Padé-Improvement of CP-odd Higgs Decay Rate into Two Gluons

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Abstract

We present an asymptotic Padé-approximant estimate for the four-loop coefficients within the linear combination of correlators entering the recently calculated decay rate of a CP-odd Higgs boson, with an assumed mass $m_A = 100 \text{ GeV}$, into two gluons. All but one of these coefficients are shown to be determined for arbitrary $m_A$ from the known 3-loop-order rate by renormalization group methods. Asymptotic Padé-approximant estimates for these coefficients are all seen to be within 12% of their correct values. The four-loop term in the decay rate for $m_A = 100 \text{ GeV}$ is estimated to be only 4.3% of its leading one-loop contribution.
The decay rate into two gluons \( (g) \) of a CP-odd Higgs boson \((A)\) occurring within a two-Higgs-doublet extension of the standard model has been calculated to three loop order by Chetyrkin, Kniehl, Steinhauser, and Bardeen [1]. Their result is expressed in terms of a linear combination of the imaginary parts of three different correlators:

\[
\Gamma(A \to gg) = \frac{\sqrt{2}G_F}{m_A} R \left( \alpha_s, q^2 = m_A^2, \mu^2 = m_A^2, m_t^2 \right),
\]

\[
R(\alpha_s, q^2, \mu^2, m_t^2) \equiv \tilde{C}_1^2 Im\langle [O_1']^2 \rangle \\
+ 2\tilde{C}_1\tilde{C}_2 Im\langle [O_1'][O_2'] \rangle \\
+ \tilde{C}_2^2 Im\langle [O_2']^2 \rangle.
\]

(2)

For \( N_c = 3 \) and \( n \) light flavours, the terms within the linear combination (2) are shown [1] to be \([x \equiv \alpha_s/\pi, L \equiv \ln(\mu^2/q^2), L' \equiv \ln(m_t^2/q^2)]\)

\[
\tilde{C}_1 = \frac{-x}{16} \left[ 1 + O(x^3) \right],
\]

(3)

\[
\tilde{C}_2 = x^2 \left[ \frac{1}{8} - (L - L')/4 \right] + O(x^3),
\]

(4)

\[
Im\langle [O_1']^2 \rangle = \frac{8q^4}{\pi} \left\{ 1 + x \left[ \left( \frac{97}{4} - \frac{7n}{6} \right) + \left( \frac{11}{2} - \frac{n}{3} \right) L \right] \\
+ x^2 \left[ 392.223 - 48.0753n + 0.887881n^2 \\
+ \left( \frac{3405}{16} - \frac{73}{3}n + \frac{7}{12}n^2 \right) L \\
+ \left( \frac{363}{16} - \frac{11}{4}n + \frac{n^2}{12} \right) L^2 \right] + O(x^3) \right\},
\]

(5)

\[
Im\langle [O_1'][O_2'] \rangle = \frac{q^4xn}{\pi} + O(x^2),
\]

(6)

\[
Im\langle [O_2']^2 \rangle = \frac{q^4x^2n^2}{8\pi} + O(x^3).
\]

(7)

One can combine these results to obtain the following series for the linear combination of correlators defined by (2):

\[
R(\alpha_s, q^2, \mu^2, m_t^2) \equiv \frac{q^4}{32\pi} S[x, L, L']
\]

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\[\begin{align*}
&= \left( \frac{q^4}{32\pi} \right) x^2 \left[ 1 + (a_0 + a_1 L)x + \left( b_0 + b_1 L + b_2 L^2 \right) x^2 \right. \\
&\quad + \left( c_0 + c_1 L + c_2 L^2 + c_3 L^3 \right) x^3 + \ldots \] \\
&\quad \quad ,
\end{align*}\] (8a)

\[a_0 = \frac{97}{4} - \frac{7n}{6},\] (8b)

\[a_1 = \frac{11}{2} - \frac{n}{3},\] (8c)

\[b_0 = 392.223 - (48.5753 + L')n + 0.887881n^2,\] (8d)

\[b_1 = \frac{3405}{16} - \frac{70n}{3} + \frac{7}{12}n^2;\] (8e)

\[b_2 = \frac{363}{16} - \frac{11n}{4} + \frac{n^2}{12}.\] (8f)

The terms listed above arise entirely from the first two terms of (2), as the final term \( (\tilde{C}_2^2 \text{Im} \langle O_2^2 \rangle) \) is \( \mathcal{O}(x^6) \). The four-loop \( \mathcal{O}(x^5) \) coefficients \( c_0 - c_3 \) in (8) are as yet undetermined. All but \( c_0 \) of these can be obtained via renormalization-group (RG) methods.

RG-invariance of the physical decay rate (1) and, consequently, the linear combination of correlators (2) implies that the function \( S[x, L, L'] \) in (8a) satisfies

\[\left[ \frac{\partial}{\partial L} + \beta(x) \frac{\partial}{\partial x} + 2\gamma_{m_t}(x) \frac{\partial}{\partial L'} \right] S[x, L, L'] = 0,\] (9)

where

\[\beta(x) = -\beta_0 x^2 - \beta_1 x^3 - \beta_2 x^4 \ldots ,\] (10)

\[\gamma_{m_t}(x) = -\gamma_0 x - \gamma_1 x^2 - \gamma_2 x^3 \ldots .\] (11)

If \( m_t \) is a pole-mass independent of the renormalization scale \( \mu \), then \( \gamma_{m_t} = 0 \). However, if \( m_t \) is a \( \mu \)-dependent running quark mass, then \( \gamma_0 = 1 \), and subsequent \( \gamma_i \)'s in (11) are as determined in ref. [2].

Substitution of (8a), (10), and (11) into (9) yields the following set of equations for the aggregate coefficient of \( x^k L^k \) to vanish:

\[x^3: a_1 - 2\beta_0 = 0,\] (12)
\[ x^4: b_1 - 3\beta_0 a_0 - 2\beta_1 = 0, \]  
\[ x^4L: 2b_2 - 3a_1\beta_0 = 0, \]  
\[ x^5: c_1 - 4b_0\beta_0 - 3a_0\beta_1 - 2\beta_2 + 2n\gamma_0 = 0, \]  
\[ x^5L: 2c_2 - 4b_1\beta_0 - 3a_1\beta_1 = 0, \]  
\[ x^5L^2: 3c_3 - 4b_2\beta_0 = 0. \]  

The coefficients \( \beta_{0-2} \) in (10) for \( n \) light flavours are given by [3]

\[ \beta_0 = \frac{11}{4} - \frac{n}{6}, \]  
\[ \beta_1 = \frac{51}{8} - \frac{19n}{24}, \]  
\[ \beta_2 = \frac{2857}{128} - \frac{5033n}{1152} + \frac{325n^2}{3456}; \]

the coefficients \( \gamma_1, \gamma_2, \ldots \) in (11) do not enter (9) until \( \mathcal{O}(x^6) \). Using eqs. (8.b,c,e,f), (18), and (19), we see that eqs. (12-14) are explicitly upheld, thereby confirming the RG invariance of (8a). The unknown coefficients \( c_1, c_2, \) and \( c_3 \) are obtained via equations (15-17):

\[ c_1 = 4822.88 - n(11L' + 884.455 + 2\gamma_0) + n^2 \left( \frac{2}{3}L' + 45.1091 \right) - 0.591921n^3, \]  
\[ c_2 = \left( \frac{11}{2} - \frac{n}{3} \right) \left( \frac{1779}{8} - \frac{1177n}{48} + \frac{7n^2}{12} \right), \]  
\[ c_3 = \frac{1}{2} \left( \frac{11}{2} - \frac{n}{3} \right)^3. \]

For the physical case of \( n = 5, m_t = 175.6 \text{ GeV} \) [the t-quark pole mass \( (\gamma_0 = 0) \)], with \( m_A \) chosen as in [1] to have a reference value of 100 GeV, we find that

\[ c_1 = 1411, \quad c_2 = 438.4, \quad c_3 = 28.16. \]  

The coefficient \( c_0 \) is RG-inaccessible to order \( x^5 \).
The four-loop correlation-function coefficients \(c_{0-3}\) can be estimated using asymptotic Padé-approximant methods as delineated in ref. [4]. Given a correlation function of the form

\[\Pi(x) = F(x) \left[1 + R_1 x + R_2 x^2 + R_3 x^3 + \ldots\right],\]  

with only coefficients \(R_1\) and \(R_2\) known, the simplified asymptotic error formula (utilized in [5] to estimate \(\beta_3\) from \(\beta_{0-2}\))

\[\delta_{N+2} \equiv \frac{R_{[N]} - R_{N+2}}{R_{N+2}} = \frac{-A}{N + 1},\]  

characterizing the \([N][1]\) Padé-approximant prediction for \(R_{N+2}\), yields the following prediction for \(R_3\) [6]:

\[R_3 = \frac{2R_2^3}{R_1^2 + R_1 R_2}.\]  

Comparing eq. (25) to (8a), we see that the coefficients \(R_1, R_2, R_3\) are necessarily functions of \(L = \ln(\mu^2/q^2)\):

\[R_1 = a_0 + a_1 L,\]  

\[R_2 = b_0 + b_1 L + b_2 L^2,\]  

\[R_3 = c_0 + c_1 L + c_2 L^2 + c_3 L^3.\]  

Consequently, we can obtain \(c_{0-3}\) from the moment integrals

\[N_k \equiv (k + 2) \int_0^1 dw w^{k+1} R_3(w),\]  

where \(w \equiv q^2/\mu^2[L = -\ln(w)].\) Explicit substitution of (28c) into (29) yields [4]

\[N_{-1} = c_0 + c_1 + 2c_2 + 6c_3,\]  

\[N_0 = c_0 + \frac{1}{2} c_1 + \frac{1}{2} c_2 + \frac{3}{4} c_3,\]  

\[\text{Such integrals characterize the } O(x^3) \text{ contributions to the } k^{th} \text{ finite-energy sum rule integral } \int_0^s \theta^k \Pi(x, \theta) d\theta \text{ over the correlator (25), where } q^2 \text{ in (29) corresponds to } t, \text{ and where } \mu^2 \text{ in (29) corresponds to the continuum threshold } s_0.\]
\[
N_1 = c_0 + \frac{1}{3}c_1 + \frac{2}{9}c_2 + \frac{2}{9}c_3, \quad (32)
\]
\[
N_2 = c_0 + \frac{1}{4}c_1 + \frac{1}{8}c_2 + \frac{3}{32}c_3. \quad (33)
\]

Numerical values of \(N_{-1}, N_0, N_1, \) and \(N_2\) can be obtained through explicit use of the Padé-motivated estimate (27) within the integrand of (29) with \(R_1\) and \(R_2\) given by (28a) and (28b). Within these latter two equations, the coefficients \(a_0, a_1, b_0, b_1, b_2\) are as given by (8.b-f). We choose \(n = 5\) and \(L' = 2 \ln(1.756)\) to facilitate comparison with the true RG values (24) for \(m_A = 100\) GeV, and find that
\[
N_{-1} = 3310.4, \quad N_0 = 1863.8, \quad N_1 = 1512.6, \quad N_2 = 1359.2 . \quad (34)
\]

We substitute these values into (30-33) to find that
\[
c_0 = 981.7, \quad c_1 = 1274, \quad c_2 = 452.3, \quad c_3 = 24.96 . \quad (35)
\]

The relative errors of the above Padé estimates for \(c_1, c_2, \) and \(c_3\) from their true values, as given in (24), are respectively -9.7%, +3.2%, and -11.3%.

An alternative method for extracting \(c_{0-3}\) is to fit \(R_3(w)\), as obtained from (27), to the form of (28c) via least-squares minimization of the following function:
\[
\chi^2(c_0, c_1, c_2, c_3) = \int_0^1 [R_3(w) - (c_0 - c_1 \ln(w) + c_2 \ln^2(w) - c_3 \ln^3(w))]^2 dw
\]
\[
= 0.2532 \cdot 10^8 + 720c_3^2 + 12c_0c_3 + 24c_2^2
\]
\[
+ 48c_1c_3 + 2c_1^2 + 240c_2c_3 + c_0^2 + 12c_1c_2
\]
\[
+ 2c_1c_0 + 4c_2c_0 - 6620.83c_0 - 13688.4c_1
\]
\[
- 46945.6c_2 - 217719c_3. \quad (36)
\]

The values of \(c_{0-3}\) which minimize \(\chi^2\) are
\[
c_0 = 978.9, c_1 = 1285, c_2 = 445.0, c_3 = 26.03, \quad (37)
\]
in very close agreement with the values (35) extracted from the moment integrals (29). Moreover, \( \chi^2 \) is equal to only 4.7 at this minimum. The near cancellation of the \( O(10^7) \) lead term in (36) is indicative of the precision of the fit obtained between (27) and (28c), as is evident from Figure 1. Relative errors of \( \{c_1, c_2, c_3\} \) obtained from (36) with respect to their true RG-determined values (24) are respectively -8.9\%, +1.5\%, and -7.6\%, confirming the usefulness of the asymptotic Padé approach in estimating four-loop order contributions to the correlation function (25).

The \( m_A = 100 \text{ GeV} \) estimate for \( c_0 \) can be improved somewhat by using the correct values (24) of \( c_{1-3} \) within equation (30), the lowest moment integral estimated in (34) by asymptotic Padé-approximant methods. We then find that

\[
c_0 = 3310.4 - 1411 - 2(438.4) - 6(28.16) = 854. \tag{38}
\]

Identically the same result is obtained by minimizing \( \chi^2 \) with respect to \( c_0 \) after explicit incorporation of the correct (RG) values (24) of \( c_{1-3} \) into (36). There is a 15\% discrepancy between (38) and the estimates for \( c_0 \) in (35) and (37), indicative of the magnitude of anticipated relative error with respect to the true value for \( c_0 \). It is hoped that these estimates can be tested against an exact 4-loop calculation in the not-too-distant future.

The 4-loop correction to the CP-odd Higgs decay into two gluons, as determined to 3-loop order in [1], is found from (1), (8) and (38):

\[
\Gamma(A \rightarrow gg) = \frac{\sqrt{2}G_F m_A^3}{32\pi} \left[ 1 + a_0 x(m_A) + b_0 x^2(m_A) + c_0 x^3(m_A) \right]. \tag{39}
\]

Given \( n = 5 \), \( m_A = 100 \text{ GeV} \), \( \alpha^{(5)}(m_A) = 0.116 \) [1], and \( m_t = 175.6 \text{ GeV} \), the square bracketed expression in (39) for successive-loop corrections is \([1 + 0.680 + 0.226 + 0.043]\). The first three numbers are as calculated in ref. [1]; the final (underlined) term is obtained from the asymptotic Padé-approximant estimate for \( c_0 \) in (38). This estimate is further indicative of a progressive decrease in the ratio of successive terms in the \( A \rightarrow 2g \) decay rate, suggesting that if such a CP-odd Higgs were discovered, a perturbative calculation of its 2-gluon decay
rate could lead to a phenomenologically testable value \[ e.g. 1 + 0.680 + 0.226 + 0.043 + \ldots \approx 2.0 \text{ for } m_A = 100 \text{ GeV} \]. Such a Higgs characterizes the two-doublet version of electroweak symmetry breaking anticipated from supersymmetric extensions of the standard model, as first noted over two decades ago [7].

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References


Figure 1: The ratio of the Padé prediction $R_{3}^\text{pade}(w)$ (27), and the $\chi^2$-minimizing fitted form $R_{3}^\text{fit}(w)$ (28c) as a function of $w$. 

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