A way of constructing mathematically correct quantum geometrodynamics of a closed universe is presented. The resulting theory appears to be gauge- noninvariant and thus consistent with the observation conditions of a closed universe, by that being considerably distinguished from the traditional Wheeler–DeWitt one. For the Bianchi-IX cosmological model it is shown that a normalizable wave function of the Universe depends on time, allows the standard probability interpretation and satisfies a gauge-noninvariant dynamical Schrödinger equation. The Wheeler–DeWitt quantum geometrodynamics is represented by a singular, BRST-invariant solution to the Schrödinger equation having no property of normalizability.

1. Introduction

The homogeneous cosmological Bianchi-IX model is traditionally used as a test polygon for various theoretical methods in cosmology. It combines mathematical simplicity and physical meaningfulness. The purpose of the present work is, making the most of the quantum Bianchi-IX model in extended phase space (EPS) as an example, to explore a possibility of construction of physically (operationally) interpreted quantum geometrodynamics (QGD) by a strict mathematical method without using any assumption failing to allow a detailed mathematical proof. As a result of our investigation, we have come to the conclusion that mathematically correct and physically well-grounded QGD of a closed universe is a gauge-noninvariant theory, radically distinguished from the Wheeler-DeWitt (WDW) QGD, by its content.
2. The many-worlds interpretation of quantum geometrodynamics

The standard QGD is based on the WDW equations

$$ T^\mu |\Psi\rangle = 0, $$

$$ T^0 = (-g^{(3)})^{-\frac{1}{2}} p^{ik} (g^{il} g^{km} - \frac{1}{2} g^{ik} g^{lm}) p^{lm} + (-g^{(3)})^{\frac{1}{2}} R^{(3)} + T^{00}_{\text{(mat)}}, $$

$$ T^i = -2 (\partial_k p^{ik} + \gamma^i_{lm} p^{lm}) + T^{0i}_{\text{(mat)}}, $$

$p^{ik}$ are the momenta conjugate to the 3-metric $g_{ik}$, $\gamma^i_{kl}$ are the three-connections, $g^{(3)} \equiv \det g_{ik}$, $R^{(3)}$ is the 3-curvature, $T^{\mu\nu}_{\text{(mat)}}$ is the energy-momentum tensor (EMT) of the material fields. Derivation of these equations has been discussed by many authors. Because of a number of reasons analyzed in details below in Sec. 6. for the Bianchi-IX model, the WDW equations are not deducible by correct mathematical methods in the framework of the ordinary quantum theory (QT). In principle, this fact itself is not sufficient to discard the WDW theory. The ordinary QT is a phenomenological theory for describing quasilocal (in a macroscopic sense) phenomena. Therefore its extrapolation to the scales of the Universe as a whole is a radical physical hypothesis that may be incompatible correctly with the existing formalism. In this situation it makes sense to analyze the WDW theory as it is, without fixing attention on whether a correct way of its construction exists or not.

The most distinctive feature of the WDW theory is that there is no quantum evolution of state vector in time. Once adopting the WDW theory, one should admit that a wave function satisfying Eqs. (1) describes the past of the Universe as well as its future with all observers being inside the Universe at different stages of its evolution, and all observations to be made by these observers. We should emphasize that the question about the status of an observer in the WDW theory is rather specific since there is no vestige of an observer in Eqs. (1). The introduction of the observer into the theory is performed by fixing boundary conditions for a universe wave function (WF); we shall return to them below.

Let us discuss the peculiarities of setting problems in the WDW QGD using the Bianchi-IX model as an example, space homogeneity of the latter reducing the set
Gauge invariance is expressed by that the choice of a time coordinate is not made when deriving (more precisely, when writing down) the WDW equation. The equation should be solved under some boundary conditions. However, carrying out this program one should bear in mind that solutions to this equation are unnormalizable. The latter is obvious from the following mathematical observations: the WDW equation coincides formally with an equation for the eigenfunction of the physical Hamiltonian $H_{ph}$, appropriate to the zero eigenvalue. Meanwhile, nothing prevents us from studying the whole spectrum of eigenvalues of the operator $H_{ph}$; then the WF satisfying Eq. (2) turns out to be normalizable only if the value $E = 0$ belongs to a discrete spectrum of the operator $H_{ph}$.

As for the operator $H_{ph}$, explicit form of which will be presented in Sec. 6. of the paper, it has a continuous spectrum. In this situation one faces the alternative: 1) to declare the Bianchi-IX model to be meaningless and to put the question about searching for such models, whose operator $H_{ph}$ has a discrete spectrum line at $E = 0$, or 2) to discard normalizing the universe WF, enlarging more, by that, the discrepancy between QGD and ordinary QT. In the WDW QGD the second way is chosen that does not contradict in principle to the statement about the status of the universe WF mentioned above.

What should the WDW theory be taken for by an individual local observer? Obviously — for a paradigm fixing a certain way of thinking that in principle cannot be verified or overthrown experimentally. The reference to the fact that in the classical limit of the WDW theory one can obtain the Einstein equations, conclusions from which can be compared with cosmological observations, is not an argument, since it is obvious in advance that there exist an infinite number of ways to make a quantum generalization of the classical theory of gravity based not on mathematically correct procedures but on adopting some paradigm.

The WDW paradigm may contain a deep sense that is inaccessible for understanding yet. However, it is clear that its existence does not deprive another approach to QGD problems of a sense; for instance, an approach based on adopting another interpretative paradigm or an approach based on procedures pretending to a greater extent of mathematical strictness than those used in the WDW theory.

\[ H_{ph}|\Psi\rangle = 0. \] (2)

\[ ^a \text{Strictly speaking, unnormalizability of the WF means that there is no probabilistic interpretation of it, which is quite natural, because the results of the continuous reduction of a wave packet are involved in the boundary conditions but not in the structure of a superposition. Nevertheless, attempts were made by many authors to retrieve a probabilistic interpretation of QGD (see, for example,}^5\text{) analyzing its physical and mathematical contents by methods of the modern gauge field theory. In such an approach mathematical correctness of the analysis, thorough study of the question, whether the used mathematical procedures do exist, take on fundamental significance. Just this very problem will be discussed in details in our paper. The conclusions of our investigation are these: mathematically correct procedures exist at least for systems with finite number of degrees of freedom, however, such procedures lead to the theory radically distinguished from the WDW one by its mathematical form as well as by its physical content.} \]
3. Our research algorithm

In our work the following research algorithm is realized.

(i) A path integral (PI) form of a transition amplitude between any two states of the Universe is adopted as a basic QGD object (the probabilistic interpretation of the theory is predetermined by that).

(ii) Making use of a nonlocal gauge completely removing gauge degeneracy, we define an amplitude as a PI not containing divergences issuing from infinite number of gauge orbit representatives.

(iii) We take notice of the circumstance that a closed universe has no asymptotical states in which splitting off the 3-scalar and 3-vector gravitons takes place. By this reason we refuse the incorrect operations supposedly singling out information about the three-dimensionally transversal modes before finding a universe WF.

(iv) Instead we state the problem of constructing universe WF containing information about a physical object as well as about a reference system (RS), fixed by a gauge, in which the object is studied. A gauge is selected by such a way that the dynamics of the integrated system “the physical object + observation means (OM)” is described in EPS. The dynamical Schrödinger equation (SE) is an unambiguous consequence of the approach; it is derived directly from the quantum Hamilton equations in EPS or by the mathematically equivalent PI method.

(v) The solution of the existence problem of a WF, containing information about a physical object as well as about OM, can be obtained in an explicit form for the Bianchi-IX model: here we prove that it is possible to ascertain the structure of the general solution to the gauge-noninvariant Schrödinger equation (SE).

The results of the realization of this program consist in the following.

(i) The general solution (GS) to the gauge-noninvariant SE is normalizable and amenable to the standard Copenhagen interpretation of QT.

(ii) The transition to WDW theory realized as singling out the BRST-invariant sector of SE solutions is not dictated by any general physical or mathematical reasons and leads to unnormalizable BRST-invariant wave functions.

4. The nonlocal gauge and a transition amplitude

The standard quantum theory of gravity is based on the assumption of existence of the gauge-invariant transition amplitude

$$\langle f|i \rangle = \frac{1}{W(g)} \int Dg^{\mu\nu} \exp \left( -\frac{i}{2\kappa} \int \sqrt{-g} R d^4 x \right) ;$$

$$Dg^{\mu\nu} = \prod_x \left( -g \right)^{\frac{\kappa}{2}} \prod_{\nu \leq \mu} dg^{|\nu\mu|}.$$
Quantum Geometrodynamics of the Bianchi-IX Model in Extended Phase Space

\( g^{\mu\nu}(x) \) is a 4-metric tensor, \( R \) is a scalar curvature, \( \kappa \) is the gravity constant.

In Eq. (3) a normalizing factor \( W(g) \) is a divergent integral over gauge group space. The numerator of (3) also contains divergences issuing from the infinite number of gauge orbit representatives. The ratio of two divergent integrals is supposed to be a finite and physically meaningful quantity.

In applications of quantum gravity theory to the quantization problem of the Universe as a whole (in QGD) the mathematical uncertainty of the amplitude (3) gains a fundamental significance. The adduced above statements and generally known procedures dealing with the amplitude (3) are just a heuristic conventions concerning carrying out formal operations with the expression which one cannot give a definite mathematical sense to. We shall refer to the theoretically unprovable assumption about existence of the amplitude (3) as the hypothesis about a gauge-invariant regularization (\( R \)-hypothesis).

To make the PI convergent one uses “the expansion of a gauge unit” followed by factoring out the integral over gauge group space. It enables to use the action extremals with gauge-fixing and ghost terms to skeletonize the PI. When carrying out factoring, one changes the order of integration in a divergent integral that in principle cannot be regularized until the described procedure is over. The assumption that such operations, firstly, are admissible and, secondly, do not change properties of the original expression, is the essence of the second unprovable \( F \)-hypothesis.

In the above procedure local gauges are used

\[ f^\mu (h^{\mu\nu}) = 0, \] (4)

where \( h^{\mu\nu} = \sqrt{-g} g^{\mu\nu}, \) which maintain residual degeneration under the diffeomorphism group transformations. The residual transformation parameters satisfy the equations

\[ \hat{M} \nu^\mu \eta^\mu = 0; \] (5)

\[ \hat{M} \nu^\mu = \delta f^\nu \delta h^{\rho\sigma} \left( -\partial_\rho h^{\nu\sigma} + \delta^\sigma_\nu h^{\rho\lambda} \partial_\lambda + \delta^\rho_\nu h^{\sigma\lambda} \partial_\lambda \right). \]

The infinitesimal functions

\[ \delta h^{\mu\nu} = -\partial_\lambda h^{\mu\nu} \eta^\lambda + h^{\mu\lambda} \partial_\lambda \eta^\nu + h^{\nu\lambda} \partial_\lambda \eta^\mu \] (6)

describe inertial fields emerging or vanishing under the transformations of RS, within the class determined by a local gauge.

We propose a method of constructing the theory without making use of the \( R \)- and \( F \)-hypotheses. The method is based on the nonlocal analog of the condition (4) completely removing gauge degeneracy,

\[ F^\mu (h^{\lambda\nu}) = \int f^\nu \left[ h^{\lambda\sigma}(x') \right] G^\nu_\sigma(x, x') \, d^4x' = 0. \] (7)

Here \( G^\nu_\sigma(x, x') \) is the retarded Green’s function satisfying the equation

\[ \hat{M}^{\nu\lambda} G^\nu_\sigma(x, x') = \delta^\nu_\lambda \delta(x - x'); \]
the operator $\hat{M}^{\nu\mu}_{\lambda}$ is Hermitian conjugate to the operator (5). The rigidity of the nonlocal gauge condition is easily revealed: its variation under infinitesimal trans-
formations of the metric may be written as

$$\delta F^\mu = \int \left[ \frac{\delta f^\nu}{\delta h^\lambda} h^\lambda (x') \right] G^\mu_\nu (x, x') d^4 x' = \int \left[ \hat{M}^\nu_\mu \eta^\lambda (x') \right] G^\mu_\nu (x, x') d^4 x' =$$

$$= \int \eta^\lambda (x') \hat{M}^{\nu\mu}_{\lambda} G^\mu_\nu (x, x') d^4 x' = \eta^\mu (x), \quad (8)$$

so the equation $\delta F^\mu = 0$ has the unique solution $\eta^\mu = 0$.

We propose to adopt the following expression for the transition amplitude

$$\langle f, t | i, t_0 \rangle = \int D_F h^\mu_\nu \exp \left[ i \int_{t_0}^{t} d\tau \int d^3 x \left( -\frac{1}{2\kappa} h^\mu_\nu R^\mu_\nu + \pi^\mu f^\mu + \bar{\theta}^\nu \hat{M}^\nu_\mu \theta^\mu \right) \right], \quad (9)$$

$$D_F h^\mu_\nu = \prod_{x} \prod_{t_0 < t} M [h^\mu_\nu (x, \tau)] \prod_{\nu \leq \mu} d\eta^\mu_\nu (x, \tau) \prod_{\mu} \delta [F^\mu (x, \tau)],$$

as a basic postulate of quantum theory of gravity enabling to avoid the operations with divergent PI.

By identity transformations, symbolically written as

$$\delta (F^\mu) = \frac{1}{\det ||G^\mu_\nu||} \delta (f^\mu) = \det ||\hat{M}^{\nu\mu}_{\lambda}|| \delta (f^\mu),$$

followed by the representation of the $\det ||\hat{M}^{\nu\mu}_{\lambda}||$ through a PI over Grassmanian variables $\theta^\mu, \bar{\theta}^\mu$, the amplitude (9) is reduced to

$$\langle f | i \rangle = \int D (h^\mu_\nu, \pi^\mu, \theta^\mu, \bar{\theta}^\mu) \exp \left[ i \int \left( -\frac{1}{2\kappa} h^\mu_\nu R^\mu_\nu + \pi^\mu f^\mu + \bar{\theta}^\nu \hat{M}^\nu_\mu \theta^\mu \right) d^4 x \right]; \quad (10)$$

$$D (h^\mu_\nu, \pi^\mu, \theta^\mu, \bar{\theta}^\mu) = \prod_{x} \left(-h\right)^{-\frac{3}{2}} \prod_{\nu \leq \mu} d\theta^\nu d\bar{\theta}^\nu.$$

The expression (10) coincides with the known Faddeev – Popov amplitude, but, as may be seen from the mathematically correct way it was obtained by, there is no ground for a statement that the quantum gravity is a gauge- invariant theory. Moreover, it is evident, that among the effects contained in (10) gauge-noninvariant effects are present inevitably. As to the question whether gauge-invariant effects are contained in the amplitude, it should be analyzed within the theory itself according to properties of the initial and the final states. Our investigation shows that in the QGD of a closed universe there exist no gauge- invariant transition amplitude.
5. The dynamical Schrödinger equation for the Bianchi-IX model

The interval in the cosmological model Bianchi-IX looks like:

$$ds^2 = N^2(t) dt^2 - \eta_{ab}(t) e^a_i e^b_k dx^i dx^k;$$

$$\eta_{ab}(t) = \text{diag}(a^2(t), b^2(t), c^2(t));$$

$$e^a_i = (\sin x^a, -\cos x^a \sin x^1, 0),$$

$$e^a_i = (\cos x^a, \sin x^a \sin x^1, 0),$$

$$e^a_i = (0, \cos x^1, 1).$$

We shall also assume that the model includes an arbitrary number $K$ of real scalar fields described by the Lagrangian

$$L_{(\text{scal})} = \frac{1}{2\sqrt{-g}} \left[ \frac{1}{2} \sum_{i=1}^{K} \phi^i_{\mu} \phi^i_{\mu} - U_s(\phi_1, \ldots, \phi_K) \right],$$

and use the following parametrization:

$$a = \frac{1}{2} r \exp \left[ \frac{1}{2} (\sqrt{3} \varphi + \chi) \right]; b = \frac{1}{2} r \exp \left[ \frac{1}{2} (-\sqrt{3} \varphi + \chi) \right]; c = \frac{1}{2} r \exp (-\chi).$$

Writing out the Einstein equations in the given parametrization it is easy to notice that the Bianchi-IX model can be considered as a model of a Friedman – Robertson – Walker closed universe on which a transversal nonlinear gravitational wave $\varphi(t), \chi(t)$ is superposed, $r(t)$ being the scale factor. It will be more convenient to use a parameter $q = 2 \ln r$.

Denoting

$$Q^a = (q, \varphi, \chi, \phi, \ldots);$$

we shall define the gauge variable $\mu$ through the “lapse function” $N$ by an arbitrary fixed function

$$\zeta(\mu, Q) = \ln \frac{r^3}{N}$$

and confine our attention to the special class of gauges not depending on time

$$\mu = f(Q) + k; \ k = \text{const},$$

or, in a differential form,

$$\dot{\mu} = f, Q \dot{Q}^a,$$

an index after a comma here and further denoting differentiation with respect to generalized coordinates: $f, Q = \partial f/\partial Q^a$. Practically, any gauge can be represented by Eq. (12) using an appropriate parametrization (11). The choice of a differential gauge condition form is dictated by our intention to construct a Hamiltonian dynamics in EPS where the Lagrangian multiplier of a gauge in the action has to be the momentum canonically conjugate to the gauge variable.
The appropriate effective action, which a PI is based on when quantizing the model, reads

\[ S_{\text{eff}} = \int dt \left\{ \frac{1}{2} \exp \left[ \zeta (\mu, Q^a) \right] \gamma_{ab} \dot{Q}^a \dot{Q}^b - \exp \left[ - \zeta (\mu, Q^a) \right] U (Q^a) + \left( \pi + \bar{\theta} \dot{\theta} \right) \left( \mu - f, a Q^a \right) + \frac{i}{\zeta_{\mu}} \dot{\bar{\theta}} \dot{\theta} \right\}, \tag{13} \]

where \( \zeta_{\mu} = \partial \zeta (\mu, Q)/\partial \mu \); \( \theta, \bar{\theta} \) are the Faddeev–Popov ghosts after replacement \( \theta \rightarrow -i\bar{\theta} \); indices \( a, b, \ldots \) are raised and lowered with the “metric” \( \gamma_{ab} = \text{diag} (-1, 1, 1, 1, \ldots) \);

\[ U(Q) = e^{2q(U_g (\phi, \chi) + e^{3q(U_s(\phi)})}, \]

The Batalin–Fradkin–Vilkovisky (BFV).\(^9\) EPS consists of the dynamical variables \( Q, \mu, \theta, \bar{\theta} \) and the appropriate canonical momenta

\[ P_a = \frac{\partial L}{\partial \dot{Q}^a}, \quad P_0 = \frac{\partial L}{\partial \dot{\mu}} = \lambda, \quad \bar{\rho} = \frac{\partial L}{\partial \dot{\bar{\theta}}}, \quad \rho = \frac{\partial L}{\partial \dot{\theta}}. \]

The corresponding Hamiltonian

\[ H = \frac{1}{2} G^{\alpha\beta} P_\alpha P_\beta - i \zeta_{\mu} \bar{\rho} \bar{\rho} + e^{-\zeta} U, \tag{14} \]

where \( \alpha = (0, a), Q^0 = \mu, \)

\[ G^{\alpha\beta} = e^{-\zeta} \left( f, a f, a \gamma_{ab} \right), \]

gives the canonical equations

\[ \dot{X} = \{H, X\}, \quad X = (P_\alpha, \rho, \bar{\rho}, Q^a, \theta, \bar{\theta}) \tag{15}. \]

The action (13) in canonical variables

\[ S = \int \left( \dot{Q}^a P_\alpha + \bar{\theta} \dot{\theta} + \bar{\rho} \dot{\rho} - H \right) dt \]

is invariant, up to dynamically insignificant terms, under BRST and anti-BRST transformations in EPS, generated by the charges

\[ \Omega = H \theta + i P_0 \rho, \quad \bar{\Omega} = H \bar{\theta} - i P_0 \bar{\rho}. \tag{16} \]
Both the charges are real, nilpotent and mutually anticommutative in the sense of Poisson brackets,
\[ \{ \Omega, \Omega \} = \{ \bar{\Omega}, \bar{\Omega} \} = \{ \Omega, \bar{\Omega} \} = 0, \]
and are integrals of motion.

The set of equations (15), which we will refer to as conditionally-classical for the presence of Grassmannian variables in it, is applied to approximate a PI by a multiple integral. An important feature of the conditionally-classical model is the presence of a specific subsystem, further being referred to as the “gravitational vacuum condensate” (GVC). In (13) the “OM Lagrangian” \( \lambda(\dot{\mu} - f_a \dot{Q}^a) \), where \( \lambda = \pi + \dot{\theta} \), corresponds to this subsystem. In a simple case, when the gauge function \( f(Q^a) \) depends only on \( Q^1 = q \) and \( \zeta = \zeta(\mu, q) \), the quasi-energy-momentum tensor (quasi-EMT) corresponding to this Lagrangian is isotropic:
\[
T^\nu_{\mu(\text{obs})} = \text{diag}(\epsilon(\text{obs}), -p(\text{obs}), -p(\text{obs}), -p(\text{obs})) ,
\]
where an index \( k \) denotes that the substitution \( \mu = f(Q^a) + k \) has been made. So, the GVC is a continuous medium with the equation of state essentially depending on a parametrization and a gauge, thus the latter two is proved to be cosmological evolution factors. Notice, as well, that investigation of the equations of motion following from the effective action (13) reveals the existence of the conserved quantity
\[
(\zeta_{\mu})^{-1}_k \dot{\lambda} = E_k .
\]
thus the latter being a quantitative description of the GVC. As it will be shown below, finding the \( E_k \) value spectrum is one of the main tasks of gauge-noninvariant QGD.

Proceeding to constructing quantum theory, it is essentially important to note that in the EPS formalism a dynamical SE is a direct and unambiguous consequence of canonical quantization procedure by no means depending on our concepts about gauge invariance or noninvariance of the theory. Really, an SE can be derived from the quantum analogue of the canonical equation set (15) alone:
\[
i \frac{\partial |\Psi\rangle}{\partial t} = H |\Psi\rangle ,
\]
where \( H \) is the Hamiltonian (14) defined in EPS.

A dynamical SE, surely, can also be obtained in PI formalism having certain advantages over operator one. In the latter, as is generally known, the operator ordering problem is not resolvable. When deriving an SE from a PI, ordering turns to be bound up with a way of a final definition of a PI as a limit of a multiple integral, and with a choice of a gauge variable parametrization. The parametrization choice determines a PI measure as well (the latter being identical with a measure of the
Quantum Geometrodynamics of the Bianchi-IX Model in Extended Phase Space

WF normalizing integral – probability measure). Let us consider a PI for a WF in the coordinate representation. Such a WF, according to the stated above, is defined on the extended configurational space with the coordinates $Q^a, \mu, \theta, \bar{\theta}$:

$$\Psi(Q^a, \mu, \theta, \bar{\theta}; t) = \int \langle Q^a(0), \mu(0), \theta(0), \bar{\theta}(0); t_0 | Q^a(0), \mu(0), \theta(0), \bar{\theta}(0); t_0 \rangle \Psi(Q^a(0), \mu(0), \theta(0), \bar{\theta}(0); t_0) \times M(Q^a(0), \mu(0)) d\theta(0) d\bar{\theta}(0) d\mu(0) \prod_b dQ^b(0).$$  (17)

The transition amplitude, appearing in (17),

$$\langle Q^a, \mu, \theta, \bar{\theta}; t | Q^a(0), \mu(0), \theta(0), \bar{\theta}(0); t_0 \rangle = C \int \exp[iS(t, t_0)] \prod_{t_0 < \tau < t} M(Q^a(\tau), \mu(\tau)) d\mu(\tau) d\theta(\tau) d\bar{\theta}(\tau) \prod_b dQ^b(\tau) d\lambda(\tau) d\lambda(t),$$

is given by the gauged action (13).

It is well known that a PI is not defined in internal terms. Proceeding from the standard treating, we shall consider it as a limit at $\epsilon_i \to 0$ of the following integral:

$$\Psi^{(N)}(Q^a, \mu, \theta, \bar{\theta}) = C \int \exp \left\{ i \sum_{i=1}^N S(t_i, t_{i-1}) \right\} \Psi^{(0)}(Q^a, \mu, \theta, \bar{\theta}) \times \prod_{i=0}^{N-1} M(Q^a(i), \mu(i)) d\mu(i) d\theta(i) d\bar{\theta}(i) \prod_b dQ^b(i) d\lambda(i) d\lambda(i+1),$$

where $t_i - t_{i-1} = \epsilon_i$,

$$S(t_i, t_{i-1}) \approx \epsilon_i \left\{ \frac{1}{2} \exp(\zeta(i)) \dot{Q}^a(i) \dot{Q}^a(i) - \exp(-\zeta(i)) U(Q^a(i)) + \lambda(i) \left[ \dot{\mu}(i) - \dot{f}(Q^a(i)) \right] + \frac{i}{\zeta(i)} \dot{\theta}(i) \dot{\bar{\theta}}(i) \right\}.$$

Operating by the standard Feynman method (see.10), in the first order one obtains the SE

$$i \frac{\partial \Psi(Q^a, \mu, \theta, \bar{\theta}; t)}{\partial t} = H \Psi(Q^a, \mu, \theta, \bar{\theta}; t)$$

the zero-order terms give the constraint between the measure $M$, step $\epsilon$ and parametrization function $\zeta$:

$$\frac{1}{\epsilon \zeta(i)} (\epsilon e^{-\zeta}) \frac{d}{d\tau} M = \text{const.}.$$

The requirement of the Hamiltonian to be Hermitian gives rise to another constraint between the measure and the parametrization,

$$M = \text{const} \cdot \zeta(i) \exp \left( \frac{K + 3}{2} \frac{1}{\zeta} \right).$$  (20)

The independence of the parameter $\epsilon$ on the variables $Q^a, \mu$ follows from(19), (20).
The Hamiltonian in the SE obtained by the PI method is presentable in the form

\[ H = -i\zeta_\mu \frac{\partial}{\partial Q^\mu} - \frac{1}{2M} \frac{\partial}{\partial Q^a} \tilde{G}^{a\beta} \frac{\partial}{\partial Q^\beta} + e^{-\zeta}(U - V), \]

where \( M \) is defined by the formula (20), \( \tilde{G}^{a\beta} = MG^{a\beta} \).

\[ V = \left( \zeta_\mu \zeta_\mu \right)_a + \left( \frac{\zeta_\mu}{3\zeta_\mu} \right)_a + \left( \frac{K + 1}{6\zeta_\mu} \right) \zeta_\mu \zeta_\mu + \left( \frac{K + 2}{6} \right) \zeta_a, \]

\[ \zeta_a = \frac{\partial \zeta}{\partial Q^a} + f_{\alpha a} \frac{\partial \zeta}{\partial \mu}. \]

The general solution (GS) to the SE in the coordinate representation is a WF

\[ \Psi = \Psi(t, Q^a, Q^0, \theta, \bar{\theta}), \]

depending on time \( t \), physical variables \( Q^a \), gauge variable \( Q^0 \equiv \mu \), and ghost variables \( \theta, \bar{\theta} \). Making use of the Hamiltonian structure only, one can ascertain the WF dependence on the variables \( Q^0, \theta, \bar{\theta} \).

To begin with, note that in the class of gauges (12), not depending on time explicitly, the GS to the SE (18) is expandable in the stationary state eigenfunctions satisfying the stationary SE

\[ H \Psi_n(Q^a, Q^0, \theta, \bar{\theta}) = E_n \Psi_n(Q^a, Q^0, \theta, \bar{\theta}). \]

One of the canonical equations, the gauge equation

\[ [H, Q^0 - f(Q^a)] = 0 \]

means the commutativeness of the Hamiltonian with the operator \( Q^0 - f(Q^a) \) and, consequently, an arbitrary solution to Eq. (22) can be presented in the form of a superposition of this operator eigenstates \( |k\rangle \),

\[ \{Q^0 - f(Q^a)\}|k\rangle = k|k\rangle. \]

The same concerns the GS to (18) as a superposition of the stationary states. In the coordinate representation

\[ |k\rangle = \delta(Q^0 - f(Q^a) - k), \]

so the GS to the SE has the structure

\[ \Psi(Q^a, Q^0, \theta, \bar{\theta}; t) = \int \Phi_k(Q^a, \theta, \bar{\theta}; t)\delta(Q^0 - f(Q^a) - k) dk. \]
Quantum Geometrodynamics of the Bianchi-IX Model in Extended Phase Space

to the dynamical SE inevitably has the structure (24). One can come to the same conclusion by making detailed analysis of the WF in the PI formalism.

The SE for \( \Phi_k(Q^a, \theta, \bar{\theta}; t) \) reads

\[
i \frac{\partial \Phi_k(Q^a, \theta, \bar{\theta}; t)}{\partial t} = H_k \Phi_k(Q^a, \theta, \bar{\theta}; t),
\]

\[H_k = -i(\zeta, \mu)_k \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \exp(-\zeta_k) \left( \frac{\partial^2}{\partial Q^a \partial Q^a} + Z^a_k \frac{\partial}{\partial Q^a} \right) \exp(-\zeta_k)(U - V),\]

\[Z^a_k = (\zeta, \mu)_k^a + K + \frac{1}{2} \zeta^a_k,
\]

\(V\) being defined by Eq. (21), \((\zeta^a)_k = \zeta^a_k \equiv \partial \zeta_k / \partial Q^a.\)

So, the WF dependence on \( \mu = Q^0 \) is determined by Eq. (24). As will be shown below, such a structure of the WF, on a certain restriction on the \( \Phi_k \) dependence on \( k \), makes the normalizing integral over the variable \( Q^0 \), transformed into the integral over \( k \), to be convergent. As for the dependence on the ghosts, it is strictly enough fixed by the SE in combination with the usual demand of the norm positivity. Indeed, in the general case the WF can be presented by the series in the Grassmannian variables,

\[
\Phi_k(Q^a, \theta, \bar{\theta}; t) = \Psi^0_k(Q^a, t) + \Psi^1_k(Q^a, t)\theta + \Psi^\bar{1}_k(Q^a, t)\bar{\theta} + \Psi^2_k(Q^a, t)\bar{\theta} \theta.
\]

After substitution into (25) one obtains the equations for the components

\[
i \frac{\partial \Psi^0_k}{\partial t} = H^0_k \Psi^0_k - i(\zeta, \mu)_k \Psi^2_k,
\]

\[i \frac{\partial \Psi^i_k}{\partial t} = H^0_k \Psi^i_k, \quad i = 1, 1, 2,
\]

\[H^0_k = H_k + i(\zeta, \mu)_k \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}},
\]

and the normalization condition puts the constraints on these components: from the norm positivity demand it follows that

\[-i \int \left( \Psi^0_k \Psi^2_k - \Psi^2_k \Psi^0_k + \Psi^1_k \Psi^1_k - \Psi^1_k \Psi^1_k \right) \partial \theta \partial \bar{\theta} \partial \bar{\theta} > 0.
\]

One of the consequences of the inequality is \( \Psi^2_k = 0, \) or \( \Psi^0_k = 0, \) or \( \Psi^2_k = i\Psi^0_k, \) and Eqs. (26), (27) reduce all the three versions to the one,

\[\Psi^0_k = \Psi^2_k = 0, \]

\[\Psi^1_k = i\Psi^1_k,
\]

so, finally,

\[\Phi_k(Q^a, \theta, \bar{\theta}; t) = \Psi_k(Q^a, t)(\bar{\theta} + i\theta),\]
where $\Psi_k(Q^a, t)$ is a solution to the equation

$$
\frac{i}{\partial t} \frac{\partial \Psi_k(Q^a, t)}{\partial t} = - \frac{1}{2M_k} \frac{\partial}{\partial Q^a} M_k \exp(-\zeta_k) \gamma_{ab} \frac{\partial \Psi_k(Q^a, t)}{\partial Q^b} \exp(-\zeta_k)(U - V_k) \Psi_k(Q^a, t),
$$

with

$$
M_k = (\zeta, \mu)_k \exp \left( \frac{K + 3}{2} - \zeta_k \right),
$$

The unitarity property of a physical state WF takes the form

$$
\int \Psi_k^*(Q^a, t) \delta(\mu - f(Q^a) - k') \Psi_k(Q^a, t) dE_k dQ^a = \int \Psi_k^*(Q^a, t) \Psi_k(Q^a, t) dk dQ^a.
$$

Thus, the GS (24), (28) to Eq. (18), under the condition the $\Psi_k(Q^a, t)$ to be a sufficiently narrow packet over $k$, is normalizable with respect to the gauge variable, as well as to the ghosts and the physical variables.

The peculiarity of the amplitude (28) lies in the fact that the theory does not control its dependence on the free parameter $k$. From the standpoint of the classical dynamic equations the parameter $k$ sets an initial condition for the variable $\mu$ and, by the same, determines an initial clock setting. In QT, however, there exist no physical state with a fixed $k$ value. Really, the unitarity requirement (see (29)) allows the existence of a physical state represented by a packet over $k$, narrow enough to fit a certain classical $\bar{k}$ value, but not by a $\delta$-shaped packet. So, in the theory appears an additional degree of freedom that could be named an observer degree of freedom. Unlike the quantum uncertainties associated with operator noncommutativeness, QGD do not control even width of a $k$ packet.

The GS structure

$$
\Psi(Q^a, Q^0, \theta, \bar{\theta}; t) = \int \Psi(E_k; Q^a) \exp(-iE_k t)(\bar{\theta} + i\theta) \delta(\mu - f(Q^a) - k) dE_k dk,
$$

where $\Psi(E_k; Q^a)$ is a solution to the stationary equation

$$
H^0_k \Psi_k(Q^a) = E_k \Psi_k(Q^a),
$$

shows that the WF carries information on 1) a physical object, 2) OM, 3) correlations between a physical object and OM. OM are represented by the factored part of the WF – by the $\delta$-function of a gauge and by the ghosts; a physical object is described by the function $\Psi_k$; correlations are present in the effective potential $V_k$, which the solution depends on, an also in the WF time dependence, or after going over to the stationary states, they are present in the effective potential and the spectrum $E_k$. 
It is of principal significance to emphasize that the construction procedure of the WF (30) is the only strict mathematical method to do it, by no way corresponding to the QGD WDW. The question arise: do Eq. (18) and its solutions have any relation to the WDW theory? In other words, whether it is possible to pick out such a physical part of the structure (30) that would satisfy the WDW equation and could be reasonably interpreted?

6. Mathematical and physical problems of the gauge-invariant QGD

To begin with, let us discuss a possibility of constructing a WF, corresponding to QGD WDW, by going over from the GS to some particular solution. As it is known, the transition to QGD WDW entails the separation of the physical variable subspace from EPS. For this purpose it is not enough to separate the physical part $\Psi_k(Q^a)$ from the GS: one also has to banish correlations between the physical object properties and those of OM. The latter are given, first of all, by the GVC parameter. Hence, to eliminate correlations, firstly, we put $E_k = 0$. Secondly, making use of the noted in Sec. 5. possibility of going over to any given gauge function $f(Q^a)$ by means of the transformation of the parametrization function $\zeta(\mu, Q^a)$, it is necessary to pass to the gauge

$$\mu = k. \quad (31)$$

And, thirdly, the measure should be factorized: $M = M_1(\mu)M_2(Q^a)$. In view of Eq. (20), the measure factorization, in turn, requires factoring the parametrization that means

$$\zeta(\mu, Q^a) = \zeta_1(\mu) + \zeta_2(Q^a), \quad (32)$$

and conservation of additivity of the function (32) when passing to the gauge (31) puts one more restriction on the parametrization function: $\zeta_1$ should be linear in $\mu$,

$$\zeta_1(\mu) = A + B\mu. \quad (33)$$

Under these conditions one obtains the WDW equation for the physical part of the WF

$$H_{ph}\Psi(Q^a) = 0, \quad (34)$$

$$H_{ph} = -\frac{1}{2M_2} \frac{\partial}{\partial Q^a} M_2 \exp(-\zeta_2) \gamma^{ab} \frac{\partial}{\partial Q^b} + \exp(-\zeta_2)U + \frac{1}{6} R, \quad (35)$$

$$R = -\exp(-\zeta_2) \left[ \frac{1}{4} (K^2 + 3K + 14) \zeta_2^a \zeta_2^a + (K + 2) \zeta_2^a \right] \quad (36)$$

being the scalar curvature constructed of the metric $G_{ab} = \exp(\zeta_2)\gamma_{ab}$.

Eq. (34) possesses all formal properties of the WDW equation including parametrization noninvariance and the lack of any visible vestige of a gauge. It however, the described above method of deriving Eq. (34) makes it apparent that in the gauge class (12), any change of the parametrization function $\zeta(\mu, Q^a)$ is mathematically equivalent to a new gauge. Hence, the generally known parametrization noninvariance of the WDW theory, as a matter of fact, is the ill-hidden gauge noninvariance.
This circumstance has to be taken into account when estimating the status of the WDW theory.

On the other hand, by origin, the single Hamiltonian eigenvalue $E_k = 0$ fixed by the equation represents the line in the continuous spectrum of the Hamiltonian (35), hence the solution is unnormalizable. In other words, on that way of formal singling out the particular solution to Eq. (18) one would fail to obtain a WF having physical meaning generally adopted in QT.

The other approach to the existence problem of gauge invariant states is based on picking out the singular, BRST-invariant solutions. The BFV scheme assigns the central part of maintaining gauge invariance to the BRST generators (16): state vectors are primordially obey the superselection rules

$$\Omega|\Psi\rangle = 0, \quad \bar{\Omega}|\Psi\rangle = 0. \quad \text{(37)}$$

These equations signify BRST and anti-BRST invariance of the quantum states, the BRST-equivalence classes of state vectors may be represented by vectors independent of ghosts and gauge degrees of freedom.\(^9\)

Notice at once that the conditions (37) are to be treated as an independent postulate: BRST invariance of the action cannot serve as a reason for them. It is known that an action invariance under some global transformations does not mean the state vector invariance; state vectors just have to be covariant, subject to the appropriate unitary transformations.

The fact that a BRST-invariant WF is unobtainable from the GS can be seen by the following: in view of the formulae (16) for the generators $\Omega, \bar{\Omega}$, the strictly defined GS (30) dependence on the ghosts and the variable $\mu$ makes impossible to put the conditions (37) on the GS. The BRST-invariant WF represents a singular solution, the quantum analogue of the mentioned above degenerate Cauchy problem. Hence, the question arises about its physical meaning, and normalizability again serves as a criterion.

For the functions independent of the ghosts the conditions (37) come to the equation

$$H_0 \Psi(Q^\alpha) = 0, \quad \text{(38)}$$

where

$$H_0 = -\frac{1}{2M} \frac{\partial}{\partial Q^\alpha} \bar{G}^{\alpha\beta} \frac{\partial}{\partial Q^\beta} + e^{-\zeta} (U - V). \quad \text{(39)}$$

This equation corresponds to the classical constraint

$$H_0^{cl} = -\frac{1}{2} G^{\alpha\beta} P_\alpha P_\beta + e^{-\zeta} U = 0.$$  

The physical part is singled out from the WF $\Psi(Q^\alpha, Q^0)$ with the help of the additional condition in the form of the quantum version of the other classical constraint,

$$P_0 = 0,$$

\(^9\)A positive solution to this problem would need, apart from a discrete $H_{ph}$ spectrum, the presence of the line $E = 0$ in it.
Quantum Geometrodynamics of the Bianchi-IX Model in Extended Phase Space

Taking account of the form of a self-conjugate momentum operator in the presence of a nontrivial measure,

$$P_\alpha = -i \left( \frac{\partial}{\partial Q_\alpha} + \frac{M_\alpha}{2M} \right),$$

the condition is represented by the equation

$$\frac{\partial \Psi}{\partial Q^0} = -\frac{1}{2M} \frac{\partial M}{\partial Q^0} \Psi. \quad (40)$$

After factoring the measure and transforming the parametrization function, required when going over to the QGD WDW (see (31) – (33)), the solution to Eq. (40) takes the form

$$\Psi(Q^a, Q^0) = \Psi(Q^a) M^{-\frac{1}{2}}(Q^0), \quad (41)$$

and Eq. (34) – (36) for the WF $\Psi(Q^a)$ (41) follows from Eq. (38) – (39). However, this time the derivation procedure of the equation does not consist in fixing a value in a Hamiltonian spectrum, but in reducing the spectrum itself to the single value by means of the conditions (37) (confining the Hamiltonian space of definition to the BRST-invariant vectors).

Nevertheless, the WF normalizability condition is not satisfied here either. In the EPS formalism a WF norm, primordially defined in the extended configurational space, contains the integral over the $\mu$ variable, divergent in the case of BRST-invariant WF. On the other hand, the integral over the ghosts vanishes, and in the standard BFV scheme the existence of an interim regularization of the normalizing integral is supposed that would result in a finite, in particular, unity norm. However, the possibility of a mathematically satisfactory interim regularization giving the $c$-valued unity made of $c$-valued infinity combined with the zero Grassmannian integral, is not obvious, in any case, it needs additional assumptions not present within the theory.

Hence, the singular BRST-invariant solutions do not satisfy mathematical correctness criteria either. Therefore the GS structure (30) presents the only possible correct WF structure in the consistent QGD, all the other solutions failing to satisfy the criteria of mathematical correctness and physical rationality.

There is still the third way to obtain WDW Eq. (34) – (36) on which the question about normalization does not arise directly, or more precisely, it does not get such a definite general answer. Without adducing appropriate calculations we will note that on this way two incorrect mathematical structures are necessarily used:

- a gauge-invariant PI on partially degenerate action extremals without pointing out a procedure of removing coordinate effects;
- a WF definition through a divergent integral over gauge variables.

But realization of this program necessarily requires to use, as its component, a procedure of eliminating coordinate effects, described by the expression (6). Availability of such a procedure is provided either by existence of local canonical gauges, completely removing degeneracy, or by dynamical splitting off the “nonphysical”
degrees of freedom from the physical ones in asymptotical states. In quantum theory of gravity, as it is known, there is no gauge with the mentioned property, and in QGD of a closed universe there exist no asymptotical state enabling a dynamical splitting operation. So, the third way to derive the WDW equation, that is met with in publications (see, e.g.\textsuperscript{5}), cannot be recognized to be correct, as well, for it ignores a procedure of removing coordinate effects.

The described methods to derive the WDW equation reveal the following.

(i) The WF not containing information about correlations between the physical object and OM is the same in all the approaches.
(ii) If the state vector allows to compute average values of observable quantities, then information about the correlations has to be contained in it inevitably. In other words, there is no physical (normalizable) quantum state without a GVC.

It is to be ascertained that there is no QGD as a gauge-invariant theory of physical states in a closed universe based on the general grounds of QT. The illusion of the existence of such a theory would arise if one forgets about the necessity of singling out gauge orbit representatives and tries to come to agreement about some special quantization rules.

As it was already mentioned in Sec. 2., the WDW theory supplemented with the many-worlds interpretation of a WF, according to its status of a global quantum theory, represents a new paradigm, but not a consequence of the existing quantum theory. Hence, the question is not whether it can be obtained correctly (we have seen above that one cannot do it), the question is, what formal grounds do we have for the introduction of the new paradigm? Usually the principle of gauge invariance of the theory is adduced as such grounds. However, we should emphasize that these grounds do not exist either: as it was shown above, the parametrization-noninvariant WDW equation (32) is, in fact, gauge-noninvariant.

7. On the physical interpretation of the gauge-noninvariant QGD

As to the correct PI method, it shows unambiguously that, being applied to a closed universe, the ordinary QT of gravity is gauge-noninvariant. And this feature of the theory is the evidence of its adequacy to the phenomena in question: it answers to the conditions of observations of a closed universe, where there is no possibility to remove an instrument for infinite distance from the object and thus to avoid influence of inertial fields, locally indistinguishable from gravitational field, on the instrument.

As it was shown by Landau and Lifshitz,\textsuperscript{8}, full information about dynamical geometry can be obtained directly in experimental way (without theoretical reconstruction) only in an RS disposed on a medium filling the whole space. Thus the medium have to be continuous. The choice of a certain RS within this class is fixed by a gauge condition that means breaking the space-time symmetry (diffeomorphism group symmetry) by the medium. The operational interpretation of a GVC
as such a medium, a carrier of a Landau – Lifshitz RS, follows from the foregoing.

Appealing to Landau – Lifshitz RS makes the statement about the gauge noninvariance of quantum theory of gravity be almost obvious. Indeed, a formal transformation of coordinates meaning a transition to another gauge, physically corresponds to removing OM from the whole space of the Universe and replacing them by other OM. From the viewpoint of the standard Copenhagen interpretation of QT that declares existence of unremovable connections between the properties of an object and OM, it seems to be incredible that such an operation performed on the whole Universe scale, would not result in changing its quantum properties.

References


\[\text{In the theory of quantum transitions between asymptotic states it is obvious that the replacement of bodies on which an asymptotically inertial RS is realized and replacement of detectors disposed on these bodies cannot change a physical situation. Therefore such a theory has to be gauge-invariant.}\]