Instantons at Strong Coupling, Averaging over Vacua, and the Gluino Condensate

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Abstract

We consider instanton contributions to chiral correlators, such as $\langle 0|\mathrm{Tr}\lambda(x)\mathrm{Tr}\lambda(x')|0\rangle$, in $\mathcal{N}=1$ supersymmetric Yang-Mills theory with either light adjoint or fundamental matter. Within the former model, extraction of the gluino condensate from a connected 1-instanton diagram, evaluated at strong coupling, can be contrasted with expectations from the Seiberg-Witten solution perturbed to an $\mathcal{N}=1$ vacuum. We observe a numerical discrepancy, coinciding with that observed previously in $\mathcal{N}=1$ SQCD. Moreover, since knowledge of the vacuum structure is complete for softly broken $\mathcal{N}=2$ Yang-Mills, this model serves as a counterexample to the hypothesis of Amati *et al.* that 1-instanton calculations at strong coupling can be interpreted as averaging over vacua. Within $\mathcal{N}=1$ SQCD, we point out that the connected contribution to the relevant correlators actually vanishes in the weakly coupled Higgs phase, despite having a nonzero value through infra-red effects when calculated in the unbroken phase.

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1 Introduction

Historically, the first arena in which supersymmetry and nonperturbative instanton effects were successfully integrated was in the direct calculation of certain chiral correlators [1]. These correlators involve the lowest components of chiral superfields, and a well-known example in $\mathcal{N}=1$ supersymmetric Yang-Mills (SYM) with gauge group $\text{SU}(2)$ is the 2-point function of gluino bilinears $\langle 0|\text{Tr} \lambda^2(x) \text{Tr} \lambda^2(x')|0 \rangle$. Chirality ensures that there can be no contribution at any order in perturbation theory, while at first sight it is also highly nontrivial that instantons can lead to a nonzero result without breaking supersymmetry. That they do, reflects one of the profound features of supersymmetric gauge dynamics, in that such correlators are essentially topological. In flat space, supersymmetry demands that they be spacetime independent constants [1], while more generally, when evaluated in nontrivial geometries, they are independent of the background metric, and reflect only specific topological features of the background [2]. This was made particularly clear by Witten’s subsequent construction of topological field theories [3].

The qualitative picture for instanton generation of these chiral correlators, and the associated superpotentials, has been clear for some time. However, the issue of the precise numerical coefficients has been rather more controversial. During the eighties, two different techniques for instanton calculations in this context were developed. The “strong-coupling instanton” (SCI) approach [1, 4, 5, 6] (see [7] for a review), applies in strongly coupled theories such as $\mathcal{N}=1$ SYM where the instanton is an exact solution to the equations of motion, and involves a direct integration over the instanton zero modes. For a given correlator, only a given instanton order can contribute, and the integration over instanton size $\rho$ is peaked at $\rho \sim |x - x'|$. Since the result is spacetime independent as a consequence of supersymmetry, by taking $|x - x'| \to 0$ one expects the calculation to be well defined, and thus the result appears, mathematically at least, to be “exact”. The loophole is that although small size instantons are under control, there is no guarantee that additional fluctuations at some larger scale, e.g. $\rho \sim \Lambda$, cannot contribute.

An additional puzzle concerns cluster decomposition. At large distances $|x - x'| \to \infty$ one expects the result to factorise. Returning to the SU(2) example, we expect $\langle 0|\text{Tr} \lambda^2(x) \text{Tr} \lambda^2(x')|0 \rangle = \langle 0|\text{Tr} \lambda^2|0 \rangle^2$, implying a nonzero gluino condensate. The double valued nature of $\langle 0|\text{Tr} \lambda^2|0 \rangle$ is consistent with the Witten index [8] for this theory, but the puzzle lies in the fact that instantons cannot contribute to $\langle 0|\text{Tr} \lambda^2|0 \rangle$ directly as $\text{Tr} \lambda^2$ can only saturate two of four fermionic zero modes. Thus in the instanton approximation $\langle 0|\text{Tr} \lambda^2|0 \rangle = 0$, and one apparently has a violation of cluster decomposition. A possible resolution to this puzzle, known as the “vacuum averaging hypothesis” was proposed by Amati et al. [7], within which the SCI calculation reflects an average over the vacua of the theory. This nicely explains why $\langle 0|\text{Tr} \lambda^2(x) \text{Tr} \lambda^2(x')|0 \rangle$ for example is nonzero, while a direct instanton calculation of $\langle 0|\text{Tr} \lambda^2|0 \rangle$ at strong coupling necessarily vanishes, since (for gauge group SU(2)) one obtains contributions of opposite sign from the two chirally asymmetric vacua.
The second approach to instanton calculations, known as the “weak-coupling instanton” (WCI) technique [9, 10] (for a recent review, see [11]), first makes use of additional fundamental matter fields to put the system at weak coupling where the instanton calculation should indeed be reliable. The WCI calculation produces a gluino condensate with a square-root dependence on the mass \( m \) of the matter fields, \( \langle \text{Tr} \lambda^2 \rangle \sim \sqrt{m} \).

The important feature, proven in [12], is that the mass dependence observed at weak coupling is exact. The proof is based on supersymmetric Ward identities [4] which demand that \( \langle \text{Tr} \lambda^2 \rangle \) be a holomorphic function of \( m \), i.e. \( \partial \langle \text{Tr} \lambda^2 \rangle / \partial m = 0 \), and on the relation \( \partial \langle \text{Tr} \lambda^2 \rangle / \partial m = \langle \text{Tr} \lambda^2 \rangle / 2m \). Therefore we can then take the limit \( m \to \infty \) to return to strong coupling and compare directly with the SCI calculation.

Comparison of the SCI and WCI results at strong coupling leads to a well-known numerical discrepancy,

\[
\langle 0 | \text{Tr} \lambda^2 | 0 \rangle^2_{\text{SC}} = \frac{4}{5} \langle 0 | \text{Tr} \lambda^2 | 0 \rangle^2_{\text{WC}}.
\] (1)

Note that in contrast to the suggestion of vacuum averaging within the SCI calculation, the WCI calculation necessarily reflects the contribution from a single vacuum, as one analytically continues from weak coupling where the vacuum state is unambiguous. However, if we assume that this model has two chirally asymmetric vacua, as implied by the Witten index [8], these contribute equally to \( \langle 0 | \text{Tr} \lambda^2 | 0 \rangle^2_{\text{SC}} \) and thus the factor of 4/5 in (1) represents an incompatibility, the origin of which has been the source of some debate over the last ten years.

The WCI calculations have the advantage that they are performed at weak coupling where the relevant contributions are well understood. Furthermore, the constraints of holomorphy have since been used by Seiberg to unearth a wealth of nonperturbative information about \( \mathcal{N}=1 \) gauge theories [13], and similar tools were also used by Seiberg and Witten in the context of \( \mathcal{N}=2 \) SYM [14]. Given these successes of the general weak coupling approach, suspicion naturally falls first on the strong coupling calculation and it is of interest to know whether the discrepancy in (1) has any physical content. In this regard, the vacuum averaging hypothesis has been utilised by Kovner and Shifman [15], who suggested that the numerical discrepancy in (1) could be explained in this way within pure \( \mathcal{N}=1 \) SYM by assuming the existence of an additional chirally symmetric vacuum, where \( \langle 0 | \text{Tr} \lambda^2 | 0 \rangle = 0 \). Such a vacuum must have rather unusual properties and there has been considerable debate as to its feasibility [16].

In this paper, we are not concerned with the chirally symmetric vacuum specifically, but in the underlying vacuum-averaging hypothesis, and consequently on the interpretation of the SCI calculation itself. Our main point will be that there is a clear counter-example to this hypothesis, namely \( \mathcal{N}=1 \) SYM with light adjoint matter (or in other words softly broken \( \mathcal{N}=2 \) SYM). The additional insight necessary to make this conclusion arises by perturbing the Seiberg-Witten solution [14], so that the exact \( \mathcal{N}=1 \) vacuum structure may be deduced. As is well-known, one finds only two chirally asymmetric vacua. This result will be briefly reviewed in Section 2. Knowledge of these vacua allows a simple deduction of the gluino condensate via use of the Konishi anomaly [17]. For comparison, we then consider the direct SCI calculation.
of $\langle 0 | \Tr \lambda^2(x) \Tr \lambda^2(x') | 0 \rangle_{SC}$ in this model, finding exactly the same numerical discrepancy as in Eq. (1). This failure of vacuum averaging leaves the interpretation of the SCI calculation rather unclear. We discuss this issue in relation to the known quantum corrections to the $\mathcal{N}=2$ moduli space in Section 3.

We turn in Section 4 to $\mathcal{N}=1$ SQCD, i.e. involving light fundamental matter, and review several aspects of the WCI and SCI calculations. In this case we have greater control over the WCI calculation, for the correlator $\langle 0 | \Tr \lambda^2(x) \phi^2(x') | 0 \rangle$ (where $\phi$ is a squark modulus) in particular, and we show how factorisation occurs naturally at weak coupling, where the SCI result (which is a connected diagram) actually gives no contribution. We also revisit an additional inconsistency (and a suggested resolution) of the SCI calculation which arises at 1-instanton order, in that the numerical discrepancy between the SCI and WCI calculations for $\langle 0 | \Tr \lambda^2(x) \Tr \lambda^2(x') | 0 \rangle$ and $\langle 0 | \Tr \lambda^2(x) \phi^2(x') | 0 \rangle$ actually differs, violating the Konishi relation [17].

In regard to inconsistencies of the SCI calculation in $\mathcal{N}=1$ SQCD, we note that recent work on correlators saturated at strong coupling by multi-instanton configurations [18] has suggested that cluster-decomposition may not hold within the SCI approach. We briefly comment on the distinction between these results and 1-instanton effects at the end of Section 4. Finally, we conclude in Section 5 with some additional remarks on the interpretation of the strong coupling calculation.

## 2 Exact Vacuum Structure vs SCI Calculations

In this section we shall present two different calculations of the gluino condensate within softly broken $\mathcal{N}=2$ SYM, one from the Seiberg-Witten solution of the $\mathcal{N}=2$ theory, and the other from a direct SCI calculation. We find a discrepancy advertised in the previous section, which we shall elaborate on in Section 3.

In terms of $\mathcal{N}=1$ multiplets, softly broken $\mathcal{N}=2$ SYM possesses one vector, and one adjoint chiral multiplet, the latter having mass $m \ll \Lambda$, where $\Lambda$ is the dynamical scale in $\mathcal{N}=2$ SYM. Throughout we shall take the gauge group to be $SU(2)$. The gluino $\lambda$, is the lowest component of the field strength $W_\alpha$ of the real vector superfield $V$, while we shall denote the lowest component of the adjoint chiral superfield $\Phi$, by $\phi$.

The gluino condensate, on which we will focus, is related to condensates of scalar fields by the Konishi anomaly [17]. This anomaly is expressed in terms of the following operator relation

$$Z_\Phi \mathcal{D}^2 (\bar{\Phi} e^V \Phi) = 4\Phi \frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} + \frac{T_R}{2\pi^2} \Tr W^2,$$

where $\mathcal{W}(\Phi)$ is the $\mathcal{N}=1$ superpotential, $R$ denotes the representation of $\Phi$, and the group factors for $SU(2)$ are $T_{\text{fund}} = 1/2$, $T_{\text{adj}} = 2$. $Z_\Phi$ is the bare field normalisation factor, and note that summation over the colour components of $\Phi$ is implied. An average of the left-hand side of this equation over a supersymmetric vacuum vanishes.
leading to the following relation for the gluino condensate,

$$\langle 0 | \text{Tr} \lambda^2 | 0 \rangle = \frac{8 \pi^2}{T_R} \left\langle 0 \left| \frac{\partial W(\phi)}{\partial \phi} \right| 0 \right\rangle.$$  \hfill (3)

In particular, when $\Phi$ is in the adjoint representation of $\text{SU}(2)$, it is natural to choose an $\mathcal{N}=2$ normalisation for $\Phi$ so that $Z_\phi = 1/g_0^2$ at the regulator scale. If we consider a particular quadratic superpotential $W(\Phi)$ of the form,

$$W(\Phi) = m \text{Tr} \Phi^2,$$  \hfill (4)

which gives a bare mass

$$m_\phi = \frac{m}{Z_\phi} = g_0^2 m$$  \hfill (5)

to the field $\Phi$, then the relation (3) reduces to

$$\langle 0 | \text{Tr} \lambda^2 | 0 \rangle = 8 \pi^2 m \langle 0 | \text{Tr} \phi^2 | 0 \rangle.$$  \hfill (6)

Knowledge of the gluino condensate thus boils down to knowledge of the vacuum state in terms of $\langle 0 | \text{Tr} \phi^2 | 0 \rangle$.

### 2.1 Exact Vacuum Structure

Before reviewing the vacuum structure which emerges quantum mechanically from the Seiberg-Witten solution, it is convenient first to recall the classical moduli space of $\mathcal{N}=2$ SYM. This space has two components,

$$\mathcal{M}_{\text{cl}} = \mathcal{M}_{a=0} + \mathcal{M}_{a \neq 0},$$  \hfill (7)

where $a$, which we identify with the scalar vev, $\langle 0 | \phi | 0 \rangle = a \tau_3/2$, parametrises the moduli space. The first component, $\mathcal{M}_{a=0}$, is simply a point $a = 0$ where the gauge group $G = \text{SU}(2)$ remains unbroken. The second component, $\mathcal{M}_{a \neq 0}$, arises for a generic vev for $\phi$ which breaks the gauge group from $G = \text{SU}(2)$ to $H = \text{U}(1)$, and is given by $\mathcal{M}_{a \neq 0} = (\mathbb{C}^*)^3/(G/H)_c$, where $(G/H)_c$ denotes the complexification of the coset, as is familiar in supersymmetric gauge theories. This manifold is simply $\mathbb{C}^*$, the complex plane with the origin removed, conveniently parametrised by $a \neq 0$. The point to emphasise here is that $\mathcal{M}_{\text{cl}}$ is actually two manifolds, the point $\mathcal{M}_{a=0}$ which, as we shall discuss is associated with SCI calculations, is distinguished by the triviality of its gauge orbit.

Turning to the quantum theory, we can focus on the low energy effective action associated, for a generic scalar vev $a$, with the unbroken $\text{U}(1)$. The light fields are described by an $\mathcal{N}=2$ $\text{U}(1)$ vector superfield $\mathcal{A}$ which contains the light multiplet associated with the modulus $a$. The action is determined by the holomorphic prepotential $\mathcal{F}(\mathcal{A})$, and in terms of $\mathcal{N}=1$ superfields the coefficient of the $\text{U}(1)$ gauge kinetic term $W^2$ is given
by \( \tau(a) \equiv F'(a) \), while the Kähler potential for the adjoint chiral fields determines the moduli space metric \( ds^2 = G_{\alpha\pi} |da|^2 \), as \( G_{\alpha\pi} = \text{Im} \tau/(4\pi) \). Classically,

\[
\tau_{\text{cl}} = \frac{4\pi i}{g_0^2} + \frac{\theta_0}{2\pi}
\]

is constant, and the metric is flat. However, quantum mechanically, the metric receives a 1-loop perturbative correction, and an infinite series of instanton corrections [19, 14].

While the vev \( a \) is a convenient coordinate for the classical moduli space, (and indeed can be given a gauge invariant meaning in terms of the \( W^\pm \)-boson mass as we shall discuss later), the natural gauge invariant modulus is given by

\[
\langle 0 | \text{Tr} \phi^2 | 0 \rangle = u .
\]

The moduli \( u \) and \( a \) are not independent, of course. While classically \( u = a^2/2 \) and thus (up to Weyl reflections) each gives an equivalent parametrisation of the classical moduli space, quantum mechanically the Seiberg-Witten solution [14] determines a more complex functional dependence and implies that only \( u \) is a good global coordinate on the moduli space.

We have focussed on the \( \mathcal{N} = 2 \) moduli space geometry, since on perturbing the system with a mass term for the adjoint chiral field given by Eq. (4), it is essentially this geometry which determines the scalar potential, \( V \). Using the definition (9) we see that in the effective low energy theory the perturbation (4) is realised in the form

\[
\mathcal{W}_{\text{eff}} = m u
\]

and thus the scalar potential is given by the reparametrisation invariant expression,

\[
V = \frac{|\partial_z \mathcal{W}_{\text{eff}}|^2}{G_{z\pi}} = \frac{|m \partial_z u|^2}{G_{z\pi}}.
\]

The natural choice \( z = u \) for the modulus leads to

\[
V = |m|^2 G^{\alpha\pi} ,
\]

which exhibits the supersymmetric vacua as the singular points where the inverse metric \( G^{\alpha\pi} \) vanishes. Classically, \( G^{\alpha\pi} \propto |u| \), i.e. the mass perturbation leads to a single vacuum state at \( u_{\text{cl}} = 0 \), corresponding to the point \( \mathcal{M}_{a=0} \) in the \( \mathcal{N} = 2 \) moduli space.

Quantum mechanically, the Seiberg-Witten solution shows that \( G^{\alpha\pi} \) vanishes at two points\(^1\),

\[
u = \pm \Lambda^2,
\]

which define two \( \mathcal{N} = 1 \) vacua. The dynamical scale \( \Lambda \) is defined here using Pauli-Villars regularisation [20] as

\[
\Lambda^4 = 4 M_{\text{PV}}^4 \exp (2\pi i \tau_{\text{cl}}).
\]

\(^1\)It is interesting that at these singular points, the metric \( G_{\alpha\pi} \) also vanishes, i.e. \( V \propto G^{\alpha\pi} \propto G_{\alpha\pi} \).
Note that at \( u = \infty \), which is the third singular point in the Seiberg-Witten solution, the metric \( G_{uu} \) does not vanish, instead it diverges.

To understand the appearance of singularities of the metric at the specific points given in (13), recall that the low energy \( \mathcal{N}=2 \) dynamics is encoded in an elliptic curve fibered over the \( u \)-plane [14],

\[
y^2 = P(x, u), \quad P(x, u) = (x - \Lambda^2)(x + \Lambda^2)(x - u).
\]

(15)

The points at which cycles of the torus can degenerate are zeros of the discriminant,

\[
\Delta = \Lambda^2(u^2 - \Lambda^4).
\]

(16)

We can see that these zeros determine the \( \mathcal{N}=1 \) vacua as follows: After perturbing with the superpotential (10) let us impose the constraint (15) via a Lagrange multiplier \( \eta \) as an addition to the superpotential, \( \Delta W_{\text{eff}} = \eta [y^2 - P(x, u)] \). One then finds that for \( m \neq 0 \) the \( F \)-flatness conditions are equivalent to the requirements: \( P = 0 \), and \( \partial_x P = 0 \). The appearance of the second order zeros in \( P(x) \) is equivalent to the vanishing of the discriminant (15). Thus the \( \mathcal{N}=1 \) vacua where \( V = 0 \) are given by the locus of points where the torus (15) degenerates. We shall see another reflection of this in the form of the scalar potential below.

Given the values in (13) for the modulus \( u \) at the \( \mathcal{N}=1 \) vacua we obtain, by use of the Konishi anomaly in the form of Eq. (6), the gluino condensate as,

\[
\langle 0 | \text{Tr} \lambda^2 | 0 \rangle_{\text{SW}} = \pm 8 \pi^2 m \Lambda^2.
\]

(17)

Note that although we have obtained this result for small \( m \ll \Lambda \), it holds for arbitrary \( m \) due to holomorphy. The proof follows in analogy with that in \( \mathcal{N}=1 \) SQCD [12]. We also note that this result may be derived without reference to the Konishi anomaly by elevating the coupling \( \tau \) to a spurion superfield, the highest component of which then acts as a source for \( \text{Tr} \lambda^2 \) (see e.g. [21, 18]).

We have emphasised the role of singularities of the metric in fixing the \( \mathcal{N}=1 \) vacua and thus the gluino condensate (17). At finite \( m \) this approach breaks down in the vicinity (\( \sim m/\Lambda \)) of the singularities and, as we will discuss in Section 3, the metric is smooth in this region. However, the superpotential is then modified to include the contribution of the gluino condensate, \( W_{\text{eff}} = \langle 0 | \text{Tr} \lambda^2 | 0 \rangle / 8 \pi^2 + mu \). This modified superpotential leads to vacua at the same points (13) as can be verified, in particular, by the “integrating in” procedure [22]. The result is thus self-consistent.

The sign ambiguity of the gluino condensate in (17) reflects the \( \mathbb{Z}_2 \) remnant of the anomalous \( U(1)_R \) symmetry. It is worth noting that this \( \mathbb{Z}_2 \) symmetry suggests that, provided the potential is smooth near \( u = 0 \), the first derivative must also vanish at this point, and it might be a meta-stable symmetric vacuum. This is not the case as may be determined by evaluating the potential (11) explicitly in terms of a complete elliptic integral \( K(k) \),

\[
V = \frac{4 \pi^3 |m \Lambda|^2}{|k|^2 \text{Re} \left( K(k') K(k) \right)}.
\]

(18)
where \( k^2 = 2\Lambda^2/(\Lambda^2 + u) \), and \((k')^2 = 1 - k^2\). The potential is nonzero at \( u = 0 \) and actually possesses a saddle point, which is a local maximum in \( \text{Re} \, u \) and a local minimum in \( \text{Im} \, u \).

Another side remark is that the scalar potential (18) is determined by periods of the curve (15). To see this, recall that the modular parameter \( \tau \) for the torus (15) is a ratio of the two periods, \( \tau = \omega_1/\omega_2 \), and these periods are given by \( \omega_1 = \partial_u F(a) = ikK(k')/\pi \) and \( \omega_2 = \partial_u a = kK(k)/\pi \). From (18), one observes that the potential is determined by the product \( \text{Im} (\omega_1 \omega_2) \), and the \( \mathcal{N}=1 \) vacua are given by the points where either the real or imaginary parts of the periods diverge in such a way that \( \text{Im} \, \tau \to 0 \), reflecting the degeneration of one of the cycles of the elliptic curve (15) as discussed above. We will discuss the physical interpretation of the scalar potential \( V \) in more detail in Section 3.

### 2.2 Strong Coupling Instanton Calculations

We shall now compare the result (17) obtained from the Seiberg-Witten solution with a 1-instanton calculation using the SCI approach. Note that this calculation is performed at the classical vacuum, \( a = 0 \), which is the unique SU(2) point in the \( \mathcal{N}=2 \) moduli space. Calculations with adjoint matter fields have previously been considered in [5].

The calculation proceeds in a manner very similar to that in pure \( \mathcal{N}=1 \) SYM, and we use the conventions of [11], to which the reader is referred for more details. In the pure \( \mathcal{N}=1 \) case, the instanton possesses four fermionic zero modes associated with \( \lambda \), which saturate the chiral correlator \( \langle 0 | \text{Tr} \, \lambda^2(x) \, \text{Tr} \, \lambda^2(x') | 0 \rangle \). The result for this correlator is given by

\[
\langle 0 | \text{Tr} \, \lambda^2(x) \, \text{Tr} \, \lambda^2(x') | 0 \rangle_{\mathcal{N}=1 \text{ SYM}} = \frac{2^{10}}{5} \pi^4 \frac{M^6_{PV}}{g_0^4} \exp(2\pi i \tau_0). \tag{19}
\]

When we add adjoint matter, there are additionally four fermionic zero modes for \( \psi \) (the superpartner of \( \phi \)). The relevant diagram is given in Fig. 1. Due to the mass

![Figure 1: One-(anti-)Instanton contribution to \( \langle 0 | \text{Tr} \, \lambda^2(x) \, \text{Tr} \, \lambda^2(x') | 0 \rangle \) in \( \mathcal{N}=1 \) SYM with an adjoint matter field of mass \( m \).](image-url)
perturbation, the additional matter zero-modes lead to a factor,

$$\left( \frac{m_\phi}{M_{PV}} \right)^2 = \left( \frac{g_0^2 m}{M_{PV}} \right)^2,$$

(20)

where $m_\phi$ is the bare mass defined in (5). Then, it is enough to multiply Eq. (19) by the factor (20) to get the result:

$$\langle 0 | \text{Tr} \lambda^2(x) \text{Tr} \lambda^2(x') | 0 \rangle_{SC} = \frac{2^8}{5} \pi^4 m^2 \Lambda^4.$$

(21)

This calculation is explicitly performed at $a = 0$, which is the classical vacuum in the presence of the mass perturbation, and we find that the result is a spacetime independent constant, as required by supersymmetry. It is also important to note that multi-instantons cannot contribute, since with $a = 0$ they will not lead to the correct $m^2 \Lambda^4$ dependence.

Since the result (21) is a constant, if we assume cluster-decomposition, we find on comparison with (17) a result for the gluino condensate satisfying,

$$\langle 0 | \text{Tr} \lambda^2 | 0 \rangle_{SC}^2 = \frac{4}{5} \langle 0 | \text{Tr} \lambda^2 | 0 \rangle_{SW}^2,$$

(22)

as advertised in Section 1. The factor $4/5$ matches the discrepancy (1) observed within instanton calculations in $\mathcal{N}=1$ SQCD. However, in contrast with the latter theories, we have in this case knowledge of the exact vacuum structure, and can verify that this numerical discrepancy cannot be explained by vacuum-averaging, which is the primary result of this paper. In particular, there are only two chirally asymmetric vacuum states (13), each contributing equally to $\langle 0 | \text{Tr} \lambda^2 | 0 \rangle^2$ according to Eq. (17).

### 3 SCI Calculations and the \(\mathcal{N}=2\) Moduli Space

Given the discrepancy between the SCI calculation and the Seiberg-Witten solution uncovered in the last section, it is natural to seek an explanation using knowledge of the exact vacuum structure. Indeed, it’s clear that the SCI calculation is performed in the SU(2) phase which, while classically the vacuum state, is lifted quantum mechanically. Since this lifting already occurs at the unperturbed level of $\mathcal{N}=2$ SYM, where only the U(1) phase on Coulomb branch is realised, we can search for an explanation within this context.

The SU(2) phase requires the presence of a massless multiplet associated with the $W^\pm$ bosons (occurring classically at $a = 0$). Quantum mechanically these states are not present within the strong coupling region of the moduli space, while the point $a = 0$ – where the SCI calculation is performed – is not present. Thus we suggest

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Instanton calculations have verified the identification of $a$ with $\langle 0 | \phi | 0 \rangle$ up to 2-instanton order.
that it is the discontinuity of the spectrum across the curve of marginal stability which is ultimately responsible for the failure of the SCI calculation in this model. In this section we shall comment on various aspects which lead to this conclusion.

**The Scalar Potential**

It is worth recalling that in \( \mathcal{N}=1 \) SQCD with one massive flavour, quantum corrections to the vacuum structure appear at first sight somewhat similar to those in softly broken \( \mathcal{N}=2 \) SYM. In particular, the unique classical vacuum \( a=0 \) corresponds to the SU(2) phase, while a 1-instanton effect generates an infinite superpotential (see section 4) at this point, resolving the classical vacuum state to two quantum vacua. Thus removal of the SU(2) phase is associated with the generation of an infinite potential at this point.

We shall see shortly that there are important distinctions between this scenario and softly broken \( \mathcal{N}=2 \). Nonetheless, we can proceed by analogy in considering the leading perturbative correction to the scalar potential \( V \). This arises directly from the analogous correction to the moduli space metric, and has the form,

\[
V = 4\pi^2 |m|^2 \frac{|a|^2}{\ln(8a^2/\Lambda^2)} \left( 1 + O \left( \frac{\Lambda^4}{a^4} \right) \right),
\]

where instantons provide the power-like corrections. Importantly, this potential exhibits a singularity at the Landau pole, beyond which, \( |a| \leq |\Lambda/2\sqrt{2}| \), it breaks down. This singularity separates the vacuum point \( a=0 \) from the region \( |a| \gg |\Lambda| \) in which the potential is physically meaningful. This presents us with a heuristic picture for the removal of \( a=0 \) from the physical moduli space. However, this singular behaviour can only be taken as a clue to what happens quantum mechanically at \( a \sim \Lambda \), since instanton corrections are numerically large in this regime, and manifestly singular at \( a=0 \), reflecting an induced “tube-like” behaviour of the metric inside the region \( |a| \leq |\Lambda/2\sqrt{2}| \). Nonetheless, we point out that the singular curve \( |a| = |\Lambda/2\sqrt{2}| \) is diffeomorphic to, and not too far from, the quantum mechanical curve of marginal stability for states such as the \( W^\pm \) bosons.

Before exploring this point in more detail, we should emphasise that this picture is at best heuristic even including the instanton corrections to (23). In particular, if we consider the behaviour of \( V \) near the vacuum \( u = \Lambda^2 \) in terms of the appropriate small variable \( a_D = \mathcal{F}'(a) \ll \Lambda \), we find a logarithmic singularity,

\[
V = 8\pi^2 \frac{|m\Lambda|^2}{\ln(8\Lambda/a_D)} + \cdots.
\]

Of course, this singularity simply reflects a breakdown of the effective theory due to the presence of additional massless states (monopoles in this case). Therefore, as noted in the weak coupling regime [20, 23, 24]. Furthermore, a gauge invariant definition of \( a \) is given by the (complex) mass of \( W \) boson supermultiplet, whose vanishing would imply restoration of SU(2). However, this is clearly true only where \( W^\pm \) bosons exist. While \( a \) may be analytically continued into the strong coupling region, it then has little to do with the SU(2) phase.
earlier, although appropriate for locating the vacua, the potential we have been using is only valid outside a small region of the singularity of size $O(m/\Lambda)$. As is well known, adding the additional light monopole (or dyon) states, $\Delta W_{\text{eff}} = \sqrt{2}M a_D M$, results in smooth vacua where these fields condense (see e.g. [25, 26, 27] for explicit studies of the potential in this case). Note that the breaking scales for $\mathbf{Z}_4 \rightarrow \mathbf{Z}_2$ (the discrete remnant of $\text{U}(1)_{R}$), and confinement, are then related: $\langle 0 | \text{Tr} \lambda^2 | 0 \rangle = -4\sqrt{2}\pi^2\Lambda(0|M M|0)$.

The Curve of Marginal Stability

As discussed above, the leading perturbative correction to the potential (23) is singular near the curve of marginal stability. Since the vacuum state at $a = 0$ lies within the singular curve, this suggests that the lifting of the SU(2) phase may be associated, not with a potential barrier as in $\mathcal{N} = 1$ SQCD, but with the marginal stability of the $W^\pm$ boson states on this curve. This is the point of view we shall now explore.

We begin by recalling that the classical singular point at $u_{\text{cl}} = a^2/2 = 0$ is degenerate in the sense that $W^\pm$ bosons and monopoles both become light in the vicinity of this point. Quantum mechanically, this singularity is “resolved” to the curve of marginal stability (CMS), given by the locus $\{u | \text{Im}(a_D/a) = 0\}$, on which BPS states, whose mass is given by the $\mathcal{N} = 2$ central charge

$$M_{\text{BPS}} = \sqrt{2} \left| n_e a + n_m a_D \right|,$$

where $n_e, n_m$ are integer electric and magnetic charges) can become marginally stable. There are then only two singular points at which charged states (monopoles and $(n_e = 1, n_m = -1)$ dyons) become massless, while $W^\pm$ bosons remain massive. Indeed, while

$W^\pm$ bosons naturally exist in the semi-classical spectrum, it is known that on crossing the CMS to the strong coupling region, they no longer exist as localised one-particle states [28, 29].

On the other hand, the relevance of the CMS curve when discussing $a = 0$ is clear when we recall that the monodromy structure of the function $a(u)$ implies that we can formally reach the point $a_{n'\text{th-sheet}} = 0$ by winding round the branch cuts (in particular, the logarithmic branch cut associated with the monopole point $u = \Lambda^2$), provided we

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure2.png}
\caption{The classical moduli space (on the left) $u_{\text{cl}} = a^2/2$ has a singular point at the origin where $W^\pm$ bosons and monopoles are massless. Quantum mechanically, this point is resolved to the CMS (on the right), where there are two distinct singularities on the real axis.}
\end{figure}
start somewhere on the first sheet where the ratio \( a_D/a \) is rational, i.e. particular points on the CMS. Thus Fig. 2 formally expresses the quantum mechanical resolution of the classical point \( a = 0 \) to a discontinuous set of points \( a_{\text{CMS}} \neq 0 \) lying above the CMS curve on higher Riemann sheets. Of course, the fact that such a point does not lie on the first sheet means that these points do not represent a restoration of the SU(2) phase. Indeed, the monodromy group is a quantum symmetry, and winding round to higher sheets simply moves us outside the fundamental domain for \( a \), which amounts to a relabelling and does not reflect a physical change.

To obtain a more physical picture for the lifting of the SU(2) phase, we note that within the U(1) effective theory, the CMS itself is not singular except at the points where monopoles and dyons become massless. Instead it reflects a degeneracy of states. However, within the theory as a whole, this degeneracy (and associated marginal stability) is reflected in a singularity of the \( Z \)-factor for massive states such as the \( W^\pm \) bosons. As noted in [28], this \( Z \)-factor is given by \( \text{Im} (F'(a)/a) \), which vanishes precisely on the CMS. The significance of this becomes apparent if we consider trying to follow a trajectory in the moduli space from the semi-classical region, reducing \( a \) toward zero. In doing so we shall necessarily reach the CMS, beyond which the \( W^\pm \) bosons are not present in the spectrum [29] and therefore its not possible to find an SU(2) phase inside this region. Therefore it is natural to conclude that the lifting of the SU(2) phase is associated physically with the marginal quantum stability of the \( W^\pm \) bosons at strong coupling whose presence would be required to restore a full massless SU(2) multiplet. We can now interpret the divergence of the potential (23) as a hint of this transition as one passes below the Landau pole.

The D3-Brane Probe Picture

Given the relation between the lifting of the SU(2) phase and the marginal stability of \( W^\pm \) boson states that we have argued for above, it is of interest to understand the decay of these states in more detail. However, while this question may be addressed in certain theories with matter, or larger gauge groups, where CMS curves are present at weak coupling [30], such techniques are not directly applicable for CMS curves present at strong coupling, as is the case here. Nonetheless, a simple geometric picture of the transition across the CMS is available within Type IIB string theory, or more precisely its extension to F-theory. Sen [31] has pointed out that the U(1) effective theory can be realised as the worldvolume theory on a D3-brane probe in the background of two mutually non-local 7-branes with the charges of monopoles and \((n_e = 1, n_m = -1)\) dyons. Within this framework, the BPS states present at strong coupling are realised as strings joining the D3-brane with one of the 7-branes [31]. However, it has been argued [32] that \( W^\pm \) boson states are realised as 4-string junctions, where the junction must lie on the CMS for the state to be supersymmetric (see Fig. 3). When the D3-brane reaches the CMS, the junction lies on the brane itself and the the two remaining “prongs” can dissociate on the the D3-brane, leaving the expected configuration of a monopole and a (1,-1) dyon, which is stable inside the CMS. Clearly, it would be of interest to understand this phenomenon within field theory.
Figure 3: The classical moduli space (on the left) with a D3-brane probe in the background of an orientifold 7-plane ($\Omega^7$). The $W^\pm$ bosons are represented by an open fundamental string ending on the D3-brane and reflecting off the $\Omega^7$-plane. Quantum mechanically (on the right), the $\Omega^7$-plane is resolved to two 7-branes with the respective charges of monopoles and $(1, -1)$ dyons, and the $W^\pm$ bosons are given by a particular 4-string junction configuration as shown.

4 Instanton Calculations in $\mathcal{N}=1$ SQCD

In this section, we turn to $\mathcal{N}=1$ SQCD with one flavour, and recall a number of well-known features, uncovered in the 1980’s [10, 6, 5], which provide additional insight into the relation between the SCI and WCI calculations.

SQCD exhibits some useful simplifications with regard to instanton calculations – namely, the 1-instanton saturation of the chiral correlators of interest, and the mass dependence of the vacuum state (see section 3.1). The model possesses chiral fields $Q^\alpha_f$, where $\alpha = 1, 2$ is the colour index and $f = 1, 2$ is a sub-flavour index. The $D$-flat directions are parametrised by a modulus superfield $X = \sqrt{Q^\alpha_f Q_{\alpha f}/2}$. Classically, the addition of a mass perturbation, $mX^2$, leads to the presence of a single vacuum at $\langle 0 | \phi | 0 \rangle = 0$ (where $\phi$ is the lowest component of $X$), i.e. the SU(2) phase, in analogy with the situation in softly broken $\mathcal{N}=2$. However, there is also a 1-instanton induced correction to the superpotential [9], leading to an expression of the form,

$$W_{\text{SQCD}} = mX^2 + \frac{\Lambda_1^5}{X^2} + \Lambda_1^5 X^2,$$  \hspace{1cm} (26)

where $\Lambda_1$ is a natural scale in the one flavour model,

$$\Lambda_1^5 = \frac{M_{PV}^5}{g_0^4} \exp(2\pi i \tau_0).$$  \hspace{1cm} (27)

We refer to bare fields and parameters, and $Z_\phi(M_{PV}) = 1$ fixes the normalization. Note that it is $m \Lambda_1^5$ which forms a renormalisation group (RG) invariant, not $\Lambda_1^5$. This superpotential implies $\langle 0 | \phi^2 | 0 \rangle \sim \Lambda_1^{5/2} m^{-1/2}$, putting the system at weak coupling in $m \ll \Lambda_1$.

WCI vs SCI Calculations

The natural chiral correlators to consider in this case are $\langle 0 | \text{Tr} \lambda^2(x) \text{Tr} \lambda^2(x') | 0 \rangle$, and $m \langle 0 | \text{Tr} \lambda^2(x) \phi^2(x') | 0 \rangle$, where we recall that the RG invariant field bilinears are
Tr $\lambda^2$ and $m\varphi^2$. In particular, within the WCI approach, the 1-instanton contribution to $\langle 0 | \text{Tr} \lambda^2(x) \varphi^2(x') | 0 \rangle$ was considered in a background with $a \gg \Lambda$ in [10]. The instanton action then has the form,

$$S_{\text{inst}} = \frac{8\pi^2}{g_0^2} + 4\pi^2 |a|^2 \rho_{\text{inv}}^2,$$

(28)

where $\rho_{\text{inv}} = \rho^2(1 + 4|\bar{\theta}_0\bar{\beta})$ is the supersymmetric instanton size, and $\bar{\theta}_0$ and $\bar{\beta}$ are fermionic collective coordinates associated respectively with supersymmetry and superconformal transformations of the instanton (see e.g. [11] for details). The large vev $a$ is associated with the instanton induced superpotential in the massless theory as discussed above. The result for $\langle 0 | \text{Tr} \lambda^2(x) \varphi^2(x') | 0 \rangle$ has the form

$$\langle 0 | \text{Tr} \lambda^2(x) \varphi^2(x') | 0 \rangle = \Lambda_5^5 I,$$

(29)

and has the special feature that it is saturated by zero size instantons. In particular, the integral $I$ in (29) has the form,

$$I = \int \frac{d\rho^2}{\rho^2} d^2\bar{\theta}_0 d^2\bar{\beta} f(\rho_{\text{inv}}^2) = 16 \left[ \rho^2 f'(\rho^2) - f(\rho^2) \right]_0^\infty,$$

(30)

where the function $f$ can be written in the form [10],

$$f(\rho_{\text{inv}}^2) = \exp(-4\pi^2 |a|^2 \rho_{\text{inv}}^2) \left[ 1 - f_{\text{sc}} \left( \frac{\rho_{\text{inv}}^2}{(x-x')^2} \right) \right].$$

(31)

Here $f_{\text{sc}}$ represents the contribution containing the dependence on the $\bar{\theta}_0$ and $\bar{\beta}$ zero modes of $\text{Tr} \lambda^2$ and $\varphi^2$,

$$f_{\text{sc}}(z) = \int_0^1 d\alpha \frac{2z^2\alpha^3}{[\alpha(1-\alpha)+z]^3} = z + O(z^2),$$

(32)

where the expansion corresponds to the limit $z = \rho_{\text{inv}}^2/(x-x')^2 \to 0$.

For nonzero $a$ both $f(\rho^2 = \infty)$ and $f'(\rho^2 = \infty)$ vanish, and the result (30) for $I$ is explicitly saturated by zero size instantons,

$$I_{\text{WC}} = 16 f(\rho^2 = 0) = 16.$$

(33)

This answer is determined by the instanton measure alone. The result displays the topological features of independence from $x - x'$, $a$, and from the detailed structure of the instanton action and the zero modes. Moreover, $f_{\text{sc}}$ which vanishes at $\rho^2 = 0$ does not contribute in the WCI result, which implies factorization since the connected contribution comes only from $f_{\text{sc}}$.

We can use the same expressions (30-32) to perform the analogous SCI calculation (see Fig. 4) by setting $a = 0$. Then,

$$I_{\text{SC}} = 16 f_{\text{sc}}(\rho^2 = \infty) = 16 \cdot \frac{1}{2}.$$

(34)

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Thus we find that the WCI and SCI results for \( \langle 0 | \text{Tr} \lambda^2(x) \varphi^2(x') | 0 \rangle \) actually arise from different sources. While the connected part does not contribute to the WCI calculation which is saturated by \( \rho = 0 \), it does contribute to the SCI calculation which is saturated at \( \rho = \infty \).

It is also possible to perform the calculation of \( I_{SC} \) directly following the connected diagram given in Fig. 4 for which one finds a result equal to (34), with the distinction that it is apparently saturated at \( \rho \sim |x - x'| \). We can see that this is in agreement with the infrared nature of the result (34) by re-introducing \( a \neq 0 \) as an infrared regulator for the calculation. In this case, for any finite \( a \), one observes an exact cancellation between two integrals saturated respectively in the ranges \( \rho \sim |x - x'| \) and \( \rho \sim 2\pi/a \), as discussed in [10]. However, within the SCI approach, one ignores the integral saturated at \( \rho \sim 2\pi/a \) leaving a finite result as in (34). Thus we see explicitly that as we send \( a \to 0 \) the discontinuity arises as an infrared effect at \( \rho = \infty \). Thus, although the details are apparently quite different, there is an analogy with softly broken \( \mathcal{N} = 2 \) SYM, in that the SCI result may be interpreted in terms of unphysical (in this case \( \rho = \infty \)) configurations.

**On the Consistency of the SCI Approach**

In addition to the arguments presented above, which serve an interpretative role for the nature of the SCI result, this calculation also reveals an additional technical inconsistency associated with the difference between the numerical factors relating the WCI and SCI calculations for \( \langle 0 | \text{Tr} \lambda^2(x) \varphi^2(x') | 0 \rangle \) and \( \langle 0 | \text{Tr} \lambda^2(x) \varphi^2(x') | 0 \rangle \). Recall that the SCI calculation for \( \langle 0 | \text{Tr} \lambda^2(x) \varphi^2(x') | 0 \rangle \) in \( \mathcal{N} = 1 \) SQCD [10] is performed in analogy with the discussion in Section 2, with the distinction that there is only one mass insertion (see [10, 11] for details). We can express the results of these calculations in the form,

\[
\langle 0 | \text{Tr} \lambda^2(x) \varphi^2(x') | 0 \rangle_{SC} = \frac{1}{2} \langle 0 | \varphi^2 | 0 \rangle_{WC} \cdot \langle 0 | \text{Tr} \lambda^2 | 0 \rangle_{WC} \tag{35}
\]

\[
\langle 0 | \text{Tr} \lambda^2(x) \text{Tr} \lambda^2(x') | 0 \rangle_{SC} = \frac{4}{5} \langle 0 | \text{Tr} \lambda^2 | 0 \rangle_{WC}^2. \tag{36}
\]
In particular, these relations imply,
\[
\langle 0 \bigg| Tr \lambda^2 \left[ 16\pi^2 m\varphi^2 - \frac{2}{5} Tr \lambda^2 \right] \bigg| 0 \rangle = 0. \tag{37}
\]
As we demonstrate below, this expression violates the Konishi relation (2) (due to the difference between the two coefficients in (35) and (36)) and is therefore inconsistent. This observation is not new (see [4, 6, 5]), but we would like to emphasise here that this conclusion follows without the imposition of the assumption of cluster-decomposition. Thus we present the argument in some detail.

The main point is that one may generalise the relation (3) to obtain,
\[
\langle 0 \bigg| O_1(x_1) \cdots O_n(x_n) \left[ \frac{8\pi^2}{T_R} \phi \frac{\partial W(\phi)}{\partial \phi} - Tr \lambda^2 \right] (y) \bigg| 0 \rangle = 0 \tag{38}
\]
where \( O_i(x_i) \) are the lowest components of chiral superfields. To prove this relation, we insert a complete set of states between \( O_n \) and the term in square brackets. The crucial point is that this sum reduces to vacuum states as an immediate consequence of supersymmetry – the correlators are spacetime independent constants, and contributions of higher states would imply \( x \)-dependence. Thus this correlator is proportional to a series of vacuum expectation values of the form,
\[
\langle 0_i \bigg| O \bigg| 0_j \rangle \propto \delta_{ij}, \tag{39}
\]
where \( |0_i\rangle \) refers to any supersymmetric vacuum state. This expression vanishes because, via the Konishi relation (2), we can rewrite it as a vacuum matrix element of a total derivative, and (38) then follows.

If we specialise (38) to the case at hand, for which it has the form,
\[
\langle 0 \bigg| Tr \lambda^2 \left[ 16\pi^2 m\varphi^2 - Tr \lambda^2 \right] \bigg| 0 \rangle = 0, \tag{40}
\]
we observe a direct contradiction with (37), implying as stated above that (37) is inconsistent with the Konishi anomaly. Although this inconsistency follows trivially if we assume cluster-decomposition, as has been noted before [4, 6, 5], we emphasise that we have not needed this additional assumption, i.e. that all chiral expectation values are diagonal in each vacuum, \( \langle 0_i | \mathcal{O} | 0_j \rangle \propto \delta_{ij} \). Of course, a direct implication is that cluster-decomposition is also violated. Given the discussion above which implies that the SCI result does not contribute to the factorisable WCI calculation, this conclusion may not be so surprising.

Possible Resolutions?

A possible resolution of this discrepancy with the Konishi relation was suggested by Amati et al. [6], in that \( \langle 0 | Tr \lambda^2 (x) \varphi^2 (x') | 0 \rangle \) may have a non-analytic mass dependence at \( m = 0 \). An explicit calculation in the limit \( m \to \infty \) indeed verified consistency with
(2), although not with the WCI results. We feel that consistency with the Konishi relation is essentially built into this latter calculation as one may see by viewing the massive field as a regulator. However, since there is no indication of mass singularities of this kind within the WCI calculation, it is not clear if this resolution suggests a helpful physical interpretation of the numerical failure of the SCI approach. We shall suggest a slightly different interpretation in Section 5.

One can ask whether the addition of a chirally symmetric vacuum state [15] would also resolve the problem. Within this model, we have less control than in softly broken $\mathcal{N}=2$ as the exact vacuum structure is not strictly determined at strong coupling. An application of cluster decomposition, with a carefully chosen weight for the symmetric vacuum, where $\langle 0_{S}|\text{Tr} \lambda^2|0_{S}\rangle = \langle 0_{S}|\varphi^2|0_{S}\rangle = 0$, apparently implies that one can explain one of the numerical factors in (35) and (36) in this way, but not both. There is a subtlety here in that the Kovner-Shifman vacuum necessarily contains massless fermions so as not to contribute to the Witten index. Thus there is no mass-gap, and the applicability of cluster-decomposition is not clear. However, we can use arguments based on supersymmetry as above, to ensure that only vacuum states survive the insertion of a complete set of states.

Cluster-Decomposition

It is interesting to contrast this 1-instanton discrepancy with recent work by Hollowood et al. [18] on correlators saturated by multi-instanton configurations. Therein, calculations based on the $n$-instanton measure at large $N_c$, and a numerical study of a particular SU(2) correlator saturated by 2-instanton configurations, led the authors to conclude that “cluster-decomposition” was violated within the SCI approach. This failure of cluster decomposition for particular correlators followed despite the fact that it holds for the $n$-instanton measure itself [33].

Note that this refers to a specific form of cluster-decomposition, namely the factorisation of an $n$-instanton saturated chiral correlator into a product of 1-instanton saturated correlators; schematically represented as, $(n$-instanton) $\rightarrow (1$-instanton)$^n$. In this regard, it is worth recalling that in simplified systems, such as certain 2D sigma models, where direct strong-coupling instanton calculations can be performed, cluster-decomposition in the sense of multi-instanton configurations follows from the structure of the collective coordinate integral (see e.g. [34]). Thus, purely at the mathematical level, the underlying reason for its apparent failure in SCI calculations in $\mathcal{N}=1$ SYM remains something of a mystery. We note only that as we emphasised earlier the classical moduli space is disconnected. Thus it may be more appropriate to consider 2D sigma models on disconnected target spaces.

In contrast, while the discrepancy we have discussed above also implies a breakdown of cluster-decomposition in a simple sense, it is quite distinct from these multi-instanton considerations. Specifically, we only consider 1-instanton effects, and thus cluster decomposition represented in the form $\langle 0|\text{Tr} \lambda^2(x) \varphi^2(x')|0\rangle \rightarrow \langle 0|\varphi^2|0\rangle \langle 0|\text{Tr} \lambda^2|0\rangle$ cannot be interpreted in terms of a factorisation of instanton contributions. Here the notion is more basic in that, as described above, the WCI result factorises trivially as it is
effectively a calculation of $\langle 0 | Tr \lambda^2 | 0 \rangle$ in a background field.

5 Concluding Remarks

In this note we have discussed various inconsistencies of the strong coupling approach to the calculation of instanton induced chiral correlators. In particular, we have argued that within a specific model – softly broken $\mathcal{N}=2$ SYM – the vacuum averaging hypothesis cannot be used to explain the numerical value of $\langle 0 | Tr \lambda^2 | 0 \rangle_{SC}$. Furthermore, within $\mathcal{N}=1$ SQCD with one flavour, although we have no clear picture of the vacuum structure at strong coupling in this case, we recalled that an apparent technical inconsistency at 1-instanton order with the Konishi relation is still perfectly consistent with the factorisability of the weak coupling calculation as the connected part associated with SCI actually vanishes.

Since are not strictly at liberty to integrate out the adjoint chiral field in softly broken $\mathcal{N}=2$, we have little to say about the possible existence (or non-existence) of the chirally symmetric Kovner-Shifman vacuum in pure $\mathcal{N}=1$ SYM. In particular, for consistency, we have always assumed when working with an adjoint chiral field that $m \ll \Lambda$. For $m \geq \Lambda$ there may be additional states to consider, and we are not guaranteed that vacuum rearrangements will not take place. However, we note that the arguments of Kovner and Shifman in favor of the new vacuum rely heavily on the vacuum averaging hypothesis [7] for SCI calculations.

The inconsistencies we have discussed suggest that the SCI approach is deficient with regard to extracting physical correlators in the specific $\mathcal{N}=1$ theories studied in this paper. In particular, within softly broken $\mathcal{N}=2$, it is apparent that the SCI calculation, saturated at 1-instanton order, is referred to the SU(2) phase which, while a classical vacuum, is lifted quantum mechanically. In contrast, the result arising from the Seiberg-Witten solution, is instead interpreted as an effect due to an infinite series of instanton corrections within the U(1) phase. Thus the discrepancy does not seem so surprising, and this raises the question of whether the SCI result has a well-defined interpretation. One possibility follows from the fact that the chiral correlators studied here apparently define particular topological invariants. Indeed they bear a close relation to expectation values calculable on generic backgrounds within Witten’s twisted $\mathcal{N}=2$ formulation of Donaldson theory. However, this picture applies quite generally to the correlators themselves, and is not specific to a particular instanton approximation. Nonetheless, given that the SCI approach determines unambiguous numerical values for these invariants, one may wonder whether they are related to a particular topological variant of $\mathcal{N}=1$ SYM.

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3Since we work in flat space, this is manifest only in independence from the spacetime coordinates. Extension to specific nontrivial geometries does however indicate dependence only on the intrinsic topology [2].
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