existence of this wide range of momenta for which (12) vanishes, indicates that the asymptotic dynamics of the system is determined by this set of points. This would suggest that problems could arise in the associated perturbation theory for massive \( \phi^4 \) theory, and this is well known to be false for this textbook example of a quantum field theory.

At this point we wish to stress that this apparent contradiction has implications beyond this simple example as there are many important quantum field theories, such as QCD and the Higgs mechanism, in which four point interactions have a major role. If there is a deficiency in the understanding of the asymptotic dynamics in this simple scalar four point theory then it is difficult to see how one might proceed on this basis, with any confidence, in other theories of the standard model. It is clear from this that the KF argument needs considerable refinement if it is to be applied to the standard model.

3 General Approach to Asymptotic Dynamics

The major flaw in the KF argument is its reliance on strong operator convergence. Instead of concentrating on the Hamiltonian itself, we shall focus our attention on matrix elements. The limits we shall consider are weak limits and this is in keeping with the LSZ formalism in quantum field theories.

If \( \Psi_{IN} \) represents an incoming wave packet in this scattering experiment and \( \Psi_{OUT} \) is an outgoing wave packet, then the matrix element of interest is \( \langle \Psi_{OUT} | H_{int} | \Psi_{IN} \rangle \). This is a time dependent c-number and the elementary notion of convergence may be adopted in the investigation of its asymptotic limit. In the case of massive scalar \( \phi^4 \) it will be shown that, under conditions which have a straightforward physical interpretation, its
asymptotic limit is zero.

This section is organised as follows. In Sect. 3.1 we shall use massive $\phi^4$ theory as a vehicle to study asymptotic dynamics. Having established the basic method and shown that it is consistent with what is already known for $\phi^4$ theory, we shall then employ it to determine the asymptotic dynamics of scalar QED.

There is a particularly interesting feature of scalar QED which makes the study of its asymptotic dynamics worthwhile. The interaction Hamiltonian in scalar QED consists of a sum of two terms, a three point interaction term, similar to that in (fermionic) QED, and a four point interaction term, which has no parallel in QED, so that the dynamics of scalar QED is richer than that of fermionic QED. From perturbation theory it is known that (fermionic) QED and scalar QED have the same infra-red problems, so their asymptotic dynamics are the same. Perturbation theory also tells us that the four point interaction term is divergence free so it must have vanishing asymptotic dynamics. We would expect this to emerge, in a natural manner, from any satisfactory theory which describes the asymptotic dynamics of scalar QED. Further, since the infra-red problem is spin independent, the asymptotic dynamics of the three point interaction will, with the obvious changes, be the same in either fermionic or scalar QED.

We shall begin our examination of scalar QED in Sect. 3.2 with the quartic term, which turns out to be the easier to deal with. We shall show that its asymptotic dynamics is similar to that of the scalar $\phi^4$ theory and has a zero limit. After this, in Sect. 3.3, we shall turn our attention to the cubic term and the infra-red problem. We shall prove that the asymptotic dynamics for the infra-red problem is exactly the same as that for a system in which the Hamiltonian is derived from a current associated with a moving charged particle with known, non-trivial asymptotic dynamics.

### 3.1 Scalar $\phi^4$ Theory

Consider the following incoming and outgoing wave packets.

\[
\Psi_{\text{IN}} = \int d^3r \, d^3w \, f(r)g(w)a^\dagger(r)a^\dagger(w)|0>,
\]

\[\Psi_{\text{OUT}} = \int d^3u \, d^3v \, h(u)i(v)a^\dagger(u)a^\dagger(v)|0>,
\]

with the functions $f, g, h, i$ being test functions for the wave packets. Referring to (8), (9), (10), and the expressions (14), one finds that $<\Psi_{\text{OUT}}|H_{\text{int}}|\Psi_{\text{IN}}>$ reduces to a single integral which is proportional to

\[
\int d^3p \, d^3q \, d^3k \, h(k)i(p + q - k)f(p)g(q)e^{-it\psi},
\]

with the exponent $\psi$ having the value $\psi = E_p + E_q - E_k - E_{p+q-k}$.

Notice that the exponent in this term has essentially the same structure as the term in (11) which appeared to cause problems when applying the methods of KF. The difference now is that (15) is a straightforward integral and not an operator, so that elementary methods may be applied to find the asymptotic limit. The machinery
we shall employ is the method of stationary phase [14]. Briefly, this says that, provided there is no point in the region of integration at which all of the first order partial derivatives of $\psi$ are zero, then the integral (15) tends to zero as $|t| \rightarrow \infty$.

The terms in $\psi$ have the form $E_l = \sqrt{|l|^2 + m^2}$ and, since $m \neq 0$, they will all have first-order partial derivatives for all values of $l$. The first order derivatives of $\psi$ are then given by

$$\frac{\partial \psi}{\partial p_i} = \frac{p_i + q_i - k_i}{E_p - \frac{E_{p+q-k}}{E_{p+q-k}}}$$

$$\frac{\partial \psi}{\partial q_j} = \frac{q_j + q_j - k_j}{E_q - \frac{E_{p+q-k}}{E_{p+q-k}}}$$

$$\frac{\partial \psi}{\partial k_n} = \frac{k_n + q_n - k_n}{E_k - \frac{E_{p+q-k}}{E_{p+q-k}}}.
$$

If at some point all of these are zero then, in particular, $\frac{\partial \psi}{\partial p_i} = \frac{\partial \psi}{\partial q_j} = 0$ for all possible values of $i, j$. This implies that $p/E_p = (p + q - k)/E_{p+q-k}$ and $q/E_q = (p + q - k)/E_{p+q-k}$ so we must also have $p/E_p = q/E_q$. If we can exclude the set of points for which this condition holds from the domain of integration, then the integral in (15) will vanish as $|t| \rightarrow \infty$.

In (15), the test functions $f, g$ for the incoming wave packet have the variables $p, q$ as arguments. The expressions $p/E_p$ and $q/E_q$ represent the velocities of the respective incoming wave packet. Experimentally, scattering is prepared by setting up the apparatus in such a way that the two beams of particles are brought together from different directions, i.e., with different velocities. This information can be incorporated into the incoming wave packet by ensuring that the supports of the test functions exclude the possibility that $p/E_p = q/E_q$. The precise statement of the requirement is that the test functions $f, g$ must have non-overlapping supports in velocity space. This condition on the test functions, of having non-overlapping supports in velocity space, is central to the construction of the $S$-matrix (see Sect. 13.4 of [15]).

To restate, if the test functions $f, g$ have non-overlapping supports in velocity space then the integral in (15) vanishes as $|t| \rightarrow \infty$. This is exactly the behaviour that one would expect for this particular scattering process but a further question remains to be answered: what constraints does this choice of test functions for the incoming wave packet impose on the outgoing wave packet? If this picture is to display all of the features of this particular scattering process then some combinations of outgoing particles must be excluded. The outgoing particles must behave as free particles at asymptotically large times which means that, as was the case for the incoming wave packets, their test functions must also have non-overlapping support in velocity space. However, this must be a consequence of the condition imposed on the test functions for the incoming wave packet and not an independently imposed condition.

The arguments of the test functions for the outgoing wave packet are $p + q - k$ and $k$, and the equality of these two variables is equivalent to $p + q = 2k$. This is simply the expression of conservation of momentum. Another principle in any scattering theory is conservation of energy. In this case this is expressed as $E_p + E_q = 2E_k$. Finally then, we must show that the two conditions

$$p + q = 2k,$
\[ E_p + E_q = 2 \ E_k, \quad (17) \]

are incompatible with the functions \( f, q \) having non-overlapping support in velocity space. Since the masses of the two incoming particles are equal, the equation \( p/E_p = q/E_q \) is equivalent to \( p = q \) and non-overlapping in velocity space is equivalent to non-overlapping in momentum space. In this case we need to show that if \( p \neq q \) then the conditions of (17) are impossible.

Again, for simplicity, let us take the mass \( m = 1 \). The first equation in (17) means that the vectors \( p, q, k \), are coplanar. In that case, we may choose a unit vector \( n \) orthogonal to this plane and write \( p' = p + n, \ q' = q + n, \ k' = k + n \). Since \( n \) is orthogonal to the plane of \( p \), we have \( |p'| = \sqrt{|p|^2 + 1} = E_p \), with similar expressions for \( q, k \). This means that (17) may now be written as

\[
\begin{align*}
|p'| + |q'| & = 2 |k'| \\
(18)
\end{align*}
\]

From the triangle inequality we know that this is only possible when \( p' \) and \( q' \) are parallel, i.e. if there is a number \( \lambda \) such that \( p' = \lambda q' \). In terms of \( p, q \) and \( n \), this can be rearranged into the form \( p - \lambda q = (1 - \lambda) n \). The only solution for this is with \( \lambda = 1 \) so that \( p = q \).

There are other possible matrix elements that are associated with the four point interaction term. There is the possibility that the incoming wave packet consists of a single field, with the outgoing wave packet made up of three fields, and there is the reverse. Both of these cases may be treated in precisely the same manner as the above case, and with precisely the same conclusions. We omit the details.

### 3.2 Scalar QED: the four point interaction.

The interaction in scalar QED is more complicated than that of QED due to the existence of the extra term representing a four point interaction. In this section we shall study the asymptotic properties associated with this term and show that it has trivial asymptotic dynamics. This will require extending our techniques since we shall have to deal with wave packets which have massless particles as an essential part of their structure.

The method of stationary phase that was used to determine the limit of (15), in Sect. 3.1, was dependent upon some of the properties of the partial derivatives (16) of the various energy eigenvalues. In the case of massless particles the energy eigenvalues, which are of the form \( \omega_k = |k| \), are not differentiable at the origin but this will not be a barrier to the application of this technique.

We begin by writing out the full, normal ordered interaction Hamiltonian for scalar QED, which is

\[
H_{\text{int}}(t) = -e \int d^3x \ : J^\mu(x)A_\mu(x) : \quad (19)
\]

with : : being normal ordering and where the current \( J^\mu \) is given by

\[
\begin{align*}
J^\mu & = i(\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi) - eg^{\mu\nu} A_\nu \phi^\dagger \phi \\
& = i J^\mu_1 - e J^\mu_2 \\
& = i (J^\mu_{11} + J^\mu_{12}) - e J^\mu_2, \quad (20)
\end{align*}
\]
with the obvious meaning given to the components of $J^\mu$ defined in (20).

We shall work in Feynman gauge and take the plane wave expansions given by

$$
\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (a(p)e^{-ip\cdot x} + b^\dagger(p)e^{ip\cdot x})
= \phi_+(x) + \phi_-(x)
$$

(21)

$$
A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} (a_\mu(k)e^{-ik\cdot x} + a_\mu^\dagger(k)e^{ik\cdot x})
= A_\mu^+(x) + A_\mu^-(x).
$$

The commutator for the photon is given by

$$
[a_\mu(p), a_\nu^\dagger(q)] = -(2\pi)^3 2\omega_k g_{\mu\nu} \delta^3(p-q).
$$

(22)

When (21) is substituted into (19) and rearranged, the quartic interaction will be found to consist of 12 terms, each of them with differing exponential terms corresponding to the different possible interactions. We shall consider the matrix element represented by the following diagram.

We shall take as our wave packets the following expressions

$$
\Psi_{\text{IN}} = \int d^3r \int d^3w \ f(r)b^\dagger(r)c^\nu(w)a_\nu^\dagger(w)|0>,
$$

$$
\Psi_{\text{OUT}} = \int d^3u \int d^3v \ g(u)b^\dagger(u)h^\mu(v)a_\mu^\dagger(v)|0>,
$$

(23)

where the $f, c^\nu, g, h^\mu$ are the respective test functions. One then finds that the amplitude $<\Psi_{\text{OUT}}|\mathcal{H}_{\text{int}}(t)|\Psi_{\text{IN}}>\) is a single term, constructed from $J^\mu_2$, which is an integral proportional to

$$
\int d^3p \int d^3q \int d^3k \ c^i(p)h_i(q)g(k)f(k+q-p)e^{i\psi t},
$$

(24)

where $\psi = \omega_p + E_{k+q-p} - E_k - \omega_q$.

Notice that the Einstein summation convention means that this integral is actually a sum of three integrals, with $i = 1, 2, 3$. The incoming and outgoing charged fields must be separated, i.e., the test functions $f$ and $g$ must have disjoint support. If not, then, by conservation of momentum, there will be no separation of the incoming and outgoing photons. In that case, no scattering will have taken place. Thus it must follow that, for each $i$, the test functions $c_i, h_i$ must have disjoint support so that at least one of them will not have the zero vector in its support. Without loss of generality, let us suppose that this is $c_i$. 

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The function $\psi$ will then have continuous partial derivatives in $p, k$ and these are given by

\[
\frac{\partial \psi}{\partial p_i} = \frac{p_i}{\omega} - \frac{k_i + q_i - p_i}{E_{k+q-p}} \\
\frac{\partial \psi}{\partial k_j} = \frac{k_j + q_j - p_j}{E_{k+q-p}} - \frac{k_j}{E_k}.
\] (25)

The vanishing of these expressions for all $i, j$ would then imply that $p/\omega_p = k/E_k$ which is impossible, since the former is a unit vector while the latter is not. The method of stationary phase (applied only to the variables $p, k$) can again be used to prove that (24) will vanish as $|t| \to \infty$.

There are other possible scattering events that are described by this four point interaction Hamiltonian, e.g., when the wave packets consist of two incoming photons and two outgoing charges (or vice versa). If the photons are separated according to our scheme (their test functions have disjoint support), then a similar exercise will show that this picture also gives rise to vanishing asymptotic dynamics.

### 3.3 Scalar QED: the infra-red approximation

The cubic term in the interaction Hamiltonian (19), which comes from the $J_{\mu 1}$ term in (20), is made up of two parts, $J_{\mu 1}^{\mu 1}$ and $J_{\mu 12}^{\mu 12}$. These two terms are similar in their structure and both contribute to the problem of infra-red divergences. We shall consider only the first of these, $J_{\mu 1}^{\mu 1}$, the results for $J_{\mu 12}^{\mu 12}$ being substantially the same.

As in QED, the infra-red problem in scalar QED occurs in relation to a scattering process in which an incoming charged particle emits a photon.

![Diagram of scattering process](image)

We shall consider the case when the wave packets are given by

\[
\Psi_{\text{IN}} = \int d^3y \ f(y) b^\dagger(y) |0> \\
\Psi_{\text{OUT}} = \int d^3u \ d^3v \ g(u) b^\dagger(u) h^\mu(v) a_{\mu}^\dagger(v) |0>. \] (26)

The matrix element will then be found to be given by

\[
<\Psi_{\text{OUT}} | H_{\text{int}}(t) | \Psi_{\text{IN}}>= -e \int d^3q \ d^3k f(q + k) g(q) q^\mu h_{\mu}(k) e^{-i\psi t} \] (27)

where $\psi = E_{q+k} - E_q - \omega_k$. Now it is easy to see why the previous method cannot be applied in this case. Since the photon is massless, the corresponding energy eigenvalue
is $\omega_k = |k|$. The expression for $\psi$, therefore, will not have partial derivatives in $k_i$ at $|k| = 0$, and the partial derivatives in $q_j$ will vanish when $|k| = 0$, for any value of $q$. It is this regime, when $|k| = 0$, which gives rise to the problem of infra-red divergences and the difficulties associated with its asymptotic dynamics cannot be avoided.

Following KF we shall compare the asymptotic dynamics of this scattering process with that governed by the scalar version of the asymptotic Hamiltonian (5).

The system that we shall take for our comparison is then the one defined the asymptotic current given by

$$J^\mu_{\text{as}}(x) = i \int \frac{d^3q}{(2\pi)^3} \left( \frac{1}{2E_q} \right)^2 b^\dagger(q) b(q) q^\mu \delta^3(x - \frac{q}{E_q}t),$$

(28)

where we omit terms which do not contribute to this matrix element. The interaction Hamiltonian is given by

$$\mathcal{H}^\text{as}_{\text{int}} = -e \int d^3x J^\mu_{\text{as}}(x) A_\mu(x).$$

(29)

If we take the incoming and outgoing wave packets (26) and contract, in the usual fashion, with $\mathcal{H}^\text{as}_{\text{int}}(t)$ then we obtain the amplitude

$$<\Psi_{\text{OUT}}^\dagger | \mathcal{H}^\text{as}_{\text{int}}(t) | \Psi_{\text{IN}} > = -e \int d^3q d^3k f(q) g(q) q^\mu h_\mu(k) e^{-i\psi t},$$

(30)

where $\psi' = q \cdot k / E_q - \omega_k$.

The asymptotic dynamics of the systems defined by the interaction Hamiltonians (19) and (29), and the wave packets (26), will then be the same if we can prove that the difference between the integrals in (27) and (30) vanish for asymptotically large time. This amounts to showing the following theorem whose proof is given in the Appendix.

**Theorem 1** Let $f, g, h_\mu, \psi, \psi'$ be defined as in (27) and (30). Then

$$\lim_{t \to \infty} \int d^3q d^3k [f(q + k)e^{-i\psi t} - f(q)e^{-i\psi' t}] g(q) q^\mu h_\mu(k) = 0.$$  

(31)

This shows that the physics of scattering in scalar QED at large times is fully described by the asymptotic Hamiltonian (29). The four point interaction vanishes and the three point one can be described using the simple current (28). We have thus extended the results of KF to scalar QED and proved the spin independence of the asymptotic dynamics in abelian gauge theories.

**4 Conclusions**

It is not necessary to assume that the coupling constant asymptotically switches off. As we have seen, one can, for theories like massive $\phi^4$, prove that the asymptotic dynamics is free or, for theories like QED, with rather more effort, determine the form of this asymptotic interaction. The arguments for determining the asymptotic properties of interactions in quantum field theories, proposed by Kulish and Fadeev [4], have been improved upon and made applicable to a more general type of interaction, including
four point couplings. The principle refinement of our approach to the asymptotic dynamics is that we examine the asymptotic properties of matrix elements corresponding to specific interactions rather than considering operators. This has the advantage of requiring only the machinery for the convergence of sequences of c-numbers rather than the more elaborate needs of operator convergence.

In the case of $\phi^4$ theory it was found that, when the incoming wave packet had test functions with non-overlapping supports in velocity space, the asymptotic interaction Hamiltonian is weakly vanishing. This condition on the test functions, of having non-overlapping support in velocity space, is exactly that which is required in the LSZ formalism and the construction of the $S$-matrix. Our result is in complete agreement with perturbation theory and shows why it works.

In the case of scalar QED we used our methods to show that the matrix elements associated with the four point interaction term are all asymptotically trivial which is again in line with the results of perturbation theory. For the three point interaction term of scalar QED, our methods show how the asymptotic dynamics associated with the event of a charged particle emitting a photon can be shown to be exactly the same as that of a charged particle with known non-trivial asymptotic dynamics, and this conformed to the approximation given by KF. The spin independence of this result immediately translates to the fermionic theory.

What can we learn from this work about QED? Firstly that the coupling does not ‘switch off’ at large times. KF further showed that this implies that the Lagrangian matter field does not asymptotically approach the free field of the plane wave expansion. Rather there is a distortion factor which expresses itself in perturbation theory in the branch cuts (instead of poles) in the matter field two point function. They drew the conclusion from this that it is not possible to describe charged particles in QED. Although this paper supports the non-vanishing of the interaction, we feel that this last conclusion is not justified. What one needs is to find the fields which do asymptotically approach the plane wave expansion and can therefore be interpreted as particles. (It is in fact clear from the start that the Lagrangian matter cannot hope to do this since it is not gauge invariant.) That such fields exist has been shown elsewhere [16] and that their Green’s functions have a good pole structure has been amply demonstrated, see, e.g., [17–19]. These are the physical fields which should be identified with the charged particles seen in experiment.

We would like to suggest here three further areas for study: massless QED is a theory with collinear divergences and as such a playground for understanding QCD. The asymptotic dynamics of this theory [20] requires further study, in particular the physical asymptotic fields need to be constructed. Finite temperature field theory is another area where infra-red divergences are important, here, of course, the residual asymptotic dynamics of zero temperature will be acerbated by excitations from the heat bath. Finally in QCD confinement shows that the interaction does not switch off and the strong interaction between quarks and gluons is indeed supposed to grow with the separation. The application of the methods of asymptotic dynamics and the construction of physical fields at short distances [21,22] could have implications for jets production. It has though been demonstrated that there is a topological obstruction to the construction of an isolated quark or gluon [22], how this relates to non-perturbative effects in the asymptotic dynamics of QCD is a topic for future work.
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A Appendix

In this section we shall provide a proof of Theorem 1. This is split into two parts: First we shall show that the two integrals
\[\int d^3q d^3k f(\mathbf{q})g(\mathbf{q})q^\mu h_\mu(\mathbf{k})e^{-i\psi' t}\] and
\[\int d^3q d^3k f(\mathbf{q} + \mathbf{k})g(\mathbf{q})q^\mu h_\mu(\mathbf{k})e^{-i\psi' t}\]
are asymptotically equivalent. Then we shall show that the two integrals
\[\int d^3q d^3k f(\mathbf{q} + \mathbf{k})g(\mathbf{q})q^\mu h_\mu(\mathbf{k})e^{-i\psi' t}\] and
\[\int d^3q d^3k f(\mathbf{q} + \mathbf{k})g(\mathbf{q})q^\mu h_\mu(\mathbf{k})e^{-i\psi' t}\]
are asymptotically equivalent. Together, they give the proof of the theorem. The first of these results is

Lemma 1 Let \( f, g, h_\mu, \psi' \) be as defined in (30). Then the two integrals
\[\int d^3q d^3k f(\mathbf{q})g(\mathbf{q})q^\mu h_\mu(\mathbf{k})e^{-i\psi' t}\] and
\[\int d^3q d^3k f(\mathbf{q} + \mathbf{k})g(\mathbf{q})q^\mu h_\mu(\mathbf{k})e^{-i\psi' t}\] have the same asymptotic limit, i.e., if \( \mathcal{I}_t \) is defined by
\[\mathcal{I}_t = \int d^3q d^3k [f(\mathbf{q} + \mathbf{k}) - f(\mathbf{q})]g(\mathbf{q})q^\mu h_\mu(\mathbf{k})e^{-i\psi' t}\] then \( \mathcal{I}_t \to 0 \) as \( |t| \to \infty \).

Proof We first take a fixed value of \( \mathbf{q} \) and consider
\[\mathcal{I}_t(\mathbf{q}) = \int d^3k [f(\mathbf{q} + \mathbf{k}) - f(\mathbf{q})]q^\mu h_\mu(\mathbf{k})e^{-i\psi' t} .\] This integral is clearly well defined and there is a a positive number \( M \), say, such that \( \int d^3k |q^\mu h_\mu(\mathbf{k})| < M \).

Now given \( \varepsilon > 0 \), choose a \( \delta > 0 \) such that \( |f(\mathbf{q} + \mathbf{k}) - f(\mathbf{q})| < \frac{\varepsilon}{2M} \) if \( |\mathbf{k}| < \delta \). Let \( U_1 = \{ \mathbf{k} : |\mathbf{k}| < \delta \} \) and \( U_2 = \{ \mathbf{k} : |\mathbf{k}| > \delta / 2 \} \) and let \( \rho_1, \rho_2 \) be a smooth partition of unity subordinate to \( U_1, U_2 \) respectively. Write
\[\mathcal{I}_t^1(\mathbf{q}) = \int d^3k \rho_1(\mathbf{k}) [f(\mathbf{q} + \mathbf{k}) - f(\mathbf{q})]q^\mu h_\mu(\mathbf{k})e^{-i\psi' t}\]
\[\mathcal{I}_t^2(\mathbf{q}) = \int d^3k \rho_2(\mathbf{k}) [f(\mathbf{q} + \mathbf{k}) - f(\mathbf{q})]q^\mu h_\mu(\mathbf{k})e^{-i\psi' t} .\]
For the first of these we have \( |\mathcal{I}_t^1(\mathbf{q})| < \varepsilon / 2 \) by construction. For the second, the integrand in \( \mathcal{I}_t^2(\mathbf{q}) \) is defined on \( U_2 \), which does not contain zero, so that \( \psi' \) is differentiable.
on the domain of integration (i.e., $U_2$). One can then apply the method of stationary phase to this integral, using the partial derivatives of $k_i$, to show that $I_t^2(q) \to 0$ as $|t| \to \infty$.

We have shown that $I_t(q) \to 0$ as $|t| \to \infty$, for every $q$. Note that $I_t(q)$ satisfies the inequality $|I_t(q)| \leq \int d^3k \, |f(q + k) - f(q)| |q^\mu h_\mu(k)|$, and the latter is in $L^1(q)$. The Lebesgue Dominated Convergence Theorem [23] may now be invoked to prove that $I_t \to 0$ as $|t| \to \infty$, and this completes the proof. □

Lemma 1 means that we have to show that the two integrals

$$\int d^3q \, d^3k \, f(q + k)g(q)q^\mu h_\mu(k) e^{-i\psi t} \quad \text{and} \quad \int d^3q \, d^3k \, f(q + k)g(q)q^\mu h_\mu(k) e^{-i\psi t}$$

have the same asymptotic limits, where $\psi, \psi'$ are defined in (30) and (27) respectively.

Before proceeding with this, let us recall the form of Taylor’s theorem for a smooth function $u$, i.e.

$$u(q + k) - u(q) - \partial_i u(q)k_i = R(q, k)$$

and $$\frac{|R(q, k)|}{|k|} \to 0 \quad \text{as} \quad |k| \to 0.$$  (37)

Thus, for a fixed value of $q$ and a given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|k| < \delta$ then $|R(q, k)|/|k| < \varepsilon$. If the choice of $k$ is restricted to the set $\{k : |k| < 1\}$, then $\exists t_0 > 0$ such that $\forall |t| > t_0$, we have $|R(q, k/t)|/|k/t| < \varepsilon$ and from this we are able to conclude that $|tR(q, k/t)| < \varepsilon$ for all $|t| > t_0$.

We shall now prove our final theorem.

**Theorem 2** Let $f, g, h_\mu, \psi, \psi'$ be defined as in (27) and (30). If we define $I_t$ as

$$I_t = \int d^3q \, d^3k \, f(q + k)g(q)q^\mu h_\mu(k)(e^{-i\psi t} - e^{-i\psi' t}),$$

then $I_t \to 0$ as $|t| \to \infty$.

**Proof** In the following discussions the value of the functions $f, g, h_\mu$ in (34) are not important and the only property that is required of them is that they are test functions. For convenience, therefore, we shall write the integral as

$$I_t = \int d^3q \, d^3k \, h(q, k)(e^{-i\psi t} - e^{-i\psi' t}).$$

with $h(q, k)$ being a test function.

Take a fixed value of $q$ and write

$$I_t(q) = \int d^3k \, h(q, k)e^{-i\psi t}(1 - e^{i(\psi - \psi')t})$$

$$= \int d \left( \frac{k}{t} \right) \, h \left( q, \frac{k}{t} \right) e^{-i\psi k/t}(1 - e^{i\psi k/t}).$$

As in Lemma 1, we shall first show that $I_t(q)$ has a zero asymptotic limit. In the latter integral we have changed the variable from $k$ to $k/t$ so that now we are writing
\[ \psi(k/t) = E_{q+k/t} - E_q - \omega(k/t) \] and \( \Phi(k/t) = E_{q+k/t} - E_q - (q \cdot k)/(tE_q) \). The latter expression is in a form that will allow the use of Taylor's theorem.

Let \( U_1, U_2 \) be the open cover of \( \mathbb{R}^3 \) given by \( U_1 = \{ k : |k| < 1 \} \), \( U_2 = \{ k : |k| > \frac{1}{2} \} \), and let \( \rho_1, \rho_2 \) be a smooth partition of unity subordinate to this cover. We write the second integral in (40) as the sum of two integrals by incorporating this partition, i.e.

\[
I_1^t(q) = \int d\left(\frac{k}{t}\right) \rho_1(k) h\left(q, \frac{k}{t}\right) e^{-i \psi(k/t)t} (1 - e^{i \Phi(k/t)t})
\]

\[
I_2^t(q) = \int d\left(\frac{k}{t}\right) \rho_2(k) h\left(q, \frac{k}{t}\right) e^{-i \psi(k/t)t} (1 - e^{i \Phi(k/t)t})
\]

\[
= \int d\left(\frac{k}{t}\right) \rho_2(k) h\left(q, \frac{k}{t}\right) (e^{-i \psi(k/t)t} - e^{-i \psi'(k/t)t}),
\]

and we shall deal with \( I_1^t(q) \) first.

Due to the presence of \( \rho_1 \), the integral \( I_1^t(q) \) is defined on \( \{ k : |k| < 1 \} \). Now given \( \varepsilon > 0, \exists \delta' > 0 \) such that if \( |\theta| < \delta' \) then \( |1 - e^{i \theta}| < \varepsilon \). From Taylor’s theorem, (see (37) and the paragraph following it) we can find a \( t_0 > 0 \) such that, for all \( |k| < 1 \), if \( |t| > t_0 \) then \( |(E_{q+k/t} - E_q - (q \cdot k)/(tE_q)| t| = |tR(q, k/t)| < \delta' \), say. Then if \( |t| > t_0 \), we have

\[
|I_1^t(q)| \leq \int d\left(\frac{k}{t}\right) \left| h\left(q, \frac{k}{t}\right) \right| \varepsilon = \varepsilon \int d^3 k |h(q, k)|,
\]

where the latter integral is well defined, is independent of \( t \) and has been obtained from the previous one by changing the variables.

The next step is to show that the asymptotic limit of \( I_2^t(q) \) is zero. For this we consider the last integral in (41) as the difference of the two obvious integrals given by their exponential terms, i.e.

\[
I_2^t(t, q) = \int d\left(\frac{k}{t}\right) \rho_2(k) h\left(q, \frac{k}{t}\right) e^{-i \psi(k/t)t}
\]

\[
= \int d^3 k \rho_2(kt) h(q, k) e^{-i \psi(k)t}
\]

\[
I_2^t(t, q) = \int d\left(\frac{k}{t}\right) \rho_2(k) h\left(q, \frac{k}{t}\right) e^{-i \psi(k/t)t}
\]

\[
= \int d^3 k \rho_2(kt) h(q, k) e^{-i \psi(k)t},
\]

with the final forms of \( I_1^t, I_2^t \) being obtained by the obvious change of variables.

We shall first examine \( I_2^t \) and we begin by changing the variable for \( k \) again, this time to \( k = \omega^k \) with \( |k| = 1 \) and \( \omega \geq 0 \), i.e. polar coordinates. We now have

\[
I_2^t(t, q) = \int d\hat{k} \int_0^\infty \omega^2 d\omega \rho_2(\hat{k}\omega t) h(q, \hat{k}\omega t) e^{-i \hat{k}\omega t}.
\]

Note that \( \rho_2(\hat{k}\omega t) = \rho_2(\hat{k}\omega t) \) and this will be zero for \( 0 \leq \omega t \leq \frac{1}{2} \). The exponent of the integral in (44) is

\[
\psi(\hat{k}\omega) = E_{q+k\omega} - E_q - \omega
\]

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so that
\[ \xi_1(\omega) \overset{\text{def}}{=} \frac{\partial \psi}{\partial \omega} = \frac{q \cdot \hat{k} + \omega}{E_{q+k}} - 1. \] (45)

As \( \omega \to 0 \), \( \xi_1(\omega) \) tends, uniformly in \( \hat{k} \), to \((q \cdot \hat{k}/E_q) - 1\), and since \(|q \cdot \hat{k}/E_q| \leq (|q|/E_q) < 1\), we have \( \xi_1(\omega) = \partial \psi/\partial \omega \) is bounded, uniformly in \( \hat{k} \), strictly away from 0 in a neighbourhood of \( \omega = 0 \). It is also easy to check that \( \xi_1(\omega) \) is non-zero for any finite value of \( \omega \).

We now have
\[ I_1^2(t, q) = \frac{1}{i t} \int d\hat{k} \int_0^\infty d\omega \rho_2(\hat{k} \omega t) h(q, \hat{k} \omega) \left( \frac{\omega^2 h(q, \hat{k} \omega)}{\xi_1(\omega)} \right) e^{-i \psi(\hat{k} \omega)t}, \] (46)
with the latter expression being obtained after integration by parts in \( \omega \), and noting that the boundary terms vanish. This can then be written as the sum of two integrals
\[ I_1^2(t, q) = \frac{1}{i t} \int d\hat{k} \int_0^\infty d\omega \frac{\partial \rho_2(\hat{k} \omega t)}{\partial \omega} \left( \frac{\omega^2 h(q, \hat{k} \omega)}{\xi_1(\omega)} \right) e^{-i \psi(\hat{k} \omega)t}, \] (47)

If \( v_1(q, \hat{k} \omega) \overset{\text{def}}{=} \omega^2 h(q, \hat{k} \omega)/\xi_1(\omega) \) then \( \partial v_1(q, \hat{k} \omega)/\partial \omega \) is rapidly decreasing at infinity. The first integral in (47) can now be disposed of since it is bounded by \((1/|t|) \int d\hat{k} \int_0^\infty d\omega \left| \partial v_1(q, \hat{k} \omega)/\partial \omega \right| \) and since this is well defined, vanishes as \(|t| \to \infty \).

Before dealing with the second integral in (47) it is worthwhile examining the properties of the derivatives of \( \rho_2 \). We can write this function as \( \rho_2(\hat{k} \omega) = \rho(\hat{k}, \omega) \), emphasising the fact that \( \rho_2 \) is a function of two variables. Now let us write
\[ \varrho(\hat{k} \omega) \overset{\text{def}}{=} \frac{\partial}{\partial \omega} \rho_2(\hat{k} \omega) = \partial \rho(\hat{k}, \omega) \] (48)

Since \( \rho_2 \) is smooth and \( \rho_2(\hat{k} \omega) = 1 \) for \( \omega > 1 \), the function \( \varrho(\hat{k} \omega) \) is a smooth function which has the important property that it is zero for \( \omega > 1 \), i.e., \( \varrho \) has its support in \( \{ \omega : 1/2 \leq \omega \leq 1 \} \).

The second integral in (47), which we denote by \( \mathcal{J}_1(q) \), is now
\[ \mathcal{J}_1'(q) = \frac{1}{i t} \int d\hat{k} \int_0^\infty \omega^2 d\omega \frac{\partial}{\partial \omega} \left( \rho_2(\hat{k} \omega t) \right) v_1(q, \hat{k} \omega) e^{-i \psi(\hat{k} \omega)t} \]
\[ = \frac{1}{i t} \int d\hat{k} \int_0^\infty \omega^2 d\omega \varrho(\hat{k} \omega t) v_1(q, \hat{k} \omega) e^{-i \psi(\hat{k} \omega)t}, \] (49)

The last integral in (49) is now written in a form which is more convenient and which is obtained by the following changes of variables. First replace \( \hat{k} \omega \) by \( k \), and then replace
The form of $J^t_1(q)$ then changes to

$$J^t_1(q) = \frac{1}{i} \int d\left(\frac{k}{t}\right) \varrho(k) v_1 \left(q, \frac{k}{t}\right) e^{-i\psi(\frac{k}{t})t} \tag{50}$$

In order to deal with (50) we shall first have to examine the second integral in (43). Applying the same methods to $I^2_2$ as we have to $I^1_2$ one can easily show that $I^2_2$ can also be written as the sum of two integrals, as in (47), but with the exponent $\psi$ replaced with $\psi'$ and $\xi_1$ replaced with $\xi_2$, where

$$\xi_2(\omega) = \frac{\partial \psi'(\hat{k}\omega)}{\partial \omega} = \frac{q \cdot \hat{k}}{E_q} - 1. \tag{51}$$

Thus

$$I^2_2(t, q) = \frac{1}{it} \int d\hat{k} \int_0^\infty d\omega \rho_2(\hat{k}\omega t) \frac{\partial}{\partial \omega} \left( \frac{\omega^2 h(q, \hat{k}\omega)}{\xi_2(\omega)} \right) e^{-i\psi'(\hat{k}\omega)t}$$

$$+ \frac{1}{it} \int d\hat{k} \int_0^\infty \omega^2 d\omega \frac{\partial}{\partial \omega} (\rho_2(\hat{k}\omega t)) \left( \frac{h(q, \hat{k}\omega)}{\xi_2(\omega)} \right) e^{-i\psi'(\hat{k}\omega)t}, \tag{52}$$

c.f. (47). The function $\xi_2$ has similar properties to $\xi_1$ and the first integral in (52) vanishes asymptotically in a similar fashion to the corresponding integral in (47). This leaves us with the latter integral in (52), which we shall write as $J^t_2(q)$. Using the same changes of variables, this can be written as

$$J^t_2(q) = \frac{1}{i} \int d^3 \left(\frac{k}{t}\right) \varrho(k) v_2 \left(q, \frac{k}{t}\right) e^{-i\psi'(\frac{k}{t})t} \tag{53}$$

The final objective is to show that $J^t_1(q) - J^t_2(q)$ is asymptotically vanishing and this can be achieved in two steps.

First define $J^t_3(q)$ as

$$J^t_3(q) = \frac{1}{i} \int d^3 \left(\frac{k}{t}\right) \varrho(k) v_1 \left(q, \frac{k}{t}\right) e^{-i\psi'(\frac{k}{t})t}. \tag{54}$$

i.e., the $v_2$ in $J^t_2(q)$ is replaced by $v_1$. Then it is straightforward to prove, along the lines of Lemma 1, that, for all $q$, we have $J^t_1(q) - J^t_3(q) \to 0$ as $|t| \to \infty$.

Secondly, we must show that $J^t_1(q) - J^t_3(q) \to 0$ as $|t| \to \infty$. This is also straightforward and can be fashioned along the lines of the proof that $I^t_1(q) \to 0$ as $|t| \to \infty$ (see (41) and the paragraph that follows it.) We omit the details.

This proves that, for every $q$, $I_t(q) \to 0$ as $|t| \to \infty$ (see(40)). The Lebesgue Dominated Convergence Theorem now implies that $I_t \to 0$ as $|t| \to \infty$, as required. □

References


