QUANTUM FIELD THEORY FOR EXPERIMENTALISTS
S. Bilenky
JINR, Dubna, Russia

Abstract
The basic principles of quantum field theory (Lagrange formalism, connection between symmetries and conservation laws, field quantization, etc.) are briefly considered. The Dyson–Wick techniques of decomposition of chronological products over normal products are presented and Feynman rules are derived. Computational techniques for the calculation of observable quantities (cross-sections, decay probabilities, etc.) are presented and several examples ($\nu_e \rightarrow \nu_e$, $Z \rightarrow \nu\bar{\nu}$ and deep-inelastic scattering in parton approximation) are considered in detail. In the last part higher order effects are discussed. The method of dimensional regularization is presented and photon vacuum polarization is calculated. Electroweak radiative corrections to the effective Fermi constant $\Delta r$ are also discussed.

1. Introduction
All observable quantities in high-energy physics are calculated in the framework of quantum field theory. Quantum field theory and Feynman diagrams are the basis of our understanding of particles and their interactions. So the knowledge of quantum field theory (at least the foundation of this theory) is absolutely necessary for everybody who works in high-energy physics.

We will begin by considering in some detail the notion of the quantized fields (scalar, spinor, vector). After that, we will introduce electromagnetic and electroweak interactions as examples of interactions that arise owing to the requirements of invariance under local gauge invariance. Later, the Dyson–Wick technique will be introduced and Feynman rules will be developed. After that we will consider the technique of calculation of observable quantities (cross-sections, probabilities of decay, polarization effects) and we will calculate, as an example, the cross-section of neutrino–electron scattering, the width of the decay of $Z$ into two neutrinos, and the cross-section of deep inelastic lepton–nucleon scattering in the parton approximation. To conclude we will discuss higher order effects. We will present the
method of dimensional regularization and calculate the loop in quantum electrodynamics. Charge and mass renormalization will be discussed.

1.1 Metric

We will use here the so-called Bjorken–Drell metric. Let \( A \) and \( B \) be some 4-vectors. Their scalar product is determined as follows

\[
AB = g_{\alpha\beta}A^\alpha B^\beta = A^\alpha B_\alpha, \quad B_\alpha = g_{\alpha\beta}B^\beta,
\]

\[
g_{00} = 1, \quad g_{i\alpha} = -1, \quad g_{\alpha\beta} = 0, \quad \alpha \neq \beta,
\]

where \( A^\alpha \) \( (A_\alpha) \) are contravariant (covariant) components of the vector \( A \), and \( \alpha, \beta \) take the values 0, 1, 2, 3. We have \( A \cdot B = A^0 B^0 - A^i B_i \) and \( B_0 = B^0, B_i = -B^i \). Dirac matrices in this metric will be introduced later.

1.2 The system \( \hbar = c = 1 \)

It is a widely accepted practice in quantum field theory to use the system in which \( \hbar = c = 1 \). We will now introduce this system. Let \( A_0 \) be some quantity in the usual system with three dimensional quantities: \( M \) (mass), \( L \) (length), \( T \) (time). Let us introduce

\[
A = \frac{A_0}{\hbar^\alpha c^\beta}
\]

and choose \( \alpha \) and \( \beta \) in such a way that the dimension of \( A \) will be \( M^n \), where \( n \) is some value (determined by the dimension of \( A \) in the initial system). Clearly we have for momentum, energy, cross-section, etc., respectively:

\[
\]

In order to obtain quantities in the usual system we simply multiply the corresponding quantities in the \( \hbar = c = 1 \) system by \( \hbar^{\alpha} c^{\beta} \), where \( \alpha \) and \( \beta \) are chosen in a way such as to restore the initial dimension. Clearly we have

\[
p_0 = p c, \quad E_0 = E c^2, \quad m_0 = m,
\]

\[
\sigma_0 = \sigma \hbar^2 c^{-2}, \quad w_0 = w \frac{c^2}{\hbar}, \ldots,
\]

(where \( w \) is the probability of decay).

For example, the cross-section of neutrino–(point)nucleon interaction is given by

\[
\sigma = \frac{G^2}{\pi} M E,
\]

where \( G \approx 10^{-5} M^{-2} \) is the Fermi constant \( (M \) is the proton mass). Using the above definitions we have

\[
\sigma_0 = \frac{1}{\pi} 10^{-10} \left( \frac{\hbar}{M_0 c} \right)^2 \frac{E}{M_0 c^2}.
\]
1.3 The Dirac equation

The Dirac equation is the quantum-mechanical equation of the motion of a relativistic particle with spin 1/2. Dirac assumed that:

i) The relativistic particle is described by the wave function $\psi_\sigma(\vec{x}, t)$, where $\sigma$ is the discrete spin index.

ii) The equation of the motion of a relativistic particle has the Schrödinger form

$$i \frac{\partial \psi(\vec{x}, t)}{\partial t} = H \psi(\vec{x}, t)$$

(this means that the evolution of the system is determined by the wave function at the initial time).

iii) The quantity $\rho(\vec{x}, t) = \psi(\vec{x}, t)\psi(\vec{x}, t)^\dagger$, as in non-relativistic quantum mechanics, is the density of the probability of finding the particle in the moment $t$ at the point $\vec{x}$.

The equation of the motion of a relativistic particle must be invariant under Lorentz transformations. From Eq. (1) it follows that, to provide Lorentz invariance, space derivatives $\partial/\partial x^k$ (where $k = 1, 2, 3$) must enter into the equation linearly. Dirac assumed that the Hamiltonian of the free equation of motion is given by

$$H = \alpha_k \frac{1}{i} \frac{\partial}{\partial x^k} + \beta m,$$

where $m$ is the mass and $\alpha_k, \beta$ are some matrices.

Let us multiply the Dirac equation by the matrix $\beta$ from the left. We obtain the Dirac equation in the form

$$i\gamma^\alpha \partial_\alpha \psi - m\psi = 0,$$

where

$$\gamma^0 = \beta, \quad \beta\alpha_k = \gamma^k,$$

and $\partial_\alpha = \partial/\partial x^\alpha$ [the derivative $(\partial/\partial x^\alpha)$ is a covariant vector]. Now let us introduce the momentum operator

$$p_\alpha = i\partial_\alpha.$$

We have

$$p^\alpha = \left( i\frac{\partial}{\partial x^\alpha}, -i\nabla \right)$$

and the Dirac equation takes the form

$$(\gamma^\alpha p_\alpha - m)\psi = 0.$$  \hspace{1cm} (4)

Now (as in the case of the Schrödinger equation) from the Dirac equation the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div} \vec{j} = 0$$

must follow (for $\rho$ to be the density of the probability, $\vec{j}$ being the current).

From Eq. (2) by Hermitian conjugation we will obtain

$$i\partial_\alpha \psi^\dagger \gamma^\alpha + m\psi^\dagger = 0.$$
Let us multiply this equation from the right by $\gamma^0$. We have

$$i\partial_\alpha\bar{\psi}\gamma^\alpha + m\bar{\psi} = 0 .$$

(6)

where $\bar{\psi} = \psi^\dagger\gamma^0$ is the conjugated spinor and $\bar{\gamma}^\alpha = \gamma^0\gamma^\alpha\gamma^0$. Now let us multiply Eq. (2) by $\bar{\psi}$ from the left, Eq. (6) by $\psi$ from the right, and add the equations obtained. We have

$$\bar{\psi}\gamma^\alpha\partial_\alpha\psi + \partial_\alpha\bar{\psi}\gamma^\alpha\psi = 0 .$$

(7)

It is clear that if $\gamma$ matrices satisfy the relation

$$\bar{\gamma}^\alpha = \gamma^0\gamma^\alpha\gamma^0 = \gamma^\alpha ,$$

(8)

Eq. (7) coincides with the continuity equation (5), the current being

$$j^\alpha = \bar{\psi}\gamma^\alpha\psi = (\psi^\dagger\gamma^\alpha\psi, \psi^\dagger\gamma^0\psi) .$$

Thus we will assume that the Dirac $\gamma$ matrices satisfy the relation (8).

Now consider the state with definite momentum

$$\psi_p(x) = u(p)e^{-ipx} ,$$

where $px = p^0x^0 - \vec{p} \cdot \vec{x}$. From the Dirac equation it follows that the spinor $u(p)$ satisfies the following equation:

$$(\gamma^\alpha p_\alpha - m)u(p) = 0 ,$$

(9)

where $\gamma^\alpha p_\alpha$. Let us multiply Eq. (9) from the left by the operator $(\gamma^\alpha + m)$. We have

$$(\gamma^\alpha p_\alpha + m^2)u(p) = 0 .$$

(10)

It is obvious that

$$\gamma^\alpha p_\alpha = \frac{1}{2}(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha)p_\alpha p_\beta .$$

Now let us assume that Dirac $\gamma$ matrices satisfy the following anticommutator relations:

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\delta^\alpha_\beta .$$

(11)

In this case Eq. (10) becomes

$$(\gamma^\alpha p_\alpha - m^2)u(p) = 0 .$$

(12)

It is clear from Eq. (12) that the Dirac equation can have a solution with momentum $p$, only if $p^2 = m^2$. From this relation it follows that $p^0 = \pm \sqrt{m^2 + \vec{p}^2}$ and we will show later that the Dirac equation has solutions both with positive and negative energies.

Thus we will assume that $\gamma$ matrices satisfy the relations (8) and (11). It could easily be demonstrated that matrices which satisfy these relations are $4 \times 4$ matrices.

Notice that from Eqs. (8) and (11) it follows that $\gamma^0$ is a Hermitian matrix and $\gamma^k$ are anti-Hermitian matrices

$$\gamma^0^\dagger = \gamma^0 , \quad \gamma^k^\dagger = -\gamma^k .$$

From the Dirac equation for $u(p)$ by Hermitian conjugation (and by multiplication by $\gamma^0$ from the right) we obtain the following equation for $\bar{u}(p) = u^\dagger(p)\gamma^0$:

$$\bar{u}(p)(\gamma^\alpha p_\alpha - m) = 0 .$$

(13)
With the help of Eqs. (9) and (13) and the commutation relations it is easy to obtain different useful relations between matrix elements. We will obtain one of them. Let us multiply Eq. (9) by \( \bar{u}(p)\gamma^0 \) from the left and Eq. (13) by \( \gamma^0 u(p) \) from the right. Now let us add the equations obtained and use commutation relations. We have

\[
p^0 \bar{u}(p)u(p) = m\bar{u}(p)\gamma^0 u(p) .
\]

Here we will normalize spinors as follows:

\[
\bar{u}(p)\gamma^0 u(p) = 2p^0 .
\]

From Eq. (14) it follows that \( \bar{u}(p)u(p) = 2m \) [Eq. (15) is the covariant normalization].

Now let us find the solution of the Dirac equation for the state with definite momentum. Let us multiply Eq. (9) by the matrix \( \beta \) from the left. We have

\[
(\bar{\alpha} \cdot \vec{p} + m\beta)u(p) = Eu(p) ,
\]

where \( E = \pm p^0, p^0 = +\sqrt{m^2 + \vec{p}^2} \). We will use the Dirac–Pauli representation

\[
\beta = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} .
\]

Let us write the 4-component spinor \( u(p) \) in the form

\[
u(p) = N \begin{pmatrix} \phi \\ \chi \end{pmatrix} ,
\]

where \( \phi \) and \( \chi \) are two component spinors. From Eqs. (16) to (18) we find

\[
\bar{\sigma} \cdot \vec{p}\chi + m\phi = E\phi ,
\]

\[
\bar{\sigma} \cdot \vec{p}\phi - m\chi = E\chi .
\]

For \( E = p^0 \), from Eq. (20) we have

\[
\begin{aligned}
\nu_+(p) &= \sqrt{p^0 + m} \begin{pmatrix} \phi \\ \sigma \end{pmatrix} \\
\nu_-(p) &= \sqrt{p^0 + m} \begin{pmatrix} -\phi \\ \sigma \end{pmatrix} .
\end{aligned}
\]

Now let us put \( \chi = \bar{\sigma} \cdot \vec{p}\phi/(p_0 + m) \) in Eq. (19). It is easy to see that this equation is satisfied for arbitrary \( \phi \). Analogously for \( E = -p^0 \), from Eq. (19) we obtain

\[
\begin{aligned}
u_-(p) &= \sqrt{p^0 + m} \begin{pmatrix} -\phi \\ \sigma \end{pmatrix} \\
u_+(p) &= \sqrt{p^0 + m} \begin{pmatrix} \phi \\ \sigma \end{pmatrix} .
\end{aligned}
\]

Equation (20) is satisfied for arbitrary \( \chi \). Thus we have found solutions of the Dirac equation with positive and with negative energies. It is clear that there are four independent solutions — two with positive energy and two with negative energy.
The matrix $\gamma_5$

Now let us introduce the matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. From Eqs. (8) and (11) it follows that

$$
\gamma_5^\dagger = \gamma_5, \quad \gamma^\alpha \gamma_5 + \gamma_5 \gamma^\alpha = 0, \quad \gamma_5 \gamma_5 = 1.
$$

(23)

Left-handed and right-handed components of a spinor $\psi$ have a very important role in theory

$$
\psi_L = \frac{1 - \gamma_5}{2} \psi, \quad \psi_R = \frac{1 + \gamma_5}{2} \psi.
$$

For any spinor we have $\psi = \psi_L + \psi_R$.

We have found solutions of the Dirac equation with definite energy and momentum.

Now let us require that these solutions also be eigenstates of the operator of the projection in the spin in some direction. Consider the operator $\gamma_5 \not{n}$, where $n$ is a space-like 4-vector that satisfies the conditions $n \cdot p = 0, n^2 = -1$. It is easy to see that this operator commutes with the operator $\not{p}$:

$$
\gamma_5 \not{n} \psi - \psi \gamma_5 \not{n} = 2\gamma_5 n \cdot p = 0.
$$

Thus the operators $\not{p}$ and $\gamma_5 \not{n}$ have common eigenstates:

$$
\not{p} u^s(p) = mu^s(p),
$$

(24)

$$
\gamma_5 \not{n} u^s(p) = su^s(p).
$$

(25)

It is easy to see that $s^2 = 1$, i.e. $s = \pm 1$.

In the rest frame $n = (0, \vec{n}_0)$ and the state with positive energy has the form (in the Dirac-Pauli representation):

$$
u^s(m) = \sqrt{2m} \left( \begin{array}{c} \phi^s \\ 0 \end{array} \right).
$$

Now Eq. (24) is reduced to $\vec{\sigma} \cdot \vec{n}_0 \phi^s = s\phi^s$. This means that the spinor $\phi^s$ describes the state with the spin projection $s$ in the direction $\vec{n}_0$ (in the rest system). In any system we have

$$
u(p) = L(p)u(m),
$$

where

$$
L(p) = \sqrt{\frac{p^0 + m}{2m}} \left(1 + \gamma^0 \frac{\vec{\gamma} \cdot \vec{p}}{p^0 + m}\right)
$$

is the Lorentz boost. Finally, for the state with energy $E = p_0$, momentum $\vec{p}$, and spin projection $s$ ($s = \pm 1$) we have

$$
u^s(p) = \sqrt{p^0 + m} \left( \begin{array}{c} \phi^s \\ \vec{\sigma} \cdot \vec{p} \phi^s \\ p^0 + m \phi^s \end{array} \right).
$$

(26)
S-matrix

Finally, let us introduce the S-matrix, which is the quantity of main interest to us. The state vector \( |\Psi(t)\rangle \) in any quantum theory satisfies the following equation of motion:

\[
i\frac{\partial |\Psi(t)\rangle}{\partial t} = (H_0 + H_I)|\Psi(t)\rangle ,
\]

(27)

where \( H \) is the free Hamiltonian and \( H_0 \) is the interaction Hamiltonian.

If \( O \) is some operator, the average value of the operator (observable value) is given by

\[
\bar{O} = \langle\Psi|O|\Psi\rangle .
\]

(28)

Now we have

\[
\langle\Psi|O|\Psi\rangle = \langle\Psi|V^\dagger VOV^\dagger V|\Psi\rangle = \langle\Psi'|O'|\Psi'\rangle ,
\]

where

\[
|\Psi'\rangle = V|\Psi\rangle ,
\]

\[
O' = VOV^\dagger
\]

(29)

and \( V \) is some unitary operator \((V^\dagger V = 1).\) Thus observables in the quantum theory are invariant under unitary transformations (29). We can use this unitary freedom and change the form of the equation of motion.

Operator \( V \) is any unitary operator. It can also depend on time. Let us assume that

\[
-i\frac{\partial V(t)}{\partial t} = V(t)H_0
\]

(30)

(such a unitary operator exists: \( V(t) = V(O)e^{iH_0t}, V^\dagger(O)V(O) = 1).\) For the new state vectors and operators we have

\[
|\Phi(t)\rangle = V(t)|\Psi(t)\rangle ,
\]

\[
O(t) = V(t)OV^\dagger(t) .
\]

(31)

From Eqs. (27), (30) and (31) it follows that new state vectors satisfy the following equation of motion:

\[
i\frac{\partial |\Phi(t)\rangle}{\partial t} = H_I(t)|\Phi(t)\rangle .
\]

(32)

So new state vectors depend on time only if there is an interaction. And (as payment for this) new operators depend on time and satisfy the equation

\[
i\frac{\partial O(t)}{\partial t} = [O(t), H_0(t)] .
\]

(33)

We have considered different representations. Our original representation was the Schrödinger one. We went over to the interaction (or Dirac) representation. This representation is very useful in field theory. We will now introduce the S-matrix in the interaction representation. Let us write down the integral equation that is equivalent to equation of motion (32) and includes the initial condition. We have

\[
|\Phi(t)\rangle = |\Phi(t_o)\rangle + (-i) \int_{t_o}^{t} dt_1 H_I(t_1)|\Phi(t_1)\rangle .
\]

(34)
We can easily solve this equation by iteration:

\[ |\Phi(t)\rangle = |\Phi(t_0)\rangle + (-i) \int_{t_0}^{t} dt_1 \ H_I(t_1)|\Phi(t_0)\rangle \]
\[ + (-i)^2 \int_{t_0}^{t} dt_1 \ H_I(t_1) \int_{t_0}^{t_1} dt_2 \ H_I(t_2)|\Phi(t_2)\rangle = ... = U(t, t_0)|\Phi(t_0)\rangle, \]

where

\[ U(t, t_0) = \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \ H_I(t_1) \int_{t_0}^{t_1} dt_2 \ H_I(t_2) ... \int_{t_0}^{t_{n-1}} dt_n \ H_I(t_n). \]

So the solution of the equation of motion (32) has the form

\[ |\Phi(t)\rangle = U(t, t_0)|\Phi(t_0)\rangle, \quad t \geq t_0. \]

We have found the operator \( u(t, t_0) \) in a perturbative expansion. Three remarks are in order:

i) The operator \( u(t, t_0) \) is a unitary operator. In fact from Eqs. (32) and (37) it follows that the operator \( u(t, t_0) \) satisfies the following equation:

\[ i \frac{\partial U(t, t_0)}{\partial t} = H_I(t)U(t, t_0). \]

By Hermitian conjugation from Eq. (38) we obtain

\[ -i \frac{\partial U^\dagger(t, t_0)}{\partial t} = U^\dagger(t, t_0)H_I(t) \]

(we have taken into account that \( H^\dagger_I = H_I \)). Now let us multiply Eq. (38) by \( U^\dagger(t, t_0) \) from the left and Eq. (39) by \( U(t, t_0) \) from the right and add the relations obtained. We have

\[ \frac{\partial}{\partial t} U^\dagger(t, t_0)U(t, t_0) = 1. \]

From this equation it follows that

\[ U^\dagger(t, t_0)U(t, t_0) = \text{const}. \]

Now taking into account that \( U(t_0, t) = 1 \) we have

\[ U^\dagger(t, t_0)U(t, t_0) = 1. \]

ii) The operator \( U(t, t_0) \) can be presented in the form

\[ U(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 ... \int_{t_0}^{t_{n-1}} dt_n \ P(H_I(t_1)H_I(t_2)...H_I(t_n)), \]

where \( P \) is the Dyson chronological operator

\[ P(H_I(t_1)H_I(t_2)...H_I(t_n)) = H_I(t_1)H_I(t_{i_2})H_I(t_{i_2}) ... H_I(t_{i_n}) \]

\[ t_{i_1} \geq t_{i_2} \geq ... \geq t_{i_n}. \]
Consider the case $n = 2$. We have

$$I_2 = \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t_2} dt_2 \ H_f(t_1) H_f(t_2) .$$

The region of the integration in $I_2$ is shown in Fig. 1.

![Fig. 1](image)

Let us change the order of the integration. We will obtain

$$I_2 = \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t} dt_2 \ H_f(t_1) H_f(t_2) = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 \ H_f(t_2) H_f(t_1) .$$

Finally for the operator $I_2$ we have

$$I_2 = \frac{1}{2} \int_{t_0}^{t} dt_1 \left[ \int_{t_0}^{t_1} dt_2 \ H_f(t_1) H_f(t_2) + \int_{t_1}^{t} dt_2 \ H_f(t_2) H_f(t_1) \right]$$

$$= \frac{1}{2} \int_{t_0}^{t_1} dt_1 \int_{t_0}^{t} dt_2 \ P(H_f(t_1) H_f(t_2)) .$$

The formula (41) for the case of arbitrary $n$ can be proved by induction.

iii) In the case of any process we are interested in (scattering, inelastic process, decay)

$$t_0 \rightarrow -\infty , \quad t \rightarrow \infty$$

and

$$|\Phi(\infty)\rangle = U(\infty, -\infty)|\Phi(-\infty)\rangle . \quad (43)$$

The operator $U(\infty, -\infty) = S$ is called the $S$-matrix. Let $|n\rangle$ be the total system of the orthogonal states (usually states that describe particles with definite momenta and spin projections). Let the initial state be one of the states $|\Phi(-\infty)\rangle = |n_0\rangle$. We have

$$|\Phi(\infty)\rangle = S|n_0\rangle = \sum_n |n\rangle \langle n| S|n_0\rangle . \quad (44)$$

Thus $\langle n| S|n_0\rangle$ is the amplitude of the transition $|n_0\rangle \rightarrow |n\rangle$. In quantum field theory the interaction Hamiltonian can be presented in the form

$$H_f(x_0) = \int H_f(x) \, d^3 x , \quad (45)$$

where $H_f(x)$ is the density of the interaction Hamiltonian (namely this quantity is usually called the Hamiltonian). For the $S$-matrix we have the following expression:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4 x_1 \int d^4 x_2 \ldots \int d^4 x_n \ P(H_f(x_1) H_f(x_2) \ldots H_f(x_n)) . \quad (46)$$

To calculate the matrix element of the $S$-matrix (and this is our main purpose):
a) We must know how to build the states $|n\rangle$.

b) We must know the interaction Hamiltonian $H_I(x)$.

c) We must develop the technique of the calculation of matrix elements of chronological products of operators (Feynman diagrams).

d) We must know the technique of the calculation of cross-sections and decay probabilities.

2. **Lagrange formalism**

Here I would like to remind you of the general principles from which the equation of motion and conserved quantities (energy, momentum, charges, etc.) can be obtained.

We will denote by $\psi(x)$ a function of space and time coordinates $x$ that describe some classical field (the index $\sigma$ runs over discrete values). The field equations follow from the **variational principle**.

Let us recall that in classical mechanics the Lagrangian is the difference between kinetic and potential energies. In the simplest case of one point we have

$$L(q, \dot{q} = \frac{m \dot{q}^2}{2} - V(q).$$

(47)

We will assume that the Lagrangian of the field depends on $\psi$ and space and time derivatives of $\psi$

$$\mathcal{L}(\psi, \partial \psi), \quad \partial \psi = (\partial_{\dot{\psi}}, \partial_{\psi}).$$

(48)

This assumption is a natural generalization of Eq. (47) for the relativistic case.

The action of the field is given by

$$S = \int_{\Omega} L \, d^4x$$

(49)

where $\Omega$ is a volume in the four-dimensional space. Now consider the variation of the field

$$\psi'(x) = \psi(x) + \delta \psi(x),$$

where $\delta \psi(x)$ is an infinitesimal arbitrary function. For the variation of the action we have

$$\delta S = S' - S = \int_{\Omega} \left( \frac{\partial L}{\partial \psi} \delta \psi + \frac{\partial L}{\partial \partial_{\psi}} \delta \partial_{\psi} \right) d^4x,$$

(50)

where

$$\delta \partial_{\psi} = \partial_{\psi} \psi' - \partial_{\psi} \psi = \partial_{\psi} \delta \psi.$$

We can rewrite the expression (50) in the form

$$\delta S = \int_{\Omega} \left( \frac{\partial L}{\partial \psi} - \partial_{\psi} \frac{\partial L}{\partial \partial_{\psi}} \right) \delta \psi d^4x + \int_{\Sigma} \frac{\partial L}{\partial \partial_{\psi}} \delta \psi d\Sigma,$$

(51)

where we have used the Gauss theorem

$$\int_{\Omega} \partial_{\sigma} \left( \frac{\partial L}{\partial \partial_{\sigma} \psi} \delta \psi \right) d^4x = \int_{\Sigma} \frac{\partial L}{\partial \partial_{\sigma} \psi} \delta \psi d\Sigma,$$

where $\Sigma$ is the surface of the four-dimensional volume considered.)

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Now let us find the minimum of the action under the condition that
\[ \delta\Psi/\Sigma = 0. \]
The minimum is reached if
\[ \frac{\partial\mathcal{L}}{\partial\psi} - \sum_{\alpha} \frac{\partial\mathcal{L}}{\partial\partial_{\alpha}\psi} = 0. \] (52)
We will accept as a postulate that these equations (Lagrange–Euler equations) are equations of motion of the field.

The laws of the conservation of total momentum, angular momentum, and charge follow from the invariance of the Lagrangian under translations, rotations, global gauge transformations.

Let us consider translations
\[ x' = x + \delta x, \]
\[ \psi'(x') = \psi(x), \] (53)
where \( \delta x \) is an infinitesimal constant.

The invariance means that
\[ \mathcal{L}(\psi'(x'), \partial'\psi'(x')) = \mathcal{L}(\psi(x), \partial\psi(x)). \] (54)

We have \( \partial'\psi'(x') = \partial\psi(x) \). Thus if the Lagrangian depends on \( \psi \) and \( \partial\psi \) (as we have assumed) translational invariance is fulfilled automatically. Now we have
\[ \psi'(x) = \psi(x - \delta x) = \psi(x) + \delta\psi(x), \]
\[ \delta\psi(x) = -\delta x^\beta \partial_\beta\psi(x). \] (55)

The invariance condition [Eq. (54)] can be written in the form
\[ \mathcal{L}(\psi'(x), ...) = \mathcal{L}(\psi(x), ...) + \delta x^\alpha \partial_\alpha\mathcal{L}(\psi(x), ...) = \mathcal{L}(\psi(x), ...) \]
or
\[ \delta\mathcal{L} + \delta x^\alpha \partial_\alpha\mathcal{L} = 0. \] (56)
Later on, taking into account the field equation (52), from Eq. (56) we obtain
\[ \partial_\alpha \left( \frac{\partial\mathcal{L}}{\partial\partial_{\alpha}\psi} \delta\psi + \delta x^\alpha\mathcal{L} \right) = 0. \] (57)

Let us determine the tensor
\[ T_\alpha^\beta = \frac{\partial\mathcal{L}}{\partial\partial_{\alpha}\psi} \partial_\beta\psi - \mathcal{L}\delta_\alpha^\beta. \] (58)

From translational invariance [Eq.(57)] it follows that
\[ \partial_\alpha T_\alpha^\beta = 0. \] (59)
This equation means that some vector is conserved (does not depend on time). In fact, let us integrate Eq. (59) over some volume \( V \) in the 3-dimensional space. We have

\[
\partial_0 \int_V T^0_\beta \, d^3x + \int_S T^i_\beta \, dS_i = 0 ,
\]

(60)

where \( S_i \) is the surface in which the volume \( V \) is contained (we have used the Gaussian theorem). Now let us consider the whole space (\( V \to \infty \)) assuming that the field is absent at infinity (natural boundary conditions). From Eq. (60) we obtain

\[
\partial_0 P_\beta = 0 ,
\]

where

\[
P_\beta = \int T^0_\beta \, d^3x
\]

(61)

is the energy-momentum vector. The tensor \( T^\alpha_\beta \) is called the canonical energy-momentum tensor.

Now we will discuss the invariance of the Lagrangian of the field under **global gauge invariance**. The field \( \psi \) is in general a complex function. This means that \( \psi \) and \( \psi^* \) are independent. So in the case of complex \( \psi \) the Lagrangian of the field is a function of \( \psi, \psi^* \) and derivatives \( \partial_\alpha \psi, \partial_\alpha \psi^* \).

Assume that the Lagrangian is invariant under the transformation

\[
\psi'(x) = e^{i\alpha} \psi(x) , \\
\psi'^*(x) = e^{-i\alpha} \psi(x) ,
\]

(62)

where \( \alpha \) is a constant. We have (invariance condition)

\[
\mathcal{L}(\psi', \psi^*, ..., ) = \mathcal{L}(\psi, \psi^*, ..., ) .
\]

(63)

The transformations (62) are called global gauge transformations. The invariance under these transformations means that phases \( \arg \psi \) and \( \arg \psi + \alpha \) are physically indistinguishable.

From the invariance under global gauge transformations it follows that a charge is conserved. In fact, for infinitesimal \( \alpha \) we have:

\[
\psi' = \psi + \delta \psi , \quad \delta \psi = i\alpha \psi , \\
\psi'^* = \psi^* + \delta \psi^* , \quad \delta \psi^* = -i\alpha \psi^* .
\]

(64)

Further, from the condition of invariance (63) it follows that

\[
\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial \psi^*} \delta \psi^* + \frac{\partial \mathcal{L}}{\partial \partial_\alpha \psi} \delta \partial_\alpha \psi + \frac{\partial \mathcal{L}}{\partial \partial_\alpha \psi^*} \delta \partial_\alpha \psi^* \\
= \partial_\alpha \left( \frac{\partial \mathcal{L}}{\partial \partial_\alpha \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial \partial_\alpha \psi^*} \delta \psi^* \right) = 0 .
\]

(65)

Notice that in Eq. (64) equations of motion were used. Let us determine the current

\[
j^\alpha = -i \left( \frac{\partial \mathcal{L}}{\partial \partial_\alpha \psi} \psi - \frac{\partial \mathcal{L}}{\partial \partial_\alpha \psi^*} \psi^* \right) .
\]

(66)
From global gauge invariance it follows that the current \( j^\alpha \) is conserved,

\[
\partial_\alpha j^\alpha = 0 ,
\]

and the charge

\[
Q = \int j^0 \, d^3x
\]

does not depend on time.

Notice that the cases that we have considered are particular cases of the general Noether theorem, which states that the conservation of physical quantities (energy, momenta, angular momenta, charges, and so on) is the consequence of the invariance of the Lagrangian of the system under continuous transformations.

### 3. Scalar (pseudoscalar) field

The 4-momentum of the free particle satisfies the condition

\[
p^2 - m^2 = 0 ,
\]

where \( m \) is the mass of the particle. Let us replace \( p_\mu \) by \( i\partial_\mu \), and assume that the field \( \psi \) satisfies the equation

\[
(\Box + m^2)\psi(x) = 0 ,
\]

where \( \Box = \partial_\mu \partial^\mu = \partial^2 - \nabla^2 \). Equation (69) is called a \textit{Klein–Gordon} equation. As we will see, this Lorentz-invariant equation provides the correct relation (69) between energy and momentum of the free particles (quanta of fields).

We will now consider a scalar (pseudoscalar) complex field \( \phi(x) \) and will assume that the field \( \phi(x) \) satisfies the Klein–Gordon equation

\[
(\Box + m^2)\phi(x) = 0 ,
\]

\[
(\Box + m^2)\phi^*(x) = 0 ,
\]

where \( m \) is a positive parameter.

As we have seen in the previous section, in order to obtain equations of motion of the field, and to obtain energy-momentum, charge, and other conserved quantities, it is necessary to know the Lagrangian of the field. Usually we postulate equations of field and build the Lagrangian that provides postulated equations and satisfies such requirements as positiveness of the energy. Let us assume now that the Lagrangian of the field \( \phi(x) \) is given by

\[
\mathcal{L} = \partial_\alpha \phi^* \partial^\alpha \phi - m^2 \phi^* \phi .
\]

It is easy to check that the equation of the motion of the field will be a Klein–Gordon equation. In fact, we have

\[
\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^* , \quad \frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi , \quad \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi} = \partial^\alpha \phi^* , \quad \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^*} = \partial^\alpha \phi .
\]

From Eqs. (52) and (73) the Klein–Gordon equation follows. It is also clear that the Lagrangian (72) is invariant under global gauge transformations. For the energy-momentum tensor, the vector of energy and momentum, current and charge, from Eqs. (58),
(66), (72), and (73), we have respectively

\[ T^\alpha_\beta = \partial^\alpha \phi^* \partial_\beta \phi + \partial^\alpha \phi \partial_\beta \phi^* - \mathcal{L} \delta^\alpha_\beta , \]  
\[ P_\beta = \int T^\alpha_\beta \, d^3x , \]  
\[ j^\alpha = -i(\partial^\alpha \phi^* \phi - \phi^* \partial^\alpha \phi) , \]  
\[ Q = \int j^0 \, d^3x . \]  

(74)  
(75)  
(76)  
(77)

As is seen from Eqs. (74) and (75), the Lagrangian (72) provides the positiveness of the energy of the field \( \phi \).

We describe the scalar field, which we are considering, by the function \( \phi \). For the purposes of the quantization it is convenient to make a Fourier transformation of the field \( \phi \). We have

\[ \phi(x) = \frac{1}{(2\pi)^{3/2}} \int \phi(q^\alpha, x^\alpha) e^{i\vec{q}\cdot\vec{x}} \, d^3q . \]  

(78)

From the Klein–Gordon equation it follows that the Fourier components \( \phi(\vec{q}, x^0) \) satisfy the following equation

\[ (\partial^2_0 + (\vec{q}^2 + m^2))\phi(\vec{q}, x^0) = 0 . \]  

(79)

(where \( \vec{q} \) is fixed)

The general solution of Eq. (79) has the following form:

\[ \phi(\vec{q}, x^0) = \phi^{(+)}(\vec{q}) e^{-i\vec{q}\cdot\vec{x}^0} + \phi^{(-)}(\vec{q}) e^{-i(-\vec{q}^0)\vec{x}^0} , \]  

\[ \begin{array}{c|c}
\text{(positive frequency)} & \text{(negative frequency)} \\
\end{array} \]

(80)

where

\[ q_0 = +\sqrt{\vec{q}^2 + m^2} . \]

Now it is convenient to introduce

\[ \phi^{(+)}(\vec{q}) = \frac{1}{\sqrt{2q^0}} a(q) , \]

\[ \phi^{(-)}(\vec{q}) = \frac{1}{\sqrt{2q^0}} a(-q) . \]  

(81)

Let us stress that \( a(q) \) and \( a(-q) \) are different functions of the 3-dimensional variable \( \vec{q} \). For the function \( \phi \) we have the following decomposition:

\[ \phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) , \]  

(82)

where the positive-frequency and negative-frequency parts of the function \( \phi(x) \) are given by the following expressions

\[ \phi^{(+)}(x) = \int N_q a(q) e^{-i\vec{q}\cdot\vec{x}} \, d^3q , \]

\[ \phi^{(-)}(x) = \int N_q a(-q) e^{i\vec{q}\cdot\vec{x}} \, d^3q . \]  

(83)
Here
\[ N_q = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2q^0}} \]
is the normalization factor.

Now it is easy to express the vector of energy-momentum and the charge of the field through the Fourier components \(a(q)\) and \(a(-q)\). We have
\[
\begin{align*}
P^\alpha &= \int (a^*(q)a(q) + a^*(-q)a(-q))q^\alpha \, d^3 q , \\
Q &= \int (a^*(q)a(q) - a^*(-q)a(-q)) \, d^3 q .
\end{align*}
\]
(84)

As is seen from these expressions, the energy of the field is a positively definite quantity while the charge of the field is not. Therefore, we obtained a physically reasonable result.

Up to now we have considered classical theory. Let us next consider quantized field. To come from usual mechanics to quantum mechanics we change canonical variables \(p\) and \(q\) by operators that satisfy the commutation relations
\[
[p, q] = \frac{1}{i} , \quad [p, p] = 0 , \quad [q, q] = 0 .
\]

To obtain an energy operator (for example) we replace in the classical expression for the energy
\[
H = \frac{p^2}{2m} + V(q)
\]
the classical quantities \(p\) and \(q\) by the operators.

In the quantum field theory \(a(q)\) and \(a(-q)\) are operators. This means that \(\phi(x)\) is also an operator which satisfies the Klein–Gordon equation
\[
(\Box + m^2)\phi(x) = 0 .
\]
(85)

There are general conditions that quantized fields must satisfy. They are determined by invariance of the Lagrangian under different transformations. Let us consider translations. We have in the classical theory
\[
\psi'(x') = \psi(x) , \quad x' = x + a ,
\]
where \(a\) is a constant vector.

In quantum theory the corresponding transformation has the form
\[
U^{-1}(a)\psi(x')U(a) = \psi(x) ,
\]
(86)
where \(U(a)\) is a unitary operator. For the infinitesimal \(a\) we have
\[
U(a) = 1 + iP_\alpha a^\alpha ,
\]
(87)
where the Hermitian operator \(P_\alpha\) is the momentum operator. From Eqs. (86) and (87) it follows that the field operator \(\psi\) must satisfy the following commutation relation:
\[
[\psi(x), P_\alpha] = i\partial_\alpha\psi(x) .
\]
(88)
Let us now consider global gauge transformations. In the classical theory we have

$$\psi'(x) = e^{i\alpha} \psi(x).$$

The corresponding relation in the quantum theory has the form

$$U^{-1}(\alpha) \psi(x) U(\alpha) = e^{i\alpha} \psi(x)$$

(89)

where $U(\alpha)$ is a unitary operator. For the infinitesimal $\alpha$ we have

$$U(\alpha) = 1 + iQ\alpha.$$  

(90)

Here $Q$ is the operator of the charge. From Eqs. (89) and (90) it follows that the operator $\psi(x)$ must satisfy the following commutation relation:

$$[\psi(x), Q] = \psi(x).$$

(91)

For the scalar (pseudoscalar) field $\phi(x)$ we have

$$[\phi(x), P_\alpha] = i\partial_\alpha \phi(x).$$

(92)

From Eqs. (82), (83), and (92) it follows that

$$[a(q), P_\alpha] = q_\alpha a(q), [a(-q), P_\alpha] = -q_\alpha a(-q).$$

(93)

From these commutation relations by Hermitian conjugation, we obtain the following relations:

$$[a^\dagger(q), P_\alpha] = -q_\alpha a^\dagger(q), \quad [a^\dagger(-q), P_\alpha] = q_\alpha a^\dagger(-q).$$

(94)

Now let $|E\rangle$ be the eigenstate of the Hamiltonian corresponding to the eigenvalue $E$

$$H|E\rangle = E|E\rangle.$$  

(95)

Let us multiply (Eq. 95) by $a(q)$ ($a^\dagger(q)$) from the left:

$$a(q)H|E\rangle = Ea(q)|E\rangle,$$

$$a^\dagger(q)H|E\rangle = Ea^\dagger(q)|E\rangle.$$ 

(96)

Taking into account Eqs. (93) and (94) we obtain

$$a(q)H = [a(q), H] + Ha(q) = q_\alpha^0 a(q) + Ha(q),$$

$$a^\dagger(q)H = -q^0 a^\dagger(q) + Ha^\dagger(q).$$

(97)

From Eqs. (96) and (97) we have

$$Ha(q)|E\rangle = (E - q_\alpha^0)a(q)|E\rangle,$$

$$Ha^\dagger(q)|E\rangle = (E + q^0)a^\dagger(q)|E\rangle.$$  

(98)

Thus if the state $a(q)|E\rangle(a^\dagger(q)|E\rangle)$ is not zero it is an eigenstate of the Hamiltonian $H$ with eigenvalues $E - q_\alpha^0(E + q^0)$. The operator $a(q)(a^\dagger(q))$ is called the absorption
operator (creation operator). It is easy to see also that $a(-q) = b^\dagger(q)$ is the creation operator and $a^\dagger(-q) = b(q)$ is the absorption operator.

To build quantum theory we must postulate commutation relations for absorption and creation operators and expressions for operators of energy, momenta, charge, etc.

We will assume that $a(q)$ and $b(q)$ satisfy the following commutation relations:

$$
\begin{align*}
\left[ a(q), a^\dagger(q') \right] &= \delta(q - q'), & \left[ a(q), a(q') \right] &= 0, \\
\left[ b(q), b^\dagger(q') \right] &= \delta(q - q'), & \left[ b(q), b(q') \right] &= 0, \\
\left[ a(q), b(q') \right] &= 0, & \left[ a(q), b^\dagger(q') \right] &= 0.
\end{align*}
$$

(99)

To obtain the Hamiltonian of the theory, as in the case of usual quantum mechanics, we will replace the classical quantities in the expression for the energy by operators. From expression (84) we have

$$
H = \int \left[ a^\dagger(q) a(q) + b(q)b^\dagger(q)\right]q^0 \, d^3q.
$$

(100)

It is clear that $H^\dagger = H$ and all eigenvalues of the operator $H$ are positive. Further it is easy to check that

$$
\begin{align*}
\left[ a(q), H \right] &= q^0 a(q), \\
\left[ b(q), H \right] &= q^0 b(q).
\end{align*}
$$

Thus commutation relations (99) and expression (100) for the Hamiltonian are in agreement with the relations (93). Now because all $E$ are positive there exists the state with the minimal energy

$$
H|E_{\text{min}}\rangle = E_{\text{min}}|E_{\text{min}}\rangle.
$$

(101)

For this state we have

$$
\begin{align*}
\langle a(q)|E_{\text{min}}\rangle &= 0, \\
\langle b(q)|E_{\text{min}}\rangle &= 0.
\end{align*}
$$

(102)

Now let us find $E_{\text{min}}$. Using Eq. (100) and commutation relations (99) we find

$$
E_{\text{min}} = \langle E_{\text{min}}| \int (a^\dagger(q)a(q) + b(q)b^\dagger(q))q^0 \, d^3q|E_{\text{min}}\rangle =
$$

$$
= \langle E_{\text{min}}| \int b(q)b^\dagger(q')b(q - q')q^0 \, d^3q \, d^3q' |E_{\text{min}}\rangle = \int \delta(q' - q)\delta(q - q') \, d^3q \, d^3q' \to \infty.
$$

Thus all eigenvalues of the Hamiltonian (100) are infinite. The origin of the problem is connected with the order of the operators in the second term of the expression (100). In the classical theory we can write quantities, say, $b(q)$ and $b^*(q)$ in any order. In quantum theory the order of operators $b(q)$ and $b^*(q)$ is important (they do not commute). So we need the rule that will determine the order of operators in the Hamiltonian and other operators that are obtained from the corresponding classical quantities.
Let us now introduce the operator of the normal product \( N \) in the following way

\[
N(a(q)a\dagger(q')) = a\dagger(q')a(q),
\]
\[
N(b(q)b(q')) = b(q)b(q') = b(q')b(q), ...
\] (103)

In the general case of \( n \) operators, operator \( N \) when acting on the product of absorption and creation operators puts them in the normal order (all creation operators are on the left of all absorption operators). Now we formulate the following rule to obtain quantum operators from classical quantities: we replace classical quantities \( a(q), b(q), \) etc., by operators, replace complex conjugation by Hermitian conjugation, and put all operators in the normal order (act by operator \( N \)).

For the operator of 4-momentum and charge, from Eq. (84) we obtain the following expressions:

\[
P_\alpha = \int [a\dagger(q) a(q) + b\dagger(q) b(q)]q_\alpha \, d^3q,
\]
\[
Q = \int [a\dagger(q) a(q) - b\dagger(q) b(q)] \, d^3q.
\] (104)

It is easy now to prove that

\[
[a(q), P_\alpha] = q_\alpha a(q), \quad [b(q), P_\alpha] = q_\alpha b(q),
\]
\[
[b(q), Q] = -b(q), \quad [a(q), Q] = a(q), \quad (105)
\]
in accordance with the relations (88) and (92).

From Eqs. (102) and (104) for the state with the minimum energy we have

\[
P_\alpha |E_{\text{min}}\rangle = 0, \quad Q |E_{\text{min}}\rangle = 0.
\] (106)

Thus the vector \( |E_{\text{min}}\rangle \) describes the vacuum state—a state with equal to zero energy, momentum and charge. We will denote the vacuum state by \( |0\rangle \).

Let us now consider the vectors \( a\dagger(q)|0\rangle, \ b\dagger(q)|0\rangle \). From Eq. (105) we have

\[
P_\alpha a\dagger(q)|0\rangle = (\{P_\alpha, a\dagger(q)\} + a\dagger(q) P_\alpha)|0\rangle = q_\alpha a\dagger(q)|0\rangle,
\]
\[
P_\alpha b\dagger(q)|0\rangle = q_\alpha b\dagger(q)|0\rangle.
\] (107)

Thus vectors \( a\dagger(q)|0\rangle \) and \( b\dagger(q)|0\rangle \) describe states with momentum \( q \) and the same mass \( m = \sqrt{q^2} \).

Now it is easy to see that the states \( a\dagger(q)|0\rangle \) and \( b\dagger(q)|0\rangle \) describe particles with different charges:

\[
Qa\dagger(q)|0\rangle = ([Q, a\dagger(q)] + a\dagger(q)Q)|0\rangle = a\dagger(q)|0\rangle,
\]
\[
Qb\dagger(q)|0\rangle = -b\dagger(q)|0\rangle.
\] (108)

Thus the vectors \( a\dagger(q)|0\rangle \) and \( b\dagger(q)|0\rangle \) describe states of a particle and an antiparticle with the same mass and opposite charges.

Let us note that we have here an example of a general consequences of the field theory. From the quantum field theory it follows that if in Nature there exists a particle
with non-zero charge (charges) the particle with the same mass and opposite charge (charges) must also exist. The latter is called the antiparticle. As is very well known, this general consequence of the quantum field theory was confirmed by all existing data.

Now let us continue the consideration of the states that describe quantized scalar field. We consider the vector \( a^\dagger(q) b^\dagger(q')|0\). We have

\[
P_\alpha a^\dagger(q) b^\dagger(q')|0\rangle = \left((P_\alpha, a^\dagger(q)) + a^\dagger(q) P_\alpha b^\dagger(q')|0\rangle = (q_\alpha + q'_\alpha)a^\dagger(q)b^\dagger(q')|0\rangle ,
\]

\[
Q_\alpha a^\dagger(q) b^\dagger(q')|0\rangle = (1 + (-1))a^\dagger(q)b^\dagger(q)|0\rangle .
\]

So the vector \( a^\dagger(q) b^\dagger(q')|0\rangle\) describes the particle with the momentum \( q \) and the antiparticle with the momentum \( q' \). It is clear that the vectors

\[
a^\dagger(q_1) \ldots a^\dagger(q_n) a^\dagger(q'_1) \ldots b^\dagger(q'_m)|0\rangle
\]

describe \( n(q)_1 \) particles with momentum \( q_1, \ldots \), \( \bar{n}(q'_1) \) antiparticles with momentum \( q'_1, \ldots \). It is possible to show that the expression (109) forms the total system of vectors that describe the quantized scalar field. Thus states of quantized scalar field are states of free particles and antiparticles with definite momentum. According to the commutation relations that we have postulated, the number of particles with the same momenta in a state can be arbitrary. This corresponds to the Bose statistics that scalar particles (particles with spin equal to zero) must satisfy. Thus we have quantized the field in accordance with the spin-statistics connection.

We will now consider the field of spin-1/2 Fermi particles.

4. Spin-1/2 (spinor) field

Let us now consider the second classical example of the field—the complex spinor field \( \psi(x) \). We will assume that the field \( \psi(x) \) satisfies the Dirac equation

\[
i\gamma^\mu \partial_\mu \psi - m\psi = 0 ,
i\partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} = 0 .
\]

Now we will apply the Lagrange formalism to this field. First of all it is easy to check that the Lagrangian

\[
\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m)\psi
\]

(111)

gives us the postulated equations of the field. In fact, we have

\[
\frac{\partial \mathcal{L}}{\partial \psi} = (i\gamma^\mu \partial_\mu - m)\psi , \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = 0 ,
\]

and from Eq. (52) we have the equation for \( \psi \). Furthermore,

\[
\frac{\partial \mathcal{L}}{\partial \psi} = -\bar{\psi}m , \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = i\bar{\psi}\gamma^\mu ,
\]
and from Eq. (52) we obtain the equation for \( \psi \). From the general formulae (58), (61), (66), and (68) for the energy-momentum, current, and charge of the field, we easily obtain

\[
P_\beta = \int T^\beta_\alpha \, d^3x = \int \tilde{\psi} \gamma^\alpha \partial_\beta \psi \, d^3x,
\]

\[
j^\alpha = \bar{\psi} \gamma^\alpha \psi, \quad Q = \int \bar{\psi} \gamma^0 \psi \, d^3x.
\]

(112)

Now we have the following Fourier decomposition of the field:

\[
\psi(x) = \int N_p [u^r(p) e^{-ipx} c_r(p) + u^r(-p) e^{ipx} c_r(-p)] \, d^3p
\]

(113)

(the state with positive energy) (the state with negative energies)

Here \( N_p = [1/(2\pi)^{3/2}] 1/\sqrt{2p^0} \) and Fourier components \( c_r(p) \) and \( c_r(-p) \) are complex functions of the 3-dimensional variable \( \vec{p} \) \( (p_0 = +\sqrt{\vec{p}^2 + m^2}) \). From Eqs. (112) and (113), for the vector of the energy-momentum and the charge of the field we easily obtain the following expressions:

\[
P_\alpha = \int [c_\alpha^*(p) c_r(p) - c_\alpha^*(p) c_r(-p)] p_\alpha \, d^3p,
\]

(114)

\[
Q = \int [c_\alpha^*(p) c_r(p) + c_\alpha^*(p) c_r(-p)] \, d^3p.
\]

(115)

The second term in expression (114) is the contribution to \( P_\alpha \) of the states with negative energy. This term gives a negative contribution to the energy of the field. So in the classical theory of the spinor field the energy is not a positively definite quantity. On the contrary, as is seen from Eq. (115) the charge in this theory is positive.

Let us now turn to the quantum theory of the spin-1/2 field, which as we will show is a consistent theory. In quantum theory \( \psi(x) \) is an operator that satisfies the Dirac equation. This means that the operator \( \psi(x) \) is given by the expression (113) where \( c_r(p) \) and \( c_r(-p) \) are operators. From the general condition

\[
[\psi(x), \, p_\alpha] = i \partial_\alpha \psi(x)
\]

and Eq. (113) it follows that the operators \( c_r(p) \) and \( c_r(-p) \) must satisfy the commutation relations:

\[
[\, c_r(p), \, P_\alpha \,] = p_\alpha c_r(p) ,
\]

\[
[\, c_r(-p), \, P_\alpha \,] = -p_\alpha c_r(-p).
\]

(116)

As we have seen in the previous section, these relations mean that \( c_r(p) \), \( c_r^\dagger(-p) = d_r(p) \) are absorption operators and \( d_r^\dagger(p) \), \( c_r(-p) = d_r^\dagger(p) \), are creation operators. Now we will postulate the following anticommutator relations for the operators \( c_r(p) \) and \( d_r(p) \)

\[
[\, c_r(p), \, c_r^\dagger(p') \,]_+ = \delta_{rr'} \delta(\vec{p} - \vec{p'}), \quad [\, c_r(p), \, c_r(p') \,]_+ = 0 ,
\]

\[
[\, d_r(p), \, d_r^\dagger(p') \,]_+ = \delta_{rr'} \delta(\vec{p} - \vec{p'}), \quad [\, d_r(p), \, d_r(p') \,]_+ = 0 ,
\]

\[
[\, d_r(p), \, c_r(p') \,]_+ = 0, \quad [\, d_r(p), \, c_r^\dagger(p') \,]_+ = 0 ,
\]

(117)
where \([a, b]_+ = ab + ba\). States of spin-1/2 particles must satisfy the Pauli principle. As we will see later, the quantization with the help of anticommutators makes sure of the fulfilment of the Pauli principle.

Now we will obtain the Hamiltonian and other operators. Let us take the classical expression (114) and replace (as in the previous scalar case) the classical quantities by operators. We have the operator

\[
H' = \int (c_r^+(p)c_r(p) - d_r(p)d_r^+(p))p^0 \, d^3p .
\]

It is clear that this operator is not a Hamiltonian. Firstly, it is not positively definite. Secondly, with this operator we have the relations

\[
[c_r(p), H'] = p^0 c_r(p) ,
\]

\[
[d_r(p), H'] = -p^0 d_r(p) .
\]

The second relation does not correspond to Eqs. (116). The reason for these problems is the minus sign in the second term of expression (118).

Let us change the sign and accept for the Hamiltonian the following expression:

\[
H = \int (c_r^+(p)c_r(p) + d_r(p)d_r^+(p))p^0 \, d^3p .
\] (118)

It clear that all eigenvalues of this operator are positive (no problems with the vacuum state) and \([d_r(p), H] = p^0 d_r(p)\) in correspondence with Eqs. (116).

In order to obtain quantum operators from corresponding classical quantities let us generalize the definition of the operator of the normal product. We determine the operator \(N\) for the case of the Fermi fields as follows:

\[
N(c_r(p)c_r^+(p')) = -c_r^+(p')c_r(p) ,
\]

\[
N(c_r(p)c_r^+(p')) = c_r(p)c_r^+(p') = -c_r(p')c_r(p) .
\] (119)

In the general case the operator \(N\) sets a product of the Fermi operators in the normal order, and multiplies the product by \((-1)\) when the number of transpositions that have been done to rearrange the operators in normal order is odd, and by \((+1)\) when it is even. Notice that this definition corresponds to the postulated anticommutator relations.

Now we can formulate the rule for obtaining operators of energy, charge and other operators from corresponding conserved classical quantities: we replace classical quantities by operators, and act on the product of operators by the operator \(N\). For the operators of energy-momentum and charge from Eqs. (114) and (115) we obtain the following expressions:

\[
P_\alpha = \int (c_r^+(p)c_r(p) + d_r(p)d_r^+(p))p_\alpha \, d^3p ,
\]

\[
Q = \int (c_r^+(p)c_r(p) - d_r(p)d_r^+(p)) \, d^3p .
\] (120)

It is easy to check with the help of commutation relations (117) that Eqs. (120) satisfy the relations (116).
All eigenvalues of the Hamiltonian (118) are positive. The state with the minimum energy satisfies

\[ c_r(p)|0\rangle = 0, \quad d_r(p)|0\rangle = 0. \tag{121} \]

For the state $|0\rangle$ we have

\[ P_\alpha|0\rangle = 0, \quad Q|0\rangle = 0. \tag{122} \]

Thus $|0\rangle$ is a vacuum state.

Now consider states

\[ c_r^\dagger(p)|0\rangle \quad \text{and} \quad d_r^\dagger(p)|0\rangle. \]

We have

\[
\begin{align*}
P_\alpha c_r^\dagger(p)|0\rangle &= ([P_\alpha c_r^\dagger(p)] + c_r^\dagger(p)P_\alpha)|0\rangle = p_\alpha c_r^\dagger(p)|0\rangle, \\
P_\alpha d_r^\dagger(p)|0\rangle &= p_\alpha d_r^\dagger(p)|0\rangle, \\
Qc_r^\dagger(p)|0\rangle &= c_r^\dagger(p)|0\rangle, \\
Qd_r^\dagger(p)|0\rangle &= -d_r^\dagger(p)|0\rangle.
\end{align*}
\tag{123}
\]

Thus $c_r^\dagger(p)$ [$d_r^\dagger(p)$] is the operator of creation of a particle (antiparticle) with momentum $p$. It is possible to show that $c_r^\dagger(p)$ [$d_r^\dagger(p)$] is an operator of the creation of a particle (antiparticle) with spin projection $r$. To obtain the general state of the fermion field it is necessary to act on the vacuum state by creation operators $c_r^\dagger(p)$ [$d_r^\dagger(p)$]. Owing to anticommutator relations (117) the states of the fermion field satisfy the Pauli principle. For example, for the two-particle state we have

\[ c_r^\dagger(p)c_{r'}^\dagger(p')|0\rangle = -c_{r'}^\dagger(p')c_r^\dagger(p)|0\rangle. \]

5. **Hermitian scalar, electromagnetic, and vector fields**

In this section we will consider very briefly other fields that are of interest to us. Let us start from the **scalar (pseudoscalar) Hermitian field**. In this case, the phase of the field is fixed and there is no global gauge invariance. As a consequence, the quanta of the Hermitian field are neutral particles that have no charges (such as $\pi^0, \eta^0$).

We have

\[ \phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x). \]

From $\phi^{(\dagger)} = \phi$ it follows that $\phi^{(-)}(x) = (\phi^{(+)}(x))^\dagger$. Thus in the case of Hermitian field antiparticles identical to particles and for the positive-frequency and the negative-frequency parts of the scalar Hermitian field we have

\[
\begin{align*}
\phi^{(+)}(x) &= \int N_\phi(q)e^{-iwx} \, d^3q, \\
\phi^{(-)}(x) &= \int N_\phi^\dagger(q)e^{iwx} \, d^3q.
\end{align*} \tag{124}
\]
where

\[
\begin{align*}
[a(q), a^\dagger(q')] &= \delta(q - q') \\
\{a(q), a(q')\} &= 0 .
\end{align*}
\] (125)

The **electromagnetic field** is described by the real vector field \(A^\alpha(x)\) that satisfies the equation

\[
\Box A^\alpha(x) = 0 .
\] (126)

From Eq. (12) it follows that

\[
A^\alpha(x) = \int N_k \left( \sum_{\lambda=0}^{3} e_\lambda^\alpha(k) a_\lambda(k) e^{-ikx} + \sum_{\lambda=0}^{3} e_\lambda^\alpha(k) a_\lambda^\dagger(k) e^{ikx} \right) d^3k ,
\] (127)

where \(k^2 = 0\) and \(e_\lambda(k)\) are vectors of polarization. Let us put \(e_0^\alpha(k) = n^\alpha\), where \(n^\alpha\) is a unit time-like vector \((n^2 = 1)\). Now we will choose

\[
e_3^\alpha(k) = \frac{k^\alpha}{k \cdot n} - n^\alpha .
\]

It is clear that \(e_3(k)n = 0, e_3^\dagger(k) = -1\). Let us finally choose \(e_i(k)(i = 1, 2)\) in such a way that

\[
e_i \cdot n = 0, \quad e_i \cdot k = 0 ,
\]

\[
e_i e_k = -\delta_{ik} .
\]

In the system where \(n = (0, 1)\) we have

\[
e_3(k) = \left( 0, \frac{\vec{k}}{|\vec{k}|} \right) , \quad e_i(k) = (0, \vec{e}_i(\vec{k})) ,
\]

\[
\vec{e}_i(\vec{k}) \cdot \vec{k} = 0 , \quad \vec{e}_i(\vec{k})\vec{e}_j(\vec{k}) = \delta_{ij} .
\]

Further canonical commutation relations for the operators \(a(k)\) have the following form:

\[
\begin{align*}
[a_\lambda(k), a^\dagger_{\lambda'}(k')] &= -g_{\lambda\lambda'} \hbar (\vec{k} - \vec{k'}) , \\
\{a_\lambda(k), a_{\lambda'}(k')\} &= 0 .
\end{align*}
\] (128)

It is possible to show that \(a_{1,2}(k)\) are operators of absorption of photons with transverse polarizations, and that \(a_0(k)\) and \(a_3(k)\) are absorption operators of scalar and longitudinal photons. Finally from gauge invariance and the Lorentz condition

\[
\partial_\alpha A^{\alpha(+)}|\phi\rangle = 0
\]

\(|\phi\rangle\) is any vector that describes a physical state, it follows that free photons can only have transverse polarizations.

**The massive vector field** is described by the vector field \(B^\alpha(x)\) that satisfies the Proca equation

\[
\partial_\alpha F^{\alpha\beta} + m^2 B^\beta = 0 .
\] (129)
Here $F^{\alpha\beta} = \partial^{\alpha} B^{\beta} - \partial^{\beta} B^{\alpha}$. It follows from Eq. (129) that the field $B^{\beta}(x)$ satisfies the condition

$$\partial^{\beta} B^{\beta} = 0 .$$

(130)

This condition means that only 3 components of $B^{\beta}(x)$ are independent (this corresponds to spin one). From Eqs. (129) and (130) it follows that the field $B^{\beta}(x)$ satisfies the Klein–Gordon equation

$$\Box B^{\beta} + m^2 B^{\beta} = 0 .$$

(131)

For the neutral vector field we have

$$B^{\alpha}(x) = \int N_q \left( \sum_{\lambda=1}^{3} e_{\lambda}^{\alpha}(q) b_{\lambda}(q) e^{-iqx} + \sum_{\lambda'=1}^{3} e_{\lambda'}^{\alpha}(q) b_{\lambda'}^{\dagger}(q) e^{iqx} \right) d^3 q ,$$

(132)

where

$$e_{\lambda}(q) e_{\lambda'}(q) = \delta_{\lambda\lambda'} .$$

The operators $b(q)$ satisfy the following canonical commutation relations:

$$\left[ b_{\lambda}(q), b_{\lambda'}^{\dagger}(q') \right] = \delta_{\lambda\lambda'} \delta(q - q') ,$$

$$\left[ b_{\lambda}(q), b_{\lambda'}(q) \right] = 0 .$$

(133)

It is easy to show that the operators $b_{\lambda}(q)$ and $b_{\lambda}^{\dagger}(q)$ are operators of absorption and creation of vector particles with momenta $q$.

6. Interaction Hamiltonians

To calculate the matrix elements of the $S$-matrix we must know the interaction Hamiltonian. In this section we will discuss very well known electromagnetic and electroweak interactions. We will recall here the principle of the local gauge invariance which generates these interactions.

Consider the Lagrangian of the free electron field $\phi(x)$:

$$\mathcal{L}_0 = \bar{\phi}(\gamma^\mu p_\mu - m) \phi .$$

(134)

This Lagrangian is invariant under global gauge transformations

$$\phi'(x) = e^{i\Lambda} \phi(x) ,$$

(135)

where $\Lambda$ is constant. As we have seen, the invariance under transformation (135) means that the current $j^\alpha = \bar{\phi} \gamma^\alpha \phi$ is conserved:

$$\partial_\alpha j^\alpha = 0 .$$

and the charge $Q = \int j^0 \, d^3 x$ does not depend on time.

We now require invariance of the Lagrangian under transformations

$$\phi'(x) = e^{i\Lambda(x)} \phi(x) ,$$

(136)
where $\Lambda(x)$ is an arbitrary function of $x$. We have
\[ \partial_\alpha e = e^{-i\Lambda}(\partial_\alpha - i\partial_\alpha\Lambda)e', \]
and it is clear that Eq. (134) is not invariant under Eq. (136). From Eq. (137) it is clear that to provide the invariance under transformations (136) it is necessary to introduce the interaction of the electron field with the vector field. The transformation of the latter should absorb $\partial_\alpha\Lambda$. Consider $(\partial_\alpha + ieA_\alpha)e$. We have
\[ (\partial_\alpha + ieA_\alpha)e = e^{-i\Lambda}(\partial_\alpha + ieA'_\alpha)e', \]
where $A'_\alpha = A_\alpha - (1/e)\partial_\alpha\Lambda$. Thus $(\partial_\alpha + ieA_\alpha)e$ transforms like a field (initial and primed quantities differ by a phase factor) and the Lagrangian
\[ \mathcal{L}_1 = \bar{e}[i\gamma^\alpha(\partial_\alpha + ieA_\alpha) + m]e \]
is invariant under local gauge transformations
\[ e'(x) = e^{i\Lambda(x)}e(x), \quad \bar{e}'(x) = \bar{e}(x)e^{-i\Lambda(x)}, \]
\[ A'_\alpha(x) = A_\alpha(x) - \frac{1}{e}\partial_\alpha\Lambda(x). \]
(139)

To build the free Lagrangian of the field $A$ consider the stress tensor
\[ F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \]
(140)

It is clear that $F'_{\alpha\beta} = F_{\alpha\beta}$. So the Lagrangian
\[ \mathcal{L}_0 = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \]
is invariant under Eqs. (139). The total Lagrangian of the electromagnetic field is equal to
\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_0. \]
(142)

and the Lagrangian of the electromagnetic interaction is
\[ \mathcal{L}_I = -\epsilon\bar{e}\gamma^\alpha eA_\alpha = -\epsilon j^\alpha A_\alpha. \]
(143)

Thus the principle of local gauge invariance fixed the form of the interaction Lagrangian. It is easy to see, however, that the requirement of the local gauge invariance does not fix the interaction unambiguously. For example, we can add to the Lagrangian the Pauli term
\[ \mathcal{L}_P = -\mu\bar{e}\sigma^{\alpha\beta}eF_{\alpha\beta} \]
that is invariant under transformations (139). The change $\partial_\alpha \rightarrow \partial_\alpha + ieA_\alpha$ in the free Lagrangian is the minimum that we must do to provide local gauge invariance. Expression (143) is the Lagrangian of the minimal electromagnetic interaction. As we know, this Lagrangian describes all existing experimental data.
The standard electroweak interaction is based on SU(2) × U(1) local gauge invariance. The gauge fields are fields of vector charged and neutral particles (W, Z, γ). Let us first discuss SU(2) Yang–Mills local gauge invariance. Let

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^{-1} \end{pmatrix}$$

be some SU(2) doublet. The Lagrangian

$$\mathcal{L}_0 = \bar{\psi} (i \gamma^\alpha p_\alpha - m) \psi$$

is invariant under global SU(2) transformations

$$\psi' = e^{i \frac{1}{2} \vec{\tau} \vec{\Lambda}} \psi$$

where \(\Lambda_i\) are constants and \(\tau_i\) are Pauli matrices.

Now let us require (in analogy with electrodynamics) that invariance under local gauge transformations

$$\psi'(x) = U(x) \psi(x)$$

be valid. Here

$$U(x) = e^{i \frac{1}{2} \vec{\tau} \vec{\Lambda}(x)} \simeq 1 + i \frac{1}{2} \vec{\tau} \cdot \vec{\Lambda}(x)$$

and \(\Lambda_i\) are infinitesimal functions of \(x\). Let us consider the covariant derivative

$$\left( \partial_\alpha + ig \frac{1}{2} \tau^i \cdot \vec{\Lambda}_\alpha \right) \psi.$$ We have

$$\left( \partial_\alpha + ig \frac{1}{2} \tau^i \cdot \vec{\Lambda}_\alpha \right) \psi = U^{-1} U \left( \partial_\alpha + ig \frac{1}{2} \tau^i \cdot \vec{\Lambda}_\alpha \right) U^{-1} \psi'$$

$$= U^{-1} \left( \partial_\alpha + ig \frac{1}{2} \tau^i \cdot \vec{\Lambda}_\alpha \right) \psi',$$

where

$$\vec{\Lambda}_\alpha = \vec{\Lambda}_\alpha + \frac{1}{g} \partial_\alpha \vec{\Lambda} - \vec{\Lambda} \times \vec{\Lambda}_\alpha$$

and we have used \( [(\frac{1}{2})\tau_i, (\frac{1}{2})\tau_k] = i \epsilon_{ikl} (\frac{1}{2})\tau_l \). It is easy to show that the stress tensor

$$\vec{F}_{\alpha\beta} = \partial_\alpha \vec{A}_\beta - \partial_\beta \vec{A}_\alpha - g \vec{A}_\alpha \times \vec{A}_\beta$$

transforms like the isovector

$$\vec{F}'_{\alpha\beta} = \vec{F}_{\alpha\beta} - \vec{A} \times \vec{F}_{\alpha\beta}.$$ (148)

From Eqs. (147) and (150) it follows that the Lagrangian

$$\mathcal{L} = \bar{\psi} [i \gamma^\alpha \left( \partial_\alpha + ig \frac{1}{2} \tau^i \cdot \vec{\Lambda}_\alpha \right) - m] \psi - \frac{1}{4} \vec{F}_{\alpha\beta} \cdot \vec{F}^\alpha_{\beta}$$

is invariant under local SU(2) transformations (145), (146). For the Lagrangian of the minimal SU(2) invariant interaction we have

$$\mathcal{L}_I = -g \vec{\partial} \cdot \vec{A}.$$

(151)
where the current is equal to \( \vec{j}_\alpha = \vec{\psi}_\alpha \frac{1}{2} \gamma^\alpha \). The unified weak and electromagnetic (electroweak) interaction is based on spontaneously broken \( \text{SU}(2) \times \text{U}(1) \) symmetry. The Hamiltonians of the interaction of quarks and leptons with vector particles have electromagnetic, charged-current, and neutral-current parts,

\[
\mathcal{H}_I = \epsilon j^\alpha A^\alpha + \left( \frac{g}{2\sqrt{2}} j^W_\alpha W^\alpha + \text{h.c.} \right) + \frac{g}{2\cos\theta_W} j^Z_\alpha Z^\alpha ,
\]

where electromagnetic current, charged current, and neutral current are given by the expressions

\[
j^\gamma_\alpha = \sum_{\ell=e,\mu,\tau} (-1) \bar{\ell}_\gamma \gamma^\alpha \ell + \sum_{q=u,\ldots} e_q \bar{q}_\gamma q , \tag{154}\]

\[
j^W_\alpha = 2 \sum_{\ell=e,\ldots} \bar{\nu}_\ell \gamma^\alpha \nu \ell + 2 \sum_{q=u,\ldots} \bar{q}_L \gamma^\alpha V_{qq'} q'_L , \tag{155}\]

\[
j^Z_\alpha = \sum_{\ell=e,\ldots} \bar{\nu}_\ell \gamma^\alpha \nu \ell - \sum_{\ell=e,\ldots} \bar{\ell}_\gamma \gamma^\alpha \ell + 2 \sin^2\theta_W \sum_{\ell=e,\ldots} \bar{\ell}_\gamma \gamma^\alpha \ell
+ \sum_{q=u,\ldots} \bar{q}_L \gamma^\alpha q \ell - 2 \sin^2\theta_W \sum_{q=u,\ldots} e_q \bar{q}_\gamma q . \tag{156}\]

Here \( V_{qq'} \) is the Cabibbo–Kobayashi–Maskawa (CKM) matrix, \( e_q \) is quark charge, \( I_q = \pm \frac{1}{2} \) for up and down quarks, respectively, and the constants \( g, \epsilon, \) and \( \sin \theta_W \) are connected by the condition \( g \sin \theta_W = \epsilon \) (unification condition).

7. Method of calculation of the matrix elements of the chronological products of field operators

The \( S \)-matrix is the sum of the chronological products of field operators. We will now consider the technique of calculation of the matrix elements of the chronological products of field operators that was developed in the 1950s by Dyson and Wick. Consider the product of two Bose operators \( \phi(x_1)\phi(x_2) \). We have

\[
\phi(x_1)\phi(x_2) = \phi^{(+)}(x_1)\phi^{(+)}(x_2) + \phi^{(-)}(x_1)\phi^{(+)}(x_2) + \phi^{(-)}(x_1)\phi^{(-)}(x_2) + \phi^{(+)}(x_1)\phi^{(-)}(x_2) . \tag{157}\]

Let us write down the product as a sum of the normal products. Taking into account that

\[
\phi^{(+)}(x_1)\phi^{(-)}(x_2) = N(\phi^{(+)}(x_1)\phi^{(-)}(x_2)) + [\phi^{(+)}(x_1), \phi^{(-)}(x_2)] , \tag{158}\]

we have

\[
\phi(x_1)\phi(x_2) = N(\phi(x_1)\phi(x_2)) + [\phi^{(+)}(x_1), \phi^{(-)}(x_2)] . \tag{159}\]

In the case of the product of two Fermi operators, \( \psi(x_1)\bar{\psi}(x_2) \), we have analogously

\[
\psi(x_1)\bar{\psi}(x_2) = N(\psi(x_1)\bar{\psi}(x_2)) + [\psi^{(+)}(x_1), \bar{\psi}^{(-)}(x_2)]_+ , \tag{160}\]

where the last term is the anticommutator of \( \psi^{(+)}(x_1) \) and \( \psi^{(-)}(x_2) \).
For the chronological product of the two Bose operators, from Eq. (159) we obtain

\[
P(\phi(x_1)\phi(x_2)) = N(\phi(x_1)\phi(x_2)) + [\phi^{(+)}(x_1), \phi^{(-)}(x_2)]_+, \quad x_{10} > x_{20}
\]
\[
= N(\phi(x_2)\phi(x_1)) + [\phi^{(+)}(x_2), \phi^{(-)}(x_2)]_+, \quad x_{20} > x_{10}
\]
\[
= N(\phi(x_1)\phi(x_2)) + \phi(x_1)\phi(x_2),
\]

(161)

where

\[
\dot{\phi}(x_1)\dot{\phi}(x_2) = [\phi^{(+)}(x_1), \phi^{(-)}(x_2)]_+, \quad x_{10} > x_{20}
\]
\[
= [\phi^{(+)}(x_2), \phi^{(-)}(x_2)]_+, \quad x_{20} > x_{10}
\]

(162)

is the contraction of operators \( \phi(x_1) \) and \( \phi(x_2) \). We will show that the contraction is a function of \( x_1 - x_2 \). This function is also called the propagation function or propagator.

The \( P \)-product of two Fermi fields is equal

\[
P(\psi(x_1)\overline{\psi}(x_2)) = N(\psi(x_1)\overline{\psi}(x_2)) + [\psi^{(+)}(x_1), \overline{\psi}^{(-)}(x_2)]_+, \quad x_{10} > x_{20}
\]
\[
= N(\overline{\psi}(x_2)\psi(x_1)) + [\overline{\psi}^{(+)}(x_2), \psi^{(-)}(x_1)]_+, \quad x_{20} > x_{10}.
\]

(163)

For Fermi fields we have

\[
N(\overline{\psi}(x_2)\psi(x_1)) = -N(\psi(x_1)\overline{\psi}(x_2)).
\]

(164)

From Eqs. (163) and (164) it follows that the chronological \( P \)-product of Fermi fields is not expressed through normal products.

Let us now introduce the Wick chronological \( T \)-products as follows:

\[
T(\psi(x_1)\overline{\psi}(x_2)) = \psi(x_1)\overline{\psi}(x_2), \quad x_{10} > x_{20}
\]
\[
= -\overline{\psi}(x_2)\psi(x_1), \quad x_{20} > x_{10}.
\]

(165)

For Bose fields \( T(\phi(x_1)\phi(x_2)) \equiv P(\phi(x_1)\phi(x_2)) \).

In general the \( T \)-operator arranges field operators in chronological order (time arguments decrease when we move from the left to the right) and multiplies the chronologically ordered product by \((-1)\) in the degree of the number of permutations of Fermi fields.

With the help of Eqs. (160) and (165) we obtain

\[
T(\psi(x_1),\overline{\psi}(x_2)) = N(\psi(x_1)\overline{\psi}(x_2)) + \dot{\psi}(x_1)\dot{\overline{\psi}}(x_2),
\]

(166)

where

\[
\dot{\psi}(x_1)\dot{\overline{\psi}}(x_2) = [\psi^{(+)}(x_1), \overline{\psi}^{(-)}(x_2)]_+, \quad x_{10} > x_{20}
\]
\[
= -[\overline{\psi}^{(+)}(x_2), \psi^{(-)}(x_1)]_+, \quad x_{20} > x_{10}.
\]

(167)

Our final aim is to calculate the matrix elements of the \( S \)-matrix. The \( S \)-matrix is given by the expression (46) where there are \( P \)-products. It is easy to see that in the expression for the \( S \)-matrix \( P \)-products can be replaced by \( T \)-products. In fact, in the interaction Hamiltonian an even number of Fermi fields are present and any permutation of Hamiltonians is an even permutation of Fermi fields.
For the $S$-matrix we finally have the following expression:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dx_1 \ldots \int dx_n \, T(\mathcal{H}_1(x_1)\mathcal{H}_1(x_2)\ldots\mathcal{H}_1(x_n)) \, .$$  \hspace{1cm} (168)

We have proved that for the product of any two fields (Bose or Fermi)

$$T(UV) = N(UV) + \hat{U}\hat{V} \, .$$  \hspace{1cm} (169)

Now let us formulate the Wick theorem, which can be proved by induction. For the product of any $n$ field operators we have

$$T(UVWR\ldots XYZ) = N(UVWR\ldots XYZ) +$$

$$+ N(\hat{U}\hat{V}\hat{W}\hat{R}\ldots XYZ) + N(\hat{U}\hat{V}\hat{W}\hat{R}\ldots XYZ) + \ldots$$  \hspace{1cm} (170)

$$+ N(\hat{U}\hat{V}\hat{W}\hat{R}\ldots XYZ) + \ldots \, .$$

The normal product with the internal contractions of operators is determined as follows:

$$N(\hat{U}\hat{V}\hat{W}\hat{R}\ldots XYZ) = \delta U\hat{W}\hat{R}\hat{X} \ldots N(V\ldots YZ) \, ,$$  \hspace{1cm} (171)

where $\delta = 1(\delta = -1)$ in the case of an even (odd) number of permutations of Fermi operators.

In the first line of Eq. (170) there is a normal product of operators, in the second line there is the sum of all possible normal products with one contraction inside them, in the third line there is the sum of all possible normal products with two contractions and so on. The interaction Hamiltonian is the normal product of the operators with the same $x$. So in the $S$-matrix the $T$-product of the normal products of operators occurs. The Wick theorem can easily be generalized for this case. It is possible to show that operators that occur in the same normal product (with the same $x$) must not be contracted.

Let us now calculate the propagator of the scalar Hermitian field. From Eqs. (124) and (125) for the commutator we have

$$[\phi^{(+)}(x_1), \phi^{(-)}(x_2)] = \int N_\gamma N_\rho e^{-iqx_1 + iq'x_2} [a(q), a^{+}(q')] \, d^3q \, d^3q' \, .$$

$$= \frac{1}{(2\pi)^3} \int \frac{1}{2q^0} e^{-iq(x_1-x_2)} \, d^3q \, .$$  \hspace{1cm} (172)

Thus the propagator is equal to

$$\phi(x_1)\phi(x_2) = \frac{1}{(2\pi)^3} \int \frac{e^{-iq(x_1-x_2)}}{2q^0} \, d^3q \, , \quad x_{10} > x_{20} \, .$$

$$= \frac{1}{(2\pi)^3} \int \frac{e^{-iq(x_2-x_1)}}{2q^0} \, d^3q \, , \quad x_{20} > x_{10} \, .$$  \hspace{1cm} (173)

These two 3-dimensional integrals can be written as one four-dimensional integral.

Let us consider

$$\Delta(x) = \frac{i}{(2\pi)^4} \lim_{\epsilon \rightarrow 0} \int \frac{e^{-iqx}}{q^2 - m^2 + i\epsilon} \, ,$$  \hspace{1cm} (174)
where $q^0$ is a variable of the integration. The poles of $1/(q^2 - m^2 + i\epsilon)$ are in the points

$$q_{1,2}^0 = \pm \left( \sqrt{m^2 + q^2} - \frac{i\epsilon}{2\sqrt{q^2 + m^2}} \right).$$

Let us consider the case $x^0 > 0$. It is clear that the integral over $q^0$ from $-\infty$ to $\infty$ is equal to the integral over the contour that is depicted in Fig. 2 at $R \to \infty$ (the integral over the semicircle disappears owing to $e^{-i\phi(\omega^2)}$).

![Fig. 2](image)

Calculating the residue we easily find

$$\Delta(x) = \frac{1}{(2\pi)^3} \int \frac{e^{-i\sqrt{m^2 + q^2} x^0 + i\phi(x)}}{2\sqrt{m^2 + q^2}} d^3 q, \quad x^0 > 0.$$  

In the case where $x^0 < 0$ the integral over $q^0$ can be changed by the integral over the contour depicted in Fig. 3 at $R \to \infty$.

![Fig. 3](image)

We have

$$\Delta(x) = \frac{1}{(2\pi)^3} \int \frac{e^{i\sqrt{m^2 + q^2} x^0 - i\phi(x)}}{2\sqrt{m^2 + q^2}} d^3 q, \quad x^0 < 0.$$  

Thus

$$\phi(x_1)\phi(x_2) = \frac{i}{(2\pi)^4} \lim_{\epsilon \to 0} \int \frac{e^{-i\phi(x_1-x_2)}}{q^2 - m^2 + i\epsilon} d^4 q. \quad (175)$$

In the case of Fermi fields it is easy to show that

$$\psi(x_1)\bar{\psi}(x_2) = \frac{i}{(2\pi)^4} \lim_{\epsilon \to 0} \int \frac{(\not{p} + m)}{(p^2 - m^2 + i\epsilon)} e^{-ip(x_1-x_2)} d^4 p = \frac{i}{(2\pi)^4} \int \frac{e^{-ip(x_1-x_2)}}{\not{p} - m} d^4 q. \quad (176)$$
8. Some examples of Feynman diagrams

Now let us calculate the matrix elements of some processes. Let us consider, as a first example, the scattering of $\gamma$-quanta on electrons—**Compton effect**:

$$\gamma + e \rightarrow \gamma + e.$$  \hspace{1cm} (177)

Let us write down the $S$-matrix in the form

$$S = \sum_n S^{(n)}$$  \hspace{1cm} (178)

where

$$S^{(n)} = \frac{(-i)^n}{n!} \int dx_1 \int dx_2 \cdots \int dx_n \, T(\mathcal{H}_1(x_1)\mathcal{H}_1(x_2)\cdots\mathcal{H}_1(x_n)).$$  \hspace{1cm} (179)

We will take into account only the electromagnetic interaction

$$\mathcal{H}_1 = e\bar{\gamma}^\alpha \gamma^\alpha A_\alpha.$$  \hspace{1cm} (180)

It is clear that in the lowest order of perturbation theory the non-zero contribution to the matrix element of the process gives the operator $S^{(2)}(2\gamma's)$. So let us calculate

$$\langle f \mid S^{(2)} \mid i \rangle,$$  \hspace{1cm} (181)

where

$$|i\rangle = c^\dagger(p)a^\dagger(k)|0\rangle,$$

$$|f\rangle = \langle 0 \mid a(k')c(p') \rangle.$$  \hspace{1cm} (182)

$k, p(k', p')$ being the momenta of the initial (final) photon and electron (spin indexes are suppressed).

In accordance with the definition of the operators of the normal product $N$ it is natural to act by $(+)$ operators on initial ket-vectors and by $(-)$ operators on final bra-vectors.

We have one electron in the initial state and one electron in the final state. Non-zero contributions to the matrix element (181) can only give normal products with one contraction of electron fields. There are two normal products of this type and it easy to see that they give the same contribution to the $S$-matrix. We have four terms in the case of two non-contracted electron operators $(++,-,-,+-,-,-)$. It is clear that only one term gives a non-zero contribution to the matrix element of the process considered:

$$T(\bar{e}(x_1)\gamma^\alpha e(x_1)\bar{e}(x_2)\gamma^\beta e(x_2)) \rightarrow \bar{e}^{(-)}(x_1)\gamma^\alpha \gamma^\beta e^{(+)}(x_2).$$  \hspace{1cm} (183)

Now let us act by the operator $c^{(+)}(x_2)$ on the initial state. We have

$$c^{(+)}(x_2)c^\dagger(p)a^\dagger(k)|0\rangle = \langle [e^{(+)}(x_2), a^\dagger(p)]_+ - c^\dagger(p)e^{(+)}(x_2)a^\dagger(k)|0\rangle$$

$$= N_{\mu \nu}(p)e^{-ipx_2}a^\dagger(k)|0\rangle.$$  \hspace{1cm} (184)

Analogously we have

$$\langle 0 \mid a(k')c(p')e^{(-)}(x_1) = \langle 0 \mid a(k')N_{\mu \nu}(p')e^{ipx_1},$$  \hspace{1cm} (185)
where we have used

$$
\langle 0 | c^\dagger(p) = 0 .
$$

Now let us consider the 'photon part' of our matrix element. It is clear that only products of (+) and (−) operators of photon fields give a non-zero contribution to the matrix element.

We have

$$
T(A_\alpha(x_1)A_\beta(x_2)) \rightarrow A_\alpha^{(-)}(x_1)A_\beta^{(+)}(x_2) + A_\beta^{(-)}(x_2)A_\alpha^{(+)}(x_1)
$$

$$
\langle 0 | A^{(+)}(x_2)a^{\dagger}(k)|0\rangle = N_k\epsilon_\beta(k)e^{-ikx_2}|0\rangle , \tag{186}
$$

$$
\langle 0 | A^{(-)}(x_1)A^{(-)} \rangle = \langle 0 | N_k\epsilon_\alpha(k')e^{ikx_1} .
$$

For the matrix element of the process, using expressions (183)–(186) after integration over \(x_1\) and \(x_2\) we easily obtain

$$
\langle f | S^{(2)} | i \rangle = (-i)^2 \int \frac{1}{(2\pi)^4 \not{p}_1 - m} \frac{1}{(2\pi)^4 \not{p}_1 - m} \not{p}_1 - m e^{2i\gamma}(2\pi)^4 \delta(p' + k' - p_1) N_p N_k \epsilon_\alpha(k') N_k \epsilon_\beta(k) + (k' \leftrightarrow -k') d^4 p_1 . \tag{187}
$$

These two terms of the matrix element of the Compton effect correspond to the Feynman diagrams shown in Fig. 4.

![Feynman diagrams](image_url)

Fig. 4: The Feynman diagrams of the process \(\gamma + e \rightarrow \gamma + e\)

The transition of the initial state into the final state is due to emission and absorption of \(\gamma\)-quanta. There are only two possibilities: the initial electron absorbs the initial photon (emits the final photon) and turns into a virtual state; the virtual electron emits the final photon (absorbs the initial photon) and turns into the final state. Both these possibilities are shown diagrammatically in Fig. 4.

Our next example will be the process of annihilation of \(e^-\) and \(e^+\) into two photons. The interaction Hamiltonian is given by Eq. (180). We will calculate

$$
\langle f | S^{(2)} | i \rangle ,
$$

where

$$
|i\rangle = c^{\dagger}(p)d^{\dagger}(p')|0\rangle ,
$$

$$
\langle f | = \langle 0 | a(k) a(k') .
$$
$p, p'$ are the momenta of the initial electron and positron; $k, k'$ are the momenta of the final photons; and the operator $S$ is given by Eq. (152). It is clear that a non-zero result gives the following operator

$$T(\bar{e}(x_1)\gamma^\alpha e(x_1)\bar{e}(x_2)\gamma^\beta e(x_2)) \rightarrow \bar{e}^{(+)}(x_1)\gamma^\alpha \bar{e}(x_1)\bar{e}(x_2)\gamma^\beta e^{(+)}(x_2).$$

Let the operator $\bar{e}^{(+)}(x_1)e^{(+)}(x_2)$ act on the initial state. We have

$$\bar{e}^{(+)}(x_1)e^{(+)}(x_2)c^\dagger(p)d^\dagger(p')|0\rangle = N_p\bar{u}(-p')e^{i(-p')x_1}N_pu(p)e^{-ipx_2}|0\rangle.$$  \hfill (188)

Let us now consider 'the photonic part' of the operator $S^{(2)}$. It is clear that a non-zero contribution gives only the operator

$$T(A_\alpha(x_1)A_\beta(x_2)) \rightarrow A^{(-)}_\alpha(x_1)A^{(-)}_\beta(x_2).$$

Further, we have

$$\langle 0|a(k)a(k')A^{(-)}_\alpha(x_1)A^{(-)}_\beta(x_2) = \langle 0|(N_ke_\alpha(k')e^{ik'x_1}N_ke_\beta(k)e^{ikx_2} + (k \leftrightarrow k')).$$  \hfill (189)

Finally with the help of expressions (188) and (189) for the matrix element of the process we obtain

$$\langle f|S^{(2)}|i\rangle = (-i)^2 \int N_p\bar{u}(-p')e\gamma^\alpha(2\pi)^4\delta(-p' - p_1 + k)$$

$$\times \frac{i}{(2\pi)^4} \frac{1}{\not{p}_1 - m} e\gamma^\beta(2\pi)^4\delta(p_1 + k' - p)u(p)N_pN_ke_\alpha(k)N_ke_\beta(k')d^4p_1 + (k \leftrightarrow k').$$  \hfill (190)

The two diagrams of Fig. 5 correspond to the matrix element of the process under consideration.

![Diagrams](image)

Fig. 5: Diagrams of the process $e^- + e^+ \rightarrow \gamma + \gamma$

New elements of these diagrams are outgoing lines with momenta $-p'$. These lines correspond to the ingoing positron. The diagrams of Fig. 5 show the following possibilities of transitions from the initial state into the final one: an electron with momentum $p$ emits a photon with momentum $k'$ ($k$) and turns into an electron in the virtual state with momentum $p_1$; a virtual electron with momentum $p$ annihilates with a positron with momentum $p'$ and emits a photon with momentum $k$ ($k'$).

Let us now consider our last example: the process $e^- + e^+ \rightarrow \mu^- + \mu^+$. The Hamiltonian is given by

$$\mathcal{H}_I = e\bar{e}\gamma^\alpha eA_\alpha + \bar{e}\gamma^\alpha\mu A_\alpha.$$

We will calculate

$$\langle f|S^{(2)}|i\rangle.$$
where

\[ |i\rangle = c_1(p_1) d_2(p_2) |0\rangle , \]

\[ \langle f | = \langle 0| d_\mu(p_2') c_\nu(p_1') . \]

\( p_1, p_2 \) are the momenta of the initial electron and positron and \( p_1', p_2' \) are the momenta of the final \( \mu^- \) and \( \mu^+ \). It is clear that a non-zero result gives only the operator

\[ T(\mu(x_1) \gamma^\alpha \mu(x_1) \bar{e}(x_2) \gamma^\beta e(x_2)) \rightarrow \mu^-(x_2) \gamma^\alpha \mu^-(x_1) \bar{e}^+(x_1) \gamma^\beta e^+(x_2) . \]

There are no photons in the initial and final states. Thus electromagnetic operators must be contracted as follows:

\[ \hat{A}_\alpha(x_1) \hat{A}_\beta(x_2) = \frac{-i}{(2\pi)^3 g_{\alpha\beta}} \int \frac{e^{-ik(x_1-x_2)}}{k^2} \, d^4k . \]

For the matrix element of the process we easily obtain

\[ \langle f | S^{(2)} | i \rangle = (-i)^2 \int N_{p_1'} \bar{u}(p') e_{\gamma^\alpha}(2\pi)^4 \delta(p_1' + p_2' - k) u(-p_2') N_{p_2} \]

\[ \times \frac{-i}{(2\pi)^3 g_{\alpha\beta}} \frac{1}{k^2} N_{p_1} \bar{u}(-p_2) e_{\gamma^\alpha}(2\pi)^4 \delta(k - p_1 - p_2) u(p_1) N_{p_1} \, d^4k . \]  

(191)

Figure 6 shows a diagram of the matrix element (191).

Fig. 6: The diagram of the process \( e^- + e^+ \rightarrow \mu^- + \mu^+ \).

A new element of this diagram is the line of the virtual photon. The diagram of Fig. 6 corresponds to the following possibility: an electron annihilates with a positron and produces a virtual photon; a virtual photon produces a pair \( \mu^- \mu^+ \).

After the examples that we have considered it is not difficult to formulate **Feynman rules:**

i) To the initial (final) fermion corresponds

\[ \begin{array}{c}
  \hline
  p \\
  \hline
\end{array} \]

\[ N \mu u(p) (N \bar{\nu} \bar{u}(p')) . \]

ii) To the propagator of a spin-1/2 particle corresponds

\[ \begin{array}{c}
  \hline
  p \\
  \hline
\end{array} \]

\[ \frac{i}{(2\pi)^4} \frac{1}{p^2 - m} . \]

iii) To the photon propagator corresponds

\[ \begin{array}{c}
  \hline
  k \\
  \hline
\end{array} \]

\[ \frac{-i}{(2\pi)^4} \frac{g_{\alpha\beta}}{k^2} . \]
iv) To a vector boson propagator corresponds

\[
\frac{-i}{q} \frac{g_{\alpha\beta} - (g_{\alpha}q_{\beta}/m_{V}^2)}{(2\pi)^4} \frac{1}{q^2 - m_{V}^2} .
\]

v) To the electromagnetic vertex corresponds

\[e\gamma^\alpha (2\pi)^4 \delta(p' + k - p) .\]

vi) To the W-boson vertex corresponds

\[
\frac{g}{2\sqrt{2}} \gamma^\alpha (1 - \gamma_5) (2\pi)^4 \delta(p' + q - p) .
\]

vii) To the Z-boson vertex corresponds

\[
\frac{g}{2\cos\theta_W} \gamma^\alpha (g_{\nu} - g_{A}\gamma_5) (2\pi)^4 \delta(p' + q - p)
\]

\[g_{\nu} = \frac{1}{2} + 2\sin^2\theta_W ,
\]

\[g_{A} = \frac{1}{2} .
\]

With the help of the Dyson–Wick technique that we considered before it is not difficult to obtain Feynman rules for any interaction. To summarize again, Feynman diagrams are an easy and clear perturbative way to take into account all possible transitions (due to interactions) from the initial state into the final state.

9. Methods of calculation of cross-sections and transition probabilities

Thus Feynman diagrams allow us to calculate the matrix elements of the processes. In this section we will briefly recall the methods of calculation of observable quantities: cross-sections, decay probabilities, and so on.
Let us consider a process with the momenta of the initial particles being \( p_1, p_2 \) and the momenta of the final particles being \( p'_1, p'_2 \), etc.

The matrix element of the process is given by

\[
\langle f | (S - 1) | i \rangle = \langle f | R | i \rangle (2\pi)^4 \delta^4(P' - P),
\]

where \( P \) and \( P' \) are the total initial and final momenta. For the probability of the transition per unit of volume and unit of time we have

\[
dw_{fi} = |\langle f | R | i \rangle|^2 (2\pi)^4 \delta^4(P' - P) \lim_{V \to \infty} \frac{1}{VT} \frac{1}{(2\pi)^4} \int e^{(P' - P)z} \, dz \, d\Gamma
\]

\[
= |\langle f | R | i \rangle|^2 (2\pi)^4 \delta^4(P' - P) \, d\Gamma,
\]

where \( d\Gamma = d^3 p' d^3 p' \ldots \).

Now consider in the laboratory system \((\vec{p}_2 = 0)\) the volume depicted in Fig. 7.

![Fig. 7: The elementary volume in the laboratory system](image)

The differential cross-section is determined in the laboratory system as follows:

\[
\rho^0_2(\Delta x \cdot 1) \, d\sigma_{fi} = \rho^0_i v^0_i \, dw_{fi} \, (\Delta x \cdot 1).
\]

Here \( \rho^0_1 \) and \( \rho^0_2 \) are the densities of the particles with momentum \( p_1 \) and \( p_2 \), and \( v^0_i \) is the velocity in the laboratory system. For the cross-section we have the following expression:

\[
d\sigma_{fi} = \frac{dw_{fi}}{j}.
\]

where the current \( j \) is given by

\[
j = \rho^0_1 \rho^0_2 v^0_1.
\]

The current \( j \) can be written in the covariant form. In fact, we have

\[
\rho^0_1 \rho^0_2 = j_1 j_2 = \rho_1 \rho_2 \frac{p_1 p_2}{p'_1 p'_2},
\]

where \( j \) is the 4-vector of current \( j_i = (\rho_i, \rho_i \vec{v}_i) \). For the velocity of the particle 1 in the laboratory system we have

\[
\vec{v}^0_1 = \frac{p^0_1}{E^0_1} = \frac{p^0_1 m_2}{E^0_1 m_2} = \frac{\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}}{p_1 p_2},
\]

where \( m_1 \) and \( m_2 \) are the masses of particles 1 and 2.
From Eqs. (194)–(196) for the flux in the covariant form (Möller flux) we obtain the following expression:

\[ j = \rho_1 \rho_2 \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1 \cdot p_2} . \]  

(197)

Finally, for the cross-section of the process we have the following expression:

\[ d\sigma_{Ri} = \frac{1}{j} |\langle f|R|i\rangle|^2 (2\pi)^4 \delta(P' - P) \, d\Gamma . \]  

(198)

Assume now that particles with momenta \( p_1 \) and \( p'_1 \) are particles with spin 1/2. In this case the matrix element of the process has the following general form:

\[ \langle f|R|i\rangle = \bar{u}'(p'_1)M u'(p_1) , \]

where \( M \) is a \( 4 \times 4 \) matrix. For the modulus squared of the matrix element in the expression (198) for the cross-section we have

\[ |\langle f|R|i\rangle|^2 = \sum_{r',r} (\bar{u}'(p'_1)M u'(p_1))\rho_r(\bar{u}'(p_1)\bar{M} u'(p'_1)) , \]

(199)

where \( \rho_r \) is the probability of finding the initial particles 1 in the state \( u'(p_1) \) and \( \bar{M} = \gamma^0 M^+ \gamma^0 \). From Eq. (199) we will obtain

\[ |\langle f|R|i\rangle|^2 = \text{Tr} \, M \rho(p_1) \bar{M} \Lambda(p'_1) . \]

(200)

Here

\[ \Lambda(p'_1) = \sum_r u'(p'_1)\bar{u}'(p'_1) = \gamma'_1 + m , \]

and

\[ \rho(p_1) = \sum_r u'(p_1)\bar{u}'(p_1)\rho_r \]

is the spin density matrix of the initial particles 1. The matrix \( \rho \) has the following general form:

\[ \rho(p_1) = \frac{1}{2}(1 + \gamma_5 \xi^\alpha)(\gamma^\alpha_1 + m) , \]

where \( \xi^\alpha \) is a 4-vector of polarization. The vector \( \xi \) is a space-like vector orthogonal to \( p \). We have \( \xi^2 = -P^2 \) where \( P \) is polarization. In the case where \( P = 0 \) the beam is unpolarized, in the case where \( P = 1 \) the beam is totally polarized.

As we see from Eq. (200), the calculation of the cross-section in the case of particles with spin requires the calculation of traces. We will recall now some simple rules of the calculation of traces of products of Dirac matrices.

These rules are based on the commutation relations:

\[ \gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2g^{\alpha\beta} , \]

\[ \gamma^\alpha \gamma_5 + \gamma_5 \gamma^\alpha = 0 , \quad \gamma_5^2 = 1 \]  

(201)

and the following property of traces:

\[ \text{Tr} \, AB = \sum_{\sigma,\sigma'} A_{\sigma\sigma'} B_{\sigma'\sigma} = \sum_{\sigma,\sigma'} B_{\sigma'\sigma} A_{\sigma\sigma'} = \text{Tr} \, BA . \]
i) The trace of the product of an odd number of Dirac matrices is equal to zero:
\[
\text{Tr} \gamma^\alpha \gamma^\beta \ldots \gamma^\sigma = \text{Tr} \gamma^\alpha \gamma^\beta \ldots \gamma^\sigma \gamma_5 \gamma_5 = -\text{Tr} \gamma_5 \gamma^\alpha \gamma^\beta \ldots \gamma^\sigma \gamma_5 = -\text{Tr} \gamma^\alpha \gamma^\beta \ldots \gamma^\sigma = 0 .
\] (202)

ii) For the product of two matrices we have
\[
\text{Tr} \gamma^\alpha \gamma^\beta = 4\epsilon^{\alpha\beta} .
\] (203)

In fact, we have
\[
\text{Tr} \gamma^\alpha \gamma^\beta = \text{Tr} (2\epsilon^{\alpha\beta} - \gamma^\beta \gamma^\alpha) = 8\epsilon^{\alpha\beta} = \text{Tr} \gamma^\alpha \gamma^\beta .
\] (204)

iii) For the product of four \(\gamma\)-matrices we have
\[
\text{Tr} \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\sigma = 4\epsilon^{\alpha\beta\mu\sigma} - 4\epsilon^{\alpha\mu\beta\sigma} + 4\epsilon^{\alpha\sigma\mu\beta} .
\] (205)

iv) From Eq. (205) it follows that
\[
\text{Tr} \gamma_5 = 0 .
\] (206)

v) Furthermore, we have
\[
\text{Tr} \gamma_5 \gamma^\alpha \gamma^\beta = 0 .
\] (207)

If \(\alpha = \beta\) the trace is equal to zero owing to Eq. (206). If \(\alpha \neq \beta\), the trace is equal to zero owing to Eq. (203).

vi) Finally we have
\[
\text{Tr} \gamma_5 \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\sigma = 4i \epsilon^{\alpha\beta\mu\sigma} ,
\] 
\((\epsilon^{0123} = -1, \ldots)
\] (208)

Now we will apply the technique we have presented and calculate the cross-sections of some processes.

10. Scattering of the neutrino (antineutrino) on the electron

Let us calculate the cross-sections of the processes
\[
\nu_e(\bar{\nu}_e) + e \rightarrow \nu_e(\bar{\nu}_e) + e .
\]

These processes are due to W and Z exchange. The Hamiltonian of the Standard Model has the following form:
\[
\mathcal{H}_I = \frac{g}{2\sqrt{2}} \bar{\nu}_e \gamma^\alpha (1 - \gamma_5) e W_\alpha + \text{h.c.} + \frac{g}{2 \cos \theta_W} \left[ \frac{1}{2} \bar{\nu}_e \gamma^\alpha (1 - \gamma_5) \nu_e Z_\alpha + \bar{e} \gamma^\nu (g_V - g_A \gamma_5) e Z_\alpha \right] ,
\] (209)

where \(g_V = -\frac{1}{2} + 2 \sin^2 \theta_W; g_A = -\frac{1}{2}\). The lowest order diagrams are shown in Fig. 8.
Fig. 8: Diagrams of the process $\nu_e + e \rightarrow \nu_e + e$

From Feynman rules it follows that the matrix element of the process $\nu_\nu e \rightarrow \nu_\nu e$ is equal to

$$
\langle f|S|i \rangle = -\frac{iG_F}{\sqrt{2}} N\bar{u}(k')\gamma^\alpha(1 - \gamma_5)u(k)\bar{u}(p')\gamma_\alpha(g_{\nu} - g_A\gamma_5)u(p)
- \bar{u}(p')\gamma^\alpha(1 - \gamma_5)u(k)\bar{t}(k')\gamma_\alpha(1 - \gamma_5)u(p))(2\pi)^4\delta(P' - P),
$$
\[ (210) \]

where

$$
\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{g^2}{8m_W^2 \cos^2 \theta_W}
$$

and $k, p$ ($k', p'$) are the momenta of the initial (final) neutrino and electron and $P = k + p$ ($P' = k' + p'$). We have assumed that $q^2 \ll m_W^2$. Let us note that the minus sign in Eq. (210) follows from normal products.

Both terms in the expression (210) can be combined into one. In fact, using the Fierz transformation, we have

$$
\bar{u}(p')\gamma^\alpha(1 - \gamma_5)u(k)\bar{u}(k')\gamma_\alpha(1 - \gamma_5)u(p) = -\bar{u}(k')\gamma^\alpha(1 - \gamma_5)u(k)\bar{u}(p')\gamma_\alpha(1 - \gamma_5)u(p). 
$$
\[ (211) \]

From Eqs. (210) and (211) it follows that

$$
\langle f|S|i \rangle = -\frac{iG_F}{\sqrt{2}} N\bar{u}(k')\gamma^\alpha(1 - \gamma_5)u(k)\bar{u}(p')\gamma_\alpha(g_{\nu} - g_A\gamma_5)u(p)(2\pi)^4\delta(P' - P),
$$
\[ (212) \]

where $g_{\nu} = g_\nu + 1, g_A = g_A + 1$. With the help of Eq. (211) for the cross-section of the process, we will obtain the following expression:

$$
\frac{d\sigma}{d^3k'd^3p'} = \frac{1}{2} \frac{G_F^2}{j} N^2 \text{Tr} \gamma^\alpha(1 - \gamma_5) j^\beta(1 - \gamma_5) j'
\times \frac{1}{2} \text{Tr} \left[\gamma^\alpha(1 - \gamma_5) j^\beta(1 - \gamma_5) j'\right]
+ g_{\nu}^2 \gamma_\alpha(1 + \gamma_5) j^\beta(1 + \gamma_5) j' + 4g_{\nu,\gamma_5} m^2 \gamma_\alpha\gamma_\beta(2\pi)^4\delta(p' - p - q) d^3k' d^3p'.
$$
\[ (213) \]

Here

$$
g_{\nu} = \frac{1}{2}(g_{\nu} + g_A), \quad g_{\nu,\gamma_5} = \frac{1}{2}(g_{\nu} - g_A), \quad q = k - k' \quad \text{and} \quad j = \frac{1}{(2\pi)^6} \frac{pk}{p^0k^0}.
$$

The calculations of the cross-section are simplified if the following matrix relations are used:

$$
\gamma^\alpha\gamma^\rho\gamma^\beta(1 - \gamma_5) = 4\delta_3(1 - \gamma_5) \quad \gamma_\alpha\gamma_\beta(1 - \gamma_5) = 4\delta_3(1 - \gamma_5),
\gamma_\alpha\gamma_\beta(1 + \gamma_5) = 4\gamma_\gamma(1 - \gamma_5) \quad \gamma^\rho(1 + \gamma_5).
$$
\[ (214) \]
These relations can easily be obtained with the help of the relation
\[ \gamma^\alpha \gamma^\nu \gamma^\beta = g^\alpha\nu \gamma^\beta - g^\alpha\beta \gamma^\nu + g^\nu\beta \gamma^\alpha - i \epsilon^{\alpha\beta\nu}\gamma_\gamma \gamma_\sigma . \] (215)

For the product of two traces in the expression (208) summed over \( \alpha \) and \( \beta \) we easily obtain the following expression:
\[ \text{Tr} \times \text{Tr} = (16)^2 [g_{L}^{2} p \cdot k p' \cdot k' + g_{R}^{2} p \cdot k p' \cdot k' - m^{2} g_{L} g_{R} k \cdot k'] . \] (216)

Let us note that from the conservation of 4-momentum it follows that \( p \cdot k = p' \cdot k' \), \( p \cdot k' = p' \cdot k \).

Now let us choose as an independent variable the energy of the neutrino in the laboratory system \( E = p k / m \) and a dimensionless variable \( y = p k / p k = T / E \) (\( T \) is the kinetic energy of the electron in the laboratory system). From the conservation of energy and momentum it follows that
\[ 0 \leq y \leq \frac{1}{1 + m / 2E} . \]

Further, taking into account that
\[ \int \delta(p' - p - q) \frac{d^3 p'}{p'^0} \frac{d^3 k'}{k'^0} = 2 \int \delta(p' - p - q) \delta(p'^2 - m^2) \frac{d^3 p'}{p'^0} \frac{d^3 k'}{k'^0} = 2 \int \delta(2pq + q^2) \frac{d^3 k'}{k'^0} , \quad \frac{d^3 k'}{k'^0} = \pi dq^2 dy , \]
for the cross-section of the process \( \nu_{e} e \rightarrow \nu_{e} e \), we obtain the following expression:
\[ \left( \frac{d \sigma}{dy} \right)_{\nu_{e} e} = 2 \frac{G_{F}^2}{\pi} mE \left( g_{L}^2 + g_{R}^2 (1 - y)^2 - g_{L} g_{R} y \frac{m}{E} \right) . \] (217)

Diagrams of the process \( \bar{\nu}_{e} e \rightarrow \bar{\nu}_{e} e \) are shown in Fig. 9

![Diagrams](image)

Fig. 9: Diagrams of the process \( \bar{\nu}_{e} e \rightarrow \bar{\nu}_{e} e \)

The matrix element of the process is given by
\[ \langle f | S | i \rangle = i \frac{G_{F}}{\sqrt{2}} \bar{N} \bar{\nu} (-k) \gamma^\alpha (1 - \gamma_\nu) u (-k') \]
\[ \times \bar{u} (p') \gamma^\nu \left( g_{L}^2 - g_{R}^2 \gamma_\gamma \right) u (p) (2\pi)^4 \delta(P' - P) , \]
where \( k, p \ (k', p') \) are the momenta of the initial (final) antineutrino and the electron.
It is clear that the product of traces is given by expression (216), in which it is necessary to replace \( k \) by \( k' \) and vice versa. For the cross-section of the process \( \bar{\nu}_e e \rightarrow \bar{\nu}_e e \) we obtain the following expression:

\[
\left( \frac{d\sigma}{dy} \right)_{\bar{\nu}_e e} = 2 \frac{G_F^2}{\pi} m E \left[ g_R^2 + g_L^2 (1 - y)^2 - g_L g_R \frac{m}{E} \right].
\]  \hfill (218)

Let us now consider the expressions (217) and (218). When \( E \gg m \), only the first two terms remain. It is easy to see that the \( y \)-dependence of cross-sections is due to helicity conservation. In fact, we consider that part of the matrix element of \( \nu_e e \) scattering that is multiplied by \( g_R \). In Fig. 10 the momenta and helicities of the neutrino and the electron in the centre-of-mass system before and after backward scattering are depicted. It is clear, from Fig. 10, that scattering at \( \theta = \pi \) is forbidden by the conservation of the projection of total momenta. This corresponds to the factor \((1 - y)^2\) in the expression (217) for the cross-section \((\theta = \pi \text{ corresponds to } y = 1)\). The helicity of the electron in that part of the matrix element that is multiplied by \( g_L \) is equal to \(-1\) and the projection of the total momenta is equal to zero (no \( y \)-dependence). Note also that owing to helicity conservation there is no \( g_L g_R \) term in the cross-section in the case \( E \gg m \).

\[
\theta = 0 \quad \begin{array}{c} \leftarrow \rightarrow \end{array} \quad \theta = \pi \quad \begin{array}{c} \leftarrow \rightarrow \end{array}
\]

\( \nu \quad \begin{array}{c} \rightarrow \quad \leftarrow \end{array} \quad e \)

Fig. 10: Momenta and helicities of \( \nu_e \) and \( e \) \((g_R \text{ part})\)
before scattering and after backward scattering

11. **The decay of the Z-boson into a neutrino–antineutrino pair**

Let us now calculate the width of the decay

\[ Z \rightarrow \nu + \bar{\nu}. \]

The Hamiltonian of the Standard Model has the form

\[ \mathcal{H}_f = \frac{g}{2 \cos \theta_W} \sum_{\ell = e, \mu, \tau} \bar{\nu}_L \gamma^\nu \nu_L Z_\nu. \]

The diagram of the process is presented in Fig. 11.

\[
\begin{array}{c}
q \\
\downarrow \\
k \\
\downarrow \\
k' \\
\end{array}
\]

Fig. 11: Feynman diagram of the process \( Z \rightarrow \nu + \bar{\nu} \)
The matrix of the process is equal to

\[
\langle f | S | i \rangle = -i \frac{g}{2 \cos \theta_W} N \frac{1}{2} \bar{u}(k) \gamma^\alpha (1 - \gamma_5) u(-k') e_\alpha(q) \times (2\pi)^4 \delta(k + k' - q) .
\]

(219)

Here \( k \) and \( k' \) are the momenta of the neutrino and antineutrino, \( q \) is the momentum of the Z-boson and \( e_\alpha(q) \) is the vector of polarization of Z.

The differential decay width \( d\Gamma_{fi} \) (probability of decay per unit time and per particle) is determined in the rest frame of the decaying particle as follows:

\[
d\Gamma_{fi} = \frac{d\omega_{fi}}{\rho_i} .
\]

(220)

Here \( \rho_i \) is the density of the initial particles in the rest frame and \( d\omega_{fi} \) is the probability of the transition. From Eqs. (219) and (220) for the width of the decay of the unpolarized Z into a neutrino–antineutrino pair we find the following expression:

\[
d\Gamma_{\nu} = (2\pi)^3 \frac{g^2}{4 \cos^2 \theta_W} N \frac{1}{4} \sum_\lambda \text{Tr} \gamma^\alpha (1 - \gamma_5) k' \gamma^\beta (1 - \gamma_5) k \times \frac{1}{3} \sum_\lambda e_\alpha^\lambda(q) e_\beta^\lambda(q) (2\pi)^4 \delta(k + k' - q) \, d^3k \, d^3k' .
\]

(221)

The sum over \( \lambda \) is equal to:

\[
\sum_\lambda e_\alpha^\lambda(q) e_\beta^\lambda(q) = -\left( g_{\alpha\beta} - \frac{g_{\alpha\beta} q_0 q_3}{m_Z^2} \right) .
\]

For the traces we easily find

\[
\text{Tr} \gamma^\alpha k' \gamma^\beta (1 - \gamma_5) k g_{\alpha\beta} = -2 \text{Tr} k' (1 - \gamma_5) k = -8k k' = -4m_Z^2 ,
\]

\[
\text{Tr} \gamma^\alpha k' \gamma^\beta (1 + \gamma_5) k g_{\alpha\beta} = 0 .
\]

Finally for the integral over \( k \) and \( k' \) we easily obtain

\[
\int \delta(k + k' - q) \frac{d^3k}{k_{0}} \frac{d^3k'}{k'_{0}} = \int \delta(2k_{0} - m_Z) \, dk_{0} \, d\Omega = 2\pi .
\]

Taking into account that

\[
\frac{g^2}{8m_Z^2 \cos^2 \theta_W} = \frac{G_F}{\sqrt{2}}
\]

(where \( G_F \) is the Fermi constant) for the total width of the decay \( Z \to \nu + \bar{\nu} \), we find the following expression:

\[
\Gamma_{\nu} = \frac{1}{2\sqrt{2\pi}} G_F m_Z^3 .
\]

(222)

It is clear that \( \Gamma_{\nu} \) must be proportional to \( G_F \). From dimensional considerations it follows that \( G_F \) must be multiplied by \( m_Z^2 \). Numerically we have \( \Gamma_{\nu} = 166.2 \pm 0.1 \text{ MeV} \).
12. Deep-inelastic processes in the parton approximation

As a last example let us calculate, in the parton approximation, the cross-sections of the deep-inelastic processes

\[
\begin{align*}
    e + N & \rightarrow e + X , \\
    \nu_\mu(\bar{\nu}_\mu) + N & \rightarrow \mu^- (\mu^+) + X , \\
    \nu_\mu(\bar{\nu}_\mu) + N & \rightarrow \nu_\mu (\bar{\nu}_\mu) + X .
\end{align*}
\]

(223) (224) (225)

The first process is due to \( \gamma \) exchange, the second one is due to \( W \) exchange and the third one is due to \( Z \) exchange. The corresponding interaction Hamiltonians have the form:

\[
\begin{align*}
\mathcal{H}_I^\gamma & = e_j^\gamma A^\alpha , \\
\mathcal{H}_I^W & = \frac{g}{2\sqrt{2}} j^W_{\alpha} W^\alpha + \text{h.c.} , \\
\mathcal{H}_I^Z & = \frac{g}{2\cos\theta_W} j^Z_{\alpha} Z^\alpha .
\end{align*}
\]

(226)

Consider, as an example, the process (223). The electromagnetic current \( j^\gamma_{\alpha} \) has the form

\[
    j^\gamma_{\alpha} = j^h_{\alpha} + j^l_{\alpha} ,
\]

where \( j^h_{\alpha} = -\sum_{\ell=e,\mu,\tau} \bar{\ell}\gamma^\alpha \ell \) is the lepton electromagnetic current and \( j^l_{\alpha} = \sum_{q=u,...} e_q \bar{q}\gamma^\alpha q \) is the quark electromagnetic current. We must also take into account strong interaction. So the total Hamiltonian of interaction is equal to

\[
    \mathcal{H}_I = \mathcal{H}_I^\gamma + \mathcal{H}_I^h ,
\]

where \( \mathcal{H}_I^h \) is the Hamiltonian of the strong interaction. In the one-\( \gamma \) exchange approximation for the matrix element of the process we have

\[
\begin{align*}
    \langle f | S | i \rangle & = \langle f | T \left( 1 - ie \int j^\alpha_{\alpha}(x_1) A^\alpha(x_1) \, dx_1 + ... \right) \left( 1 - ie \int j^h_{\alpha}(x_2) A^\beta(x_2) \, dx_2 \right) e^{-i \int \mathcal{H}_I^h(y) dy} | i \rangle \\
    & = (-i)^2 e^2 \langle f | T \left( \int j^\alpha_{\alpha}(x_1) \hat{A}^\alpha(x_1) \hat{A}^\beta(x_2) j^h_{\beta}(x_2) \right) e^{-i \int \mathcal{H}_I^h(y) dy} | i \rangle \\
    & = - ie^2 N_{k'} N_k \bar{u}(k') u(k) \frac{1}{q^2} \int e^{-i q \cdot x} \langle f | T | j^h_{\alpha}(x) e^{-i \int \mathcal{H}_I^h(y) dy} | p \rangle \, dx .
\end{align*}
\]

(227)

Here \( k \) and \( k' \) are the momenta of the initial and final electrons, \( q = k - k' \), \( p \) is the momentum of the initial nucleon, and \( p' \) is the total momentum of the final hadrons. It is clear that in Eq. (227) the strong interaction is taken exactly into account. The diagram of the process is shown in Fig. 12.

![Fig. 12: The diagram of the process \( e + N \rightarrow e + X \)](image)
Let us note that the hadronic part of the matrix element can be presented in the form
\[
\langle p'| T(j^h_\alpha(x) e^{-i \int \mathcal{L}_y dy}) | p \rangle =_{\text{out}} \langle p'| J^h_\alpha(x) | p \rangle_{\text{in}}, \tag{228}
\]
where \( J^h_\alpha(x), | p \rangle_{\text{in}} \) and \( | p \rangle_{\text{out}} \) are respectively the hadron electromagnetic current, the initial and the final state vectors in the **Heisenberg representation**. From the translational invariance it follows that
\[
J^h_\alpha(x) = e^{i P x} J^h_\alpha(0) e^{-i P x}, \tag{229}
\]
where \( P \) is the operator of the total momentum. From Eqs. (227), (228), and (229) for the matrix element of the process we obtain the following expression:
\[
\langle f | S | i \rangle = -i e^2 N c_k \bar{u}(k') \gamma^\alpha u(k) \frac{1}{q^2_{\text{out}}} \langle p'| J^h_\alpha(0) | p \rangle_{\text{in}} (2\pi)^4 \delta(p' - p - q). \tag{230}
\]

It is clear that the first multiplier in expression (230) comes from the lepton current. The second one is the photon propagator, and the third ones come from the hadron current and takes into account strong interaction. Similar structures have matrix elements of other deep-inelastic processes.

Now let us calculate the cross-sections of the processes (223) to (225) in **parton approximation**. In this approximation deep-inelastic processes are due to lepton–quark scattering
\[
e + q \rightarrow e + q', \quad \nu_\mu(\bar{\nu}_\mu) + q \rightarrow \mu^- (\mu^+) + q', \quad \nu_\tau(\bar{\nu}_\tau) + q \rightarrow \nu_\tau(\bar{\nu}_\tau) + q. \tag{231}
\]

Let us write down the Hamiltonians of interaction (226) in the form
\[
\mathcal{H}_\gamma = g_a j^a_\gamma X^a, \tag{267}
\]
where the index \( a \) is equal to \( \gamma \) or \( W \) or \( Z \). The matrix element of any of the processes (231) has the form
\[
\langle f | S | i \rangle_{\gamma} = i N \bar{u}'(k') \gamma^\alpha (c_\gamma^a - c_\alpha^a \gamma_5) u'(k) \frac{g_a^2}{q^2 - m_a^2} \times \bar{u}'(p'_q) \gamma_\alpha (c_\gamma^a(q) - c_\gamma^a(q) \gamma_5) u'(p_q) (2\pi)^4 \delta(p'_q - p_q - q). \tag{232}
\]

Here \( k \) and \( k' \) are the momenta of the initial and final leptons, \( p_q \) and \( p'_q \) are the momenta of the initial and final quarks, \( q = k - k' \), and \( N \) is the product of the normalization factors. In Eq. (232) the masses of all fermions are neglected (parton approximation). We have
\[
\gamma_5 u' = r u', \tag{233}
\]
where \( r \) is helicity. From Eqs. (226) and (227) for the matrix element we will obtain the following expression:
\[
\langle f | S | i \rangle_{\gamma} = i 4 N \bar{u}'(k') \gamma^\alpha u'(k) c_\alpha^a \frac{g_a^2}{q^2 - m_a^2} \times \bar{u}'(p'_q) \gamma_\alpha u'(p_q) c_\alpha^a(q) (2\pi)^4 \delta(p'_q - p_q - q), \tag{234}
\]
where
\[ c_V - r c_A = 2 c_r . \]  

(235)

From this relation it follows that
\[ c_V = c_{-1} + c_1 , \]
\[ c_A = c_{-1} - c_1 . \]

From Eq. (234) for the contribution of the q quark to the cross-sections of deep-inelastic scattering we will obtain the following expression:
\[
\frac{d\sigma_q^a}{d^4k} = \frac{1}{j} N^2 \int \sum_{\tilde{r}} \bar{u}^r(k') \gamma^\alpha u^r(k) \bar{u}^r(k) \gamma^\beta u^r(k') (c_q^a)^2 \rho_{rr} \\
\times \left( \frac{g_s^2}{q^2 - m_q^2} \right)^2 \sum_{\tilde{r}'} \bar{u}^{r'}(p_q) \gamma_\alpha u^{r'}(p_q) \bar{u}^{r'}(p_q) \gamma_\beta u^{r'}(p'_q) (c_q^a)^2 (q^2 (2\pi)^4 \delta(p_q - p_q - q) \\
\times d^3p_q' d^3k' f_q'(x) dx . \]  

(236)

Here \( \rho_{rr} \) is the probability of finding the initial lepton in the state with helicity \( r \) and \( f_q'(x) \) is the density of the probability of finding in the nucleon the initial q quark with momentum \( p_q = x q_p \) (\( p \) is the momentum of the initial nucleon). The kinematical factor in Eq. (236) is equal to
\[
\frac{1}{j} N^2 (2\pi)^4 \delta(p_q' - p_q - q) d^3k' d^3p_q' = \frac{1}{(2\pi)^2} \frac{1}{16 pkx} \frac{1}{pq} \delta(x_q - x) \frac{d^3k'}{k^{ao}} , \]  

(237)

where \( x = -q^2/2pq \) and we have used
\[ \frac{d^3p_q'}{p_q'^a} = 2\delta(p_q'^2) d^4p_q' . \]

Thus, as is seen from Eq. (237), owing to the conservation of 4-momentum only quarks with momentum \( p_q = xp \) can interact with intermediate vector particles (\( \gamma, W, Z \)).

As independent variables let us choose
\[ x, y = \frac{pq}{pk} , \quad \text{and} \quad pk . \]

We have
\[
\left( \frac{d\sigma_q^a}{dxdy} \right)_q = \frac{1}{2\pi pkx} \sum_{\tilde{r}} \bar{u}^r(k') \gamma^\alpha u^r(k) \bar{u}^r(k) \gamma^\beta u^r(k') \\
\times (c_q^a)^2 \rho_{rr} \left( \frac{g_s^2}{q^2 - m_q^2} \right)^2 \sum_{\tilde{r}'} \bar{u}^{r'}(p_q) \gamma_\alpha u^{r'}(p_q) (c_q^a)^2 f_q'(x) , \]  

(238)

where \( p_q = xp \) and \( p_q' = p_q + q \).

In Eq. (238) we have used
\[ \frac{d^3k'}{k^{ao}} = 2\pi pq dx dy . \]
Consider the spin factors in the expression (238). We have
\[ u^* (k) \bar{u}^* (k) = \frac{1 + r \gamma_5}{2} \sum_{r'} u^{r'} (k) \bar{u}^{r'} (k) = \frac{1 + r \gamma_5}{2} \Lambda (k) . \]

With the help of this relation we will find
\begin{align*}
\bar{u}^* (k') \gamma^\alpha u^* (k) \bar{u}^* (k) \gamma^\beta u^* (k') \bar{u}^* (p_q') \gamma_\alpha u^{r'} (p_q) \bar{u}^{r'} (p_q) \gamma_\beta u^{r'} (p_q') \\
= \text{Tr} \gamma^\alpha \frac{1 + r \gamma_5}{2} \gamma^\beta \frac{1 + r' \gamma_5}{2} k' \cdot \text{Tr} \gamma_\alpha \frac{1 + r' \gamma_5}{2} \bar{u}^{r'} (p_q) \gamma_\beta \frac{1 + r \gamma_5}{2} \bar{u}^{r'} (p_q') \\
= 16 \left[ p_q \cdot k' p_q' \frac{1 + r r'}{2} + p_q \cdot k' p_q' \frac{1 - r r'}{2} \right].
\end{align*}
(239)

Note that the calculations are strongly simplified if relations (214) are used. Note also that, owing to momentum conservation,
\[ p_q' k' = p_q k , \quad p_q k' = p_q' k . \]

From Eqs. (238) and (239) for the contribution of the q quark to the cross-section of deep-inelastic scattering we will find \(^1\)
\[ \frac{d \sigma^2}{d x d y} = \frac{8}{\pi} \frac{1}{p k x} \left( \frac{g^2}{q^2 - m^2} \right)^2 \sum_{r r'} \left[ \frac{1 + r r'}{2} + \frac{1 - r r'}{2} (1 - y)^2 \right] \times (c_r^2)^2 (c_{r'}^2 (q))^2 f_q (x) \rho_{rr} . \]
(240)

As is seen from expression (240), if helicities of initial particles are the same there is no y-dependence of the cross-section. In the case of opposite helicities y-dependence of the cross-section is determined by the factor \((1 - y)^2\).

Up to now we have considered the scattering of leptons on quarks. To take into account antiquarks inside the nucleon let us express quark currents through charge-conjugated fields
\[ q_c = C q^T , \quad \bar{q}_c = -q^T C^{-1} . \]

where
\[ C \gamma^\alpha T C^{-1} = -\gamma_\alpha , \quad C^T = -C . \]

We have
\[ \bar{q}_c (c_V - c_A \gamma_5) q_c = -q^T (c_V \gamma^T - c_A (\gamma_5 \gamma_5)^T) q^T \\
= -\bar{q}_c (c_V + c_A \gamma_5) q_c = \bar{q}_c (\bar{c}_V - \bar{c}_A \gamma_5) q_c , \]
where
\[ \bar{c}_V = -c_V , \quad \bar{c}_A = c_A . \]
(241)

From Eqs. (229) and (235) we find
\[ \bar{c}_r = \frac{1}{2} (\bar{c}_V - r \bar{c}_A) = -c_{-r} . \]

\(^1\) In the case of charged currents it is necessary to take into account also the element of the CKM mixing matrix.
Thus to obtain the contribution of antiquarks to the cross-section of deep-inelastic scattering we must substitute \( \bar{c}_r = -c_r \) and \( f'_q(x) \) for \( f_q(x) \) in the expression (240).

An analogous change must be made in the case of antileptons.

Let us now consider \textbf{charged-current induced deep-inelastic processes}:

\[
\nu_\mu (\bar{v}_\mu) + N \rightarrow \mu^- (\mu^+) + X .
\]

The Hamiltonian of the processes has the following form:

\[
\mathcal{H}_t = \frac{g}{2\sqrt{2}} \bar{\nu}_\mu \gamma^\alpha (1 - \gamma_5) \mu + \sum_{q' = u,...} V_{q'q} \bar{c}_r \gamma_{\alpha} (1 - \gamma_5) q | W_\alpha + \text{h.c.} ,
\]

where \( V_{q'q} \) is the element of the CKM mixing matrix. Consider first the elementary process

\[
\nu_\mu + q \rightarrow \mu^- + q' .
\]

where \( q = d, ... \) and \( q' = u, ... \) We have \( c_V = c_V(q) = 1, c_A = c_A(q) = 1, \) and \( c_{-1} = c_{-1}(q) = 1, c_1 = c_1(q) = 0. \) Thus \( r = r' = -1. \) Let us sum over the final quarks \( q'. \) Taking into account the unitarity of the mixing matrix \( V \) for the kinematical factor in Eq. (240) we have

\[
\frac{8}{\pi} \left( \frac{g^2}{8 m_W^2} \right)^2 \sum_{q'} |V_{q'q}|^2 = \frac{4 G_F^2}{\pi} ,
\]

where we assumed that \( |q|^2 \ll m_W^2. \)

In the unpolarized nucleon \( f_q^{-1}(x) = f_q^1(x) \) and from Eqs. (240) and (242) for the cross-section we obtain

\[
\left( \frac{d\sigma}{dx dy} \right)_q = 2 \sigma_0 x f_q(x) , \quad q = d, ... .
\]

Here

\[
\sigma_0 = \frac{G_F^2}{\pi} \frac{p_k}{p}
\]

and \( f_q(x) = f_q^1(x) + f_q^{-1}(x) \) is the density probability of finding a \( q \)-quark inside the nucleon.

Consider now the process

\[
\nu_\mu + \bar{q} \rightarrow \mu^- + \bar{q}' .
\]

In this case \( \bar{c}_1(q) = -1, \bar{c}_{-1}(q) = 0, r = -1, r' = 1, \) and from Eqs. (240) and (242) we obtain

\[
\left( \frac{d\sigma}{dx dy} \right)_{\bar{q}} = 2 \sigma_0 x (1 - y)^2 f_{\bar{q}}(x) , \quad \bar{q} = \bar{u}, ... .
\]

Summing up the contributions of quarks and antiquarks for the cross-section of the process \( \nu_\mu N \rightarrow \mu^- X \) we have in the parton approximation the following expression:

\[
\left( \frac{d\sigma}{dx dy} \right)^\nu = 2 \sigma_0 x \left( \sum_{q=d,...} f_q(x) + (1 - y)^2 \sum_{\bar{q} = \bar{u},...} f_{\bar{q}}(x) \right) .
\]

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Consider now the process $\bar{\nu}_\mu + N \rightarrow \mu^+ + X$. In the case of

$$\bar{\nu}_\mu + q \rightarrow \mu^+ + q', \quad q = u, ..., \quad q' = d, ...,$$

we have $r = 1, r' = -1$, and from Eq. (234) for the cross-section (summed over $q$) we obtain

$$\left( \frac{d\sigma}{dx dy} \right)^{\bar{\nu}_\mu}_q = 2\sigma_0 x (1 - y)^2 f_q(x), \quad q = u, .... \quad (246)$$

The cross-section of the process

$$\bar{\nu}_\mu + q \rightarrow \mu^+ + q', \quad q = d, ..., \quad q' = \bar{u}, ...,$$

summed over all possible $q$ ($r = 1, r' = 1$), is given by

$$\left( \frac{d\sigma}{dx dy} \right)^{\bar{\nu}_\mu}_{\bar{q}} = 2\sigma_0 x f_{\bar{q}}(x), \quad \bar{q} = \bar{d}, ... \quad (247)$$

For the cross-section of the process $\bar{\nu}_\mu + N \rightarrow \mu^+ + X$ we have, from Eqs. (240) and (241), the following expression:

$$\left( \frac{d\sigma}{dx dy} \right)^{\bar{\nu}_\mu} = 2\sigma_0 x [(1 - y)^2 \sum_{q = u, ...} f_q(x) + \sum_{q = d, ...} f_{\bar{q}}(x)]. \quad (248)$$

Let us now consider the neutral-current induced deep-inelastic processes

$$\nu_\mu + N \rightarrow \nu_\mu + X \quad (249)$$

$$\bar{\nu}_\mu + N \rightarrow \bar{\nu}_\mu + X \quad (250)$$

The Hamiltonian of the standard theory has the form

$$\mathcal{H}_I = \frac{y}{2 \cos \theta_W} \left[ \frac{1}{2} \bar{\nu}_\mu \gamma^\alpha (1 - \gamma_5) \nu_\mu + \sum_q \bar{q} \gamma^\alpha (c_{V}(q) - c_{A}(q) \gamma_5) q \right] Z_\alpha,$$

where for quarks

$$c_{-1}(q) = I_3^q - \sin^2 \theta_W \epsilon_q,$$

$$c_1(q) = -\sin^2 \theta_W \epsilon_q,$$

$$I_3^u = \frac{1}{2}, \quad I_3^d = -\frac{1}{2},$$

and for neutrinos

$$c_{-1} = \frac{1}{2}, \quad c_1 = 0.$$

From the general formula (240) for the cross-section of the process (249) we easily find

$$\left( \frac{d\sigma}{dx dy} \right)^{\nu}_\nu = 2\sigma_0 x \left\{ \sum_q c_{-1}^2(q) f_q(x) + \sum_q c_1^2(q) f_{\bar{q}}(x) \right\}$$

$$+ (1 - y)^2 \left[ \sum_q c_{-1}^2(q) f_q(x) + \sum_q c_{1}^2(q) f_{\bar{q}}(x) \right]. \quad (251)$$
The cross-section of the process (250) is given by

\[
\left( \frac{d\sigma}{dxdy} \right)_{\nu}^{NC} = 2\sigma_0 x \left\{ (1 - y)^2 \left[ \sum_q c_{-1}^2(q) f_{q}(x) + \sum_q c_{1}^2(q) f_{q}(x) \right] \\
+ \left[ \sum_q c_{1}^2(q) f_{q}(x) + \sum_q c_{-1}^2(q) f_{q}(x) \right] \right\}.
\] (252)

The first term of these expressions (251) and (252) is the contribution to the cross-sections of left-handed quarks and antiquarks and the second term is the contribution of the right-handed particles.

13. One-loop effects, regularization, and renormalization

Up to now we have considered different processes in the lowest order of perturbation theory (tree approximation). Here we will discuss higher order corrections. We will start with the calculation of the one-loop correction to the photon propagator. In the lowest order for the photon propagator we have

\[
-\frac{i}{(2\pi)^4} g^{\alpha\beta} \frac{1}{q^2},
\]

where \( q \) is the momentum of the photon. The virtual photon can produce a fermion–antifermion pair; the pair can annihilate and again produce the virtual photon. The corresponding Feynman diagram is shown in Fig. 13.

![Fig. 13: One-loop contribution to the photon propagator](image)

From Feynman rules it follows that the contribution of a fermion loop to the photon propagator is given by

\[
-\frac{i}{(2\pi)^4} \left[ \frac{1}{q^2} g^{\alpha\beta} (-1) \Sigma_{\rho\sigma}(q) \frac{1}{q^2} g^{\rho\sigma} \right],
\]

where \( \Sigma_{\rho\sigma} \) (the photon self-energy tensor) is equal to

\[
\Sigma_{\rho\sigma}(q) = -ie^2 \int \frac{1}{2} Tr \frac{1}{\not{f} + \not{k} - m} \gamma_\rho - \frac{1}{2} \frac{d^4k}{(2\pi)^4}.
\]

Let us note that the loop in Fig. 13 is due to the term

\[
N(\bar{\psi}(x_1)\gamma_\rho\psi(x_2)\gamma_\sigma\bar{\psi}(x_2))
\]

in the Wick expansion. This term gives the trace with an additional minus sign.
Let us calculate the trace. We have
\[ \text{Tr} \gamma_{\rho}(\not f + \not k + m)\gamma_{\sigma}(\not k + m) = 4[(q + k)_{\rho}k_{\sigma} - g_{\rho\sigma}(q + k) \cdot k + k_{\rho}(q + k)_{\sigma} + m^2g_{\rho\sigma}] . \]

The term \( g_{\rho\sigma}k_{\rho} + k_{\rho}g_{\sigma} \) can be dropped owing to current conservation (a photon interacts with the conserved current). For the self-energy tensor we have the following expression:
\[ \Sigma_{\rho\sigma}(q) = -4ie^2 \int \frac{2k_{\rho}k_{\sigma} - g_{\rho\sigma}(qk + k^2 - m^2)}{[(q + k)^2 - m^2][(k^2 - m^2)]} \ d^4k . \]  

These integrals are divergent.

When higher order corrections are taken into account the procedure of renormalization must be carried out: charges, masses, and other physical quantities that enter into the Lagrangian (bare quantities) must be expressed through measurable charges, masses, etc. After this procedure higher order corrections will be finite (in renormalizable theories). Before renormalization we must regularize divergent integrals (make them finite). The standard procedure of regularization is dimensional regularization.

Let us consider integrals in a space of dimension \( d < 4 \). As we will see, all integrals will become finite. After renormalization we will put \( d = 4 \). In \( d \)-dimensional space an element of volume \( d^dx \) has dimension
\[ [d^dx] = M^{-d} \]

(\( M \) is mass). The action \( S = \int L \ d^dx \) is a dimensionless quantity. So the Lagrangian has the dimension
\[ [L] = M^d . \]

The dimension of the fermion field \( \psi \) and the boson field \( \phi \) are equal\(^2\)
\[ [\psi] = M^{(d-1)/2} , \quad [\phi] = M^{(d-3)/2} . \]

Now consider the Lagrangian of the electromagnetic interaction
\[ H_I = e\bar{\psi}\gamma^\alpha eA_\alpha . \]

From Eqs. (255) and (256) we will obtain
\[ [e] = M^{(4-d)/2} . \]

Thus electromagnetic charge is a dimensionless quantity only in 4-dimensional space.

In the space of the dimension \( d < 4 \) let us write down the charge in the form \( e\mu^{(4-d)/2} \), where \( e \) is the dimensionless electric charge and \( \mu \) is a parameter of dimension \( M \).

The factor \((2\pi)^4\) in the expression (254) is due to the integration of an exponent over \( d^dx \). In the space of dimension \( d \) instead of \((2\pi)^4\) we will have \((2\pi)^d\). Taking into account all these remarks for the polarization tensor in a space of dimension \( d \) we have the following expression:
\[ \Sigma_{\rho\sigma}(q) = -4e^2i \int \frac{2k_{\rho}k_{\sigma} - g_{\rho\sigma}(qk + k^2 - m^2)}{[(q + k)^2 - m^2][(k^2 - m^2)]} \ d\Gamma , \]  

\(^2\) Consider, for example, the mass terms of the Lagrangians. Let us write down \( [\psi] = M^x, [\phi] = M^y \). We have \([m^2\bar{\psi}\psi] = M^{2x+1} = M^d, [m^2\phi^*\phi] = M^{2y+2} = M^d\). From these relations Eq. (256) follows.

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where
\[ d\Gamma = \mu^{4-d} \frac{d^d k}{(2\pi)^d} . \]

The calculation of the tensor \( \Sigma_{\rho\sigma} \) is reduced to the calculation of the integrals
\[ \int \frac{d\Gamma}{k^2 - m^2} = \frac{i}{16\pi^2} A , \]
\[ \int \frac{d\Gamma}{[k^2 - m^2][(k + q)^2 - m^2]} = \frac{i}{16\pi^2} B . \]  
(258)

Let us first consider the term of Eq. (257) that is proportional to \( g_{\rho\sigma} \). It is evident that
\[ -4ie^2 \int \frac{d\Gamma}{(q + k)^2 - m^2} = \frac{\alpha}{\pi} A , \]  
(259)
where \( \alpha = e^2/4\pi \). Further, taking into account that \( qk = \frac{1}{2}[(q + k)^2 - m^2] - (k^2 - m^2) - q^2 \) we will obtain
\[ -4ie^2 \int \frac{qk \ d\Gamma}{[k^2 - m^2][(q + k)^2 - m^2]} = \frac{\alpha}{\pi} \left( -\frac{q^2}{2} \right) B . \]  
(260)

The first term of the integral (254) has the following form:
\[ -4ie^2 \int \frac{k_\rho k_\rho \ d\Gamma}{[k^2 - m^2][(q + k)^2 - m^2]} = a g_{\rho\sigma} + b q_\rho q_\sigma , \]  
(261)
where \( a \) and \( b \) are functions of \( k^2 \). To find \( a \) let us multiply Eq. (261) by \( g^{\rho\sigma} \) and \( q^\rho q^\sigma \). Taking into account that \( g_{\rho\sigma} g^{\rho\sigma} = d \) we will find
\[ a(d - 1) = -4ie^2 \int \frac{[k^2 - (1/q^2)(qk)^2] \ d\Gamma}{[(q + k)^2 - m^2][k^2 - m^2]} . \]  
(262)

The \( b \) term does not give a contribution due to current conservation. Further, we have
\[ -4ie^2 \int \frac{k^2 \ d\Gamma}{[k^2 - m^2][(q + k)^2 - m^2]} = \frac{\alpha}{\pi} [A + m^2 B] . \]  
(263)

Now taking into account that
\( (qk)^2 = \frac{1}{2} q \cdot k [(q + k)^2 - m^2] - (k^2 - m^2) - q^2 \),
we will easily obtain
\[ -4ie^2 \int \frac{(qk)^2 \ d\Gamma}{[k^2 - m^2][(q + k)^2 - m^2]} = \frac{\alpha}{2\pi} [q^2 A + \frac{1}{2} (q^2)^2 B] . \]  
(264)

Let us note that in obtaining Eq. (264) we have used
\[ \int \frac{qk \ d\Gamma}{(k^2 - m^2)} = 0 . \]

Thus from Eqs. (262) to (264) we have
\[ a(d - 1) = \frac{1}{2} A + m^2 B - \frac{1}{4} q^2 B . \]  
(265)
The self-energy tensor \( \Sigma_{\rho\sigma} \) has the following general form:

\[
\Sigma_{\rho\sigma}(q) = \Sigma_{\gamma} q_{\rho\sigma} + \Sigma_{\gamma}^q q_{\rho\sigma} \quad . \tag{266}
\]

We are not interested in the second term of Eq. (266). For the first term we have

\[
\Sigma_{\gamma} = \frac{\alpha}{\pi} [2a + \frac{1}{2} q^2 B - A] \quad . \tag{267}
\]

Now let us calculate the integrals \( A \) and \( B \). It is useful to unite the product of two brackets in the denominator of \( B \). To do this we will use the identity

\[
\int_0^1 \frac{dx}{[ax + b(1 - x)]^2} = \frac{1}{ab} \quad . \tag{268}
\]

We have

\[
\int \frac{d\Gamma}{[k^2 - m^2][(k + q)^2 - m^2]} = \int_0^1 dx \int \frac{d\Gamma}{(k^2 + q^2 x - m^2 - q^2 x^2)^2} \quad .
\]

It is assumed in all the integrals we are considering that \( m^2 \to m^2 - i\epsilon \) (propagators).

The positions of corresponding poles in the complex \( k^0 \) plane are shown in Fig. 14.

Fig. 14: The positions of poles of propagators.

This means that the real \( k^0 \) axis can be turned by \( 90^\circ \) (Wick rotation) and integrals over the real \( k^0 \) axis are equal to the integrals over the imaginary axis. We have

\[
k^0 = ik_E^0 \quad , \quad k^2 = -k_E^0 \quad , \quad d^dk = id^dk_E
\]

and integrals \( A \) and \( B \) are equal

\[
\frac{i}{16\pi^2} A = \frac{-i}{(2\pi)^{d/4-d}} \int \frac{d^dk_E}{k_E^2 + m^2} \quad ,
\]

\[
\frac{i}{16\pi^2} B = \frac{i}{(2\pi)^{d/4-d}} \int_0^1 dx \int \frac{d^dk_E}{(k_E^2 + Q)^2} \quad , \tag{269}
\]

where \( Q = m^2 + q^2 x^2 - q^2 x \). Let us put \( k_E^2 = R \). We have

\[
d^dk_E = R^{(d-1)/2} dR^{1/2} d\Omega_d = \frac{1}{2} R^{(d/2)-1} dR d\Omega_d \quad .
\]

Now we will show that the integrals \( A \) and \( B \) are given by the product of \( \Gamma \) functions. The function \( \Gamma(x) \) is determined by the integral

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad . \tag{270}
\]
From Eq. (270) by integration by parts we have

\[ \Gamma(x) = (x - 1)\Gamma(x - 1) . \]

Let us now consider the product \( \Gamma(x)\Gamma(y) \). We have

\[
\begin{align*}
\Gamma(x)\Gamma(y) &= \int_0^\infty t^{x-1} e^{-t} dt \int_0^\infty u^{y-1} e^{-u} du \\
&= \int_0^\infty t^{x+y-1} e^{-t} dt \int_0^\infty u^{y-1} e^{-u} du \\
&= \int_0^\infty u^{y-1} du \int_0^\infty t^{x+y-1} e^{-t(1+\nu)} dt \\
&= \int_0^\infty u^{y-1} (1 + \nu)^{x+y} du \int_0^\infty t^{x+y-1} e^{-t'} dt' \\
&= (u = tv, \quad t' = (1 + \nu)t) .
\end{align*}
\]

Thus we have

\[ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{u^{y-1} du}{(1 + \nu)^{x+y}} . \]

(272)

As a very simple application of Eq. (272) let us consider the case \( x = y = 1/2 \). Taking into account that \( \Gamma(1) = 1 \) we will obtain

\[ \Gamma^2 \left( \frac{1}{2} \right) = \int_0^\infty \frac{v^{-1/2} dv}{1 + v} = \pi . \]

Thus we have \( \Gamma(1/2) = \sqrt{\pi} \).

Now using Eq. (272) for the integrals we are interested in we will obtain

\[
\begin{align*}
\frac{\mu^{4-d}}{(2\pi)^d} \int \frac{d^d k_E}{(k_E^2 + Q)^n} &= \mu^{4-d} \int \frac{R^{(d/2)-1} dR}{(R + Q)^n} \int \frac{d\Omega_d}{2(2\pi)^d} = \\
&= \mu^{4-d} Q^{(d/2)-n} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \int \frac{\Gamma(\frac{d}{2}) d\Omega_d}{2(2\pi)^d} \quad (R = Qt) .
\end{align*}
\]

(273)

The integrals over the angles are equal\(^3\)

\[ \int \frac{\Gamma(\frac{d}{2}) d\Omega_d}{2(2\pi)^d} = \frac{1}{(4\pi)^{d/2}} . \]

(274)

Finally, for the integral we have

\[ \frac{\mu^{4-d}}{(2\pi)^d} \int \frac{d^d k_E}{(k_E^2 + Q)^n} = \mu^{4-d} Q^{(d/2)-n} \frac{1}{(4\pi)^{(d/2)}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} . \]

(275)

\(^3\) Let us consider the case \( d = 3 \). We have

\[ \int \frac{\Gamma(\frac{3}{2}) d\Omega_3}{2(2\pi)^3} = \frac{\sqrt{\pi}}{2} \frac{4\pi}{2(2\pi)^3} = \frac{1}{(4\pi)^{3/2}} , \]

which is in agreement with the general formula (274).
Let us write down \( d = 4 - \epsilon \), where \( \epsilon \) is positive and a small parameter. The integrals \( A \) and \( B \) are expressed through \( \Gamma(n - 2 + \epsilon/2) \) with \( n = 1, 2 \). Let us consider \( \Gamma(1 + \epsilon) \) and let us expand this quantity in powers of \( \epsilon \). We have
\[
\Gamma(1 + \epsilon) = \Gamma(1) + \Gamma'(1)\epsilon + ... = 1 - \gamma\epsilon + ..., \tag{276}
\]
where \( \Gamma'(1) = -\gamma, \gamma = 0.577 \) is the Euler constant. From Eq. (276) we obtain the following expansion over \( \epsilon \) of \( \Gamma(\epsilon) \):
\[
\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon). \tag{277}
\]
Further we have
\[
\Gamma(-1 + \epsilon) = -\frac{\Gamma(\epsilon)}{1 - \epsilon} = -\frac{1}{\epsilon} + (\gamma - 1) + O(\epsilon). \tag{278}
\]
Thus the function \( \Gamma(-n + \epsilon) \) has \( 1/\epsilon \) singularity at \( n = 0, 1, 2, ... \).

The idea of the method of dimensional regularization is the following. Let us work in a space of dimension \( d < 4(\epsilon > 0) \) All integrals are finite. After renormalization we will put \( d = 4(\epsilon = 0) \).

Let us return to the integrals \( A \) and \( B \). Taking into account Eqs. (275), (277), and (278) we have
\[
A = m^2 \left[ \frac{2}{\epsilon} - \gamma + \ln 4\pi + 1 - \ln \frac{m^2}{\mu^2} + O(\epsilon) \right],
\]
\[
B = \frac{2}{\epsilon} - \gamma + \ln 4\pi - \int_0^1 dx \ln \frac{x}{\mu^2} + O(\epsilon), \tag{279}
\]
where \( Q = m^2 + q^2x^2 - q^2x \). Let us note that the integral over \( x \) in Eqs. (279) is due to the second term in the expansion \( Q^{-\epsilon/2} = 1 - (\epsilon/2) \ln Q + ... \). From Eqs. (265), (267), and (279) for \( \Sigma^\gamma(q^2) \) we obtain
\[
\Sigma^\gamma(q^2) = \frac{\alpha}{3\pi} \left[ (\Delta - \ln \frac{m^2}{\mu^2}) q^2 + (q^2 + 2m^2)B - \frac{1}{3} q^2 \right], \tag{280}
\]
where
\[
\Delta = \frac{2}{\epsilon} - \gamma + \ln 4\pi, \\
\Bar{B} = -\int_0^1 \ln \left( 1 + \frac{q^2}{m^2}(x^2 - x) \right) dx. \tag{281}
\]

Let us first consider the case \( q^2 \ll m^2 \). Taking into account a quadratic over \( (q^2/m^2) \) terms from (280) and (281) we easily find
\[
\Sigma^\gamma(q^2) = \frac{\alpha}{3\pi} q^2 \left[ (\Delta - \ln \frac{m^2}{\mu^2}) + \frac{1}{5} \frac{q^2}{m^2} \right]. \tag{282}
\]
Let us now consider the case \( |q^2| \gg m^2 \). Integrating by parts \( \Bar{B} \), we have
\[
\Bar{B} = 2 + \frac{1}{2} \left( 1 - \frac{4m^2}{q^2} \right) \int_0^1 \frac{dx}{x^2 - x + (m^2/q^2)}. \tag{283}
\]
From Eqs. (280) and (283) we easily obtain

\[ \Sigma^\gamma(q^2) = \frac{\alpha}{3\pi q^2} \left[ \left( \Delta - \ln \frac{m^2}{\mu^2} \right) + \frac{5}{3} - \ln \frac{|q^2|}{m^2} + i\pi \theta(q^2) \right], \]  

(284)

where

\[ \theta(x) = \begin{cases} 0 & (x > 0) \\ 1 & (x < 0) \end{cases}. \]

Let us note that in both cases considered \( \Sigma \) is proportional to \( q^2 \). This is a general property of \( \Sigma \) that is connected with \( 1/q^2 \) singularity of the photon propagator.

Next we introduce vacuum polarization of the photon

\[ \Pi^\gamma(q^2) = \frac{1}{q^2} \Sigma^\gamma(q^2). \]  

(285)

For the photon propagator we have

\[ \mathcal{D}^{\alpha\beta} = \frac{-i}{(2\pi)^4} g^{\alpha\beta} \frac{1}{q^2} (1 - \Pi^\gamma(q^2)). \]

Now we will discuss charge renormalization. Physical charge characterizes the interaction of a photon with a charged particle, say, an electron at \( q = 0 \). If higher order corrections are taken into account, besides the direct interaction with the electrons there is an additional contribution of a fermion–antifermion pair to the interaction (see Fig. 15)

![Fig. 15: Diagrams that show the interaction of a photon and an electron](image)

The Hamiltonian of electromagnetic interaction is given by

\[ \mathcal{H}_I = e_0 \bar{e} \gamma^\alpha e A_\alpha, \]  

(286)

where \( e_0 \) is the so-called bare charge. The physical charge can be written in the form

\[ e = e_0 - \delta e, \]  

(287)

where \( \delta e \) is the contribution of pairs to the physical charge. From Fig. 15 it follows that

\[ \delta e = \frac{1}{2} e_0 \Pi^\gamma(0). \]  

(288)

Apart from the diagram in Fig. 15 there are also additional diagrams (Fig. 16) that could contribute to the physical charge. Note that their contributions are cancelled owing to the Ward identity.
Now we will obtain the same result [Eq. (288)] by the **counterterm method**. Using Eqs. (286) and (287) for the Hamiltonian of the electromagnetic interaction we have

\[
\mathcal{H}_I = e\bar{e}\gamma^0 \gamma e A_\alpha + \delta e\bar{e}\gamma^\alpha e A_\alpha .
\]  

(289)

The last term of Eq. (289) (counterterm) will be taken into account in the lowest order of perturbation theory and \( \delta e \) will be considered as a parameter. With the help of the first term of Eq. (282) in which a physical charge is involved, higher order corrections are calculated. The interaction of a photon and an electron is shown in Fig. 17.

Thus owing to a counterterm [the last term of interaction (289)] we have additional diagrams. For the electric charge from Fig. 17 we obtain

\[
e - \frac{1}{2}e\Pi' (0) + \delta e = e .
\]  

(290)

This condition fixes \( \delta e \). We have

\[
\delta e = \frac{1}{2}e\Pi' (0) .
\]  

(291)

This condition coincides with Eq. (288) (up to \( \alpha \)-terms).

Let us now consider, as an example, elastic \( \mu + e \to \mu + e \) scattering. The diagrams of the process are given in Fig. 18.

Fig. 18: Diagrams of the process \( \mu + e \to \mu + e \)
From Fig. 18 for the ‘photonic part’ of the matrix element we will obtain

\[
\frac{e^2}{q^2} - \frac{e^2}{q^2} \Pi^\gamma(q^2) + \frac{2e\gamma \epsilon}{q^2} = \frac{e^2}{q^2} - \frac{e^2}{q^2} \hat{\Pi}^\gamma(q^2) .
\] (292)

Here

\[
\hat{\Pi}^\gamma(q^2) = \Pi^\gamma(q^2) - \Pi^\gamma(0)
\] (293)

is the **renormalized vacuum polarization** (finite function of \(q^2\)). The second term in the expression (292) is a **finite** one-loop contribution to the matrix element of the process. We have [see Eq. (282)]:

\[
\Pi^\gamma(0) = \frac{e^2}{3\pi} \left[ \Delta - \ln \frac{m^2}{\mu^2} \right] ,
\] (294)

and for the renormalized vacuum polarization from Eqs. (282) and (284) we consequently obtain

\[
\hat{\Pi}^\gamma(q^2) = \frac{\alpha}{3\pi} \frac{1}{5} \frac{q^2}{m^2} , \quad q^2 \ll m^2 ,
\]

\[
\hat{\Pi}^\gamma(q^2) = \frac{\alpha}{3\pi} \left[ \frac{5}{3} - \ln \frac{|q^2|}{m^2} + i \pi \theta(q^2) \right] , \quad |q^2| \gg m^2 .
\] (295)

It is useful to obtain the same result [Eq. (293)] in another way, without a counterterm. In this case the matrix element of \(\mu - e\) scattering (up to \(\alpha^\prime\) terms) is given by the first two diagrams of Fig. 18. We have for the ‘photonic part’

\[
\frac{e^2}{q^2} - \frac{e^2}{q^2} \Pi^\gamma(q^2) = \frac{e^2}{q^2} + \frac{2e\gamma \epsilon}{q^2} - \frac{e^2}{q^2} \Pi^\gamma(q^2) = \frac{e^2}{q^2} (1 - \hat{\Pi}^\gamma(q^2)) ,
\] (296)

where the relation (288) was used.

Let us now make the following **remark**. The renormalized one-loop contribution to the matrix element of \(\mu - e\) scattering is given by \(-\frac{e^2}{q^2} \hat{\Pi}^\gamma(q^2)\). Correspondingly, a two-loop contribution is equal to \((e^2/q^2)(\hat{\Pi}^\gamma(q^2))^2\) and so on. Summing up contributions of loops for the matrix element we have

\[
\frac{e^2}{q^2} (1 - \hat{\Pi}^\gamma(q^2) + (\hat{\Pi}^\gamma(q^2))^2 + ...) = \frac{e^2}{q^2(1 + \hat{\Pi}^\gamma(q^2))} .
\] (297)

This result can be justified with the help of the renormalization group method.

The modification of the photon propagator due to the contribution of loops is usually interpreted in terms of the **running charge**

\[
e^2(q^2) = \frac{e^2}{1 + \text{Re} \hat{\Pi}^\gamma(q^2)} .
\] (298)

Let us now consider the **renormalization of the vector boson mass**. The propagator of the vector boson is\(^4\)

\[
-\frac{i}{(2\pi)^4} g_{\alpha\beta} \frac{1}{q^2 - m_v^2} . \quad V = W, Z .
\] (299)

\(^4\) The propagator we are considering connects two fermions lines. The \(g_{\alpha\beta}\) term of the propagator gives a contribution proportional to \(m_F^2/m_F^2\), where \(m_F\) is the fermion mass. In the case of light fermions this term can be omitted.
Here $m^0_V$ is the bare mass of the vector boson. The one-fermion-loop contribution to the vector boson propagator has the form

$$-i \frac{g^{a\beta}}{(2\pi)^4 \frac{q^2 - m_V^2}{q^2 - m_V^2}} \left( -\Sigma_{\rho\sigma}(q) \frac{g^{\rho\sigma}}{q^2 - m_V^2} \right), \quad (300)$$

where the tensor $\Sigma_{\rho\sigma}$ has the following general form:

$$\Sigma_{\rho\sigma}(q) = \Sigma_V(q^2)g_{\rho\sigma} + \Sigma'_V(q^2)q_{\rho}q_{\sigma}. \quad (301)$$

For the total propagator [omitting the $g_{\rho\sigma}q_{\rho\sigma}$ term of Eq. (301)] we have

$$\frac{-i g^{a\beta}}{(2\pi)^4 \frac{q^2 - m_V^2}{q^2 - m_V^2}} \left( 1 - \Sigma_V(q^2) \frac{1}{q^2 - m_V^2} + ... \right)$$

$$= \frac{-i g^{a\beta}}{(2\pi)^4 \frac{q^2 - m_V^2}{1 + \Sigma_V(q^2)1/(q^2 - m_V^2)} = \frac{-i g^{a\beta}}{(2\pi)^4 \frac{q^2 - m_V^2}{1 + \Sigma_V(q^2)}}, \quad (302)$$

The physical mass of the vector boson $m_V$ is determined by the condition

$$(q^2 - m_V^2 + \text{Re} \Sigma_V(q^2))|_{q^2 = m_V^2} = 0. \quad (303)$$

From this condition it follows that

$$m_V^2 = m^2_V + \delta m_V^2, \quad (304)$$

where $\delta m_V^2 = \text{Re} \Sigma_V(m_V^2)$. From Eqs. (302) and (303) for the vector boson propagator we have

$$\frac{-i g^{a\beta}}{(2\pi)^4 \frac{q^2 - m^2_V + \Sigma(q^2) - \text{Re} \Sigma'(m_V^2)}}, \quad (305)$$

where $\Sigma(q^2) - \text{Re} \Sigma'(m_V^2)$ is the final quantity.

We will finish with some remarks on a parameter $\Delta r$ that is a radiative correction to the effective Fermi constant. In the standard electroweak theory, the bare parameters are connected by the unification condition

$$g_0 \sin \theta_W^0 = \epsilon_0 \quad (306)$$

and the bare Fermi constant is given by

$$\frac{G_F^0}{\sqrt{2}} = \frac{g_0^2}{8m_W^2}. \quad (307)$$

From Eqs. (306) and (307) it follows that

$$\frac{G_F^0}{\sqrt{2}} = \frac{\pi \alpha_0}{2 \sin^2 \theta_W^0 m_W^2}, \quad (308)$$

where $\alpha_0 = \epsilon_0^2/4\pi$.

Further, in the Standard Model with a Higgs doublet:

$$\sin^2 \theta_W^0 = 1 - \frac{m_W^2}{m_Z^2}. \quad (309)$$
If radiative corrections are taken into account, instead of expressions (308) and (309) we have

\[ G_F = \frac{\pi \alpha}{\sqrt{2} \sin^2 \theta_W m_W^2 (1 - \Delta r)} \quad (310) \]

\[ \sin^2 \theta_W = 1 - \frac{m_W^2}{m_Z^2} \quad (311) \]

Here \( G_F, \alpha, m_W, \) and \( m_Z \) are the measurable Fermi constant, the fine structure constant, and the masses of \( W \) and \( Z \), and \( \Delta r \) is the radiative correction.

The most precise value of the Fermi constant can be obtained from the measurement of the width of the decay

\[ \mu^- \to e^- + \bar{\nu}_e + \nu_\mu . \]

In the lowest order of the perturbation theory the diagram of this process is presented in Fig. 19.

Fig. 19: The lowest order diagram of the process \( \mu^- \to e^- + \bar{\nu}_e + \nu_\mu \).

Some of the higher order diagrams are depicted in Fig. 20.

Fig. 20: Diagrams of the process \( \mu^- \to e^- + \bar{\nu}_e + \nu_\mu \).

In the case under consideration \( q^2 \ll m_W^2 \) and the process \( \mu^- \to e^- + \bar{\nu}_e + \nu_\mu \) can be described by the effective Hamiltonian

\[ \mathcal{H}_I = \frac{G_F}{\sqrt{2}} 4 \nu_{\mu L} \gamma^\nu \mu_L \bar{e}_L \gamma_\nu \nu_{e L} + \text{h.c.} \quad (312) \]

where \( G_F \) is the effective Fermi coupling that takes into account contributions from higher order electroweak corrections. This constant is determined from the width of the decay \( \mu \to e + \bar{\nu}_e + \nu_\mu \),

\[ \Gamma_\mu = \frac{G_F^2 m_\mu^2}{192 \pi^5} \left( 1 - \frac{m_\mu^2}{m_e^2} \right) \left( 1 + \frac{\alpha}{2\pi} \left( \frac{25}{4} - \pi^2 \right) \right) \quad (313) \]

in which QED radiative corrections are taken into account. From the experimental data it follows that \( G_F = 1.16637(2) \times 10^{-5} \text{ GeV}^{-2} \). Now, from the diagrams in Figs. 19 and
we have

\[ \frac{G_F}{\sqrt{2}} = \frac{e_0^2}{8m_W^2 s_W} \left( 1 + \frac{\Sigma^W(0)}{m_W^2} + \ldots \right) \]

Here

\[ s_W^0 = 1 - \frac{m_W^0}{m_W^2} \]

and we have used the notations \( s_W^0 = \sin \theta_W^0 \), \( c_W^0 = \cos \theta_W^0 \).

For the bare masses of W and Z we have

\[ m_W^0 = m_W^2 \left( 1 + \frac{\delta m_W^2}{m_W^2} \right), \]

\[ m_Z^0 = m_Z^2 \left( 1 + \frac{\delta m_Z^2}{m_Z^2} \right). \]

From these relations for \( s_W^0 \) we obtain

\[ s_W^0 = 1 - \frac{m_W^0}{m_Z^2} \left( 1 + \frac{\delta m_W^2}{m_Z^2} - \frac{\delta m_Z^2}{m_W^2} \right). \]

Let us determine the renormalized parameter \( s_W^2 \) as follows:

\[ \sin^2 \theta_W = 1 - \frac{m_W^0}{m_Z^2}. \]

From Eqs. (316) and (317) we have

\[ s_W^0 = \sin^2 \theta_W + \cos^2 \theta_W \left( \frac{\delta m_Z^2}{m_Z^2} - \frac{\delta m_W^2}{m_W^2} \right). \]

Thus the bare parameter \( s_W^0 \) is equal to the physical one, determined by Eq. (310), plus a counterterm. Further, for the square of the bare charge we have

\[ e_0^2 = e^2 \left( 1 + 2 \frac{\delta e}{e} \right). \]

From Eqs. (314), (318), and (319) we obtain

\[ \frac{G_F}{\sqrt{2}} = \frac{e^2}{8m_W^2 \sin^2 \theta_W} (1 + \Delta r), \]

where

\[ \Delta r = 2 \frac{\delta e}{e} - \frac{\cos^2 \theta_W}{\sin^2 \theta_W} \left( \frac{\delta m_Z^2}{m_Z^2} - \frac{\delta m_W^2}{m_W^2} \right) + \frac{\Sigma^W(0)}{m_W^2} + \ldots. \]

Thus a measurable effective Fermi constant is connected with the constant \( \alpha \), the masses of W and Z (\( \sin^2 \theta_W = 1 - m_W^2/m_Z^2 \)), and the parameter \( \Delta r \), which is determined by higher order electroweak corrections.
The main contribution to $\Delta r$ comes from the first term of Eq. (321). We have

$$2 \frac{\delta e}{e} = \Pi'(0) - 2 \frac{\sin \theta_W \Sigma'(0)}{\cos \theta_W m_Z^2},$$

where the second term is due to the contribution of the diagram of Fig. 21

![Diagram](image)

Fig. 21

We have

$$\Pi'(0) = \text{Re} \ \Pi'(m_Z^2) - \text{Re} \ \Pi'(m_Z^2) + \Pi'(0) = \text{Re} \ \Pi'(m_Z^2) - \text{Re} \ \hat{\Pi}'(m_Z^2),$$

(322)

where $\hat{\Pi}(m_Z^2)$ is a renormalized vacuum polarization function at the point $m_Z^2$. The quantity $\text{Re} \ \hat{\Pi}'(m_Z^2)$ is related to the running of $\alpha$ up at the point $m_Z^2$:

$$e^2(m_Z^2) = \frac{e^2}{1 + \text{Re} \ \hat{\Pi}'(m_Z^2)}.$$  

(323)

We have from Eq. (323):

$$- \text{Re} \ \hat{\Pi}'(m_Z^2) = \frac{\alpha(m_Z^2) - \alpha(0)}{\alpha(m_Z^2)} = \Delta \alpha.$$  

For the light fermions ($m_f \ll m_Z^2$) from Eq. (295) we obtain

$$- \text{Re} \ \hat{\Pi}'(m_Z^2) = \frac{\alpha}{3\pi} \sum_f Q_f^2 \left( \ln \frac{m_Z^2}{m_f^2} - \frac{5}{3} \right).$$

(324)

The final result of the calculation of $\Delta r$ is the following:

$$\Delta r = \Delta \alpha - \frac{\cos^2 \theta_W}{\sin^2 \theta_W} \Delta \rho + \ldots \text{ (small Higgs contribution)},$$

(325)

$$\Delta \alpha = 0.0602 \pm 0.0009.$$  

Here

$$\Delta \rho = \frac{3\alpha}{16\pi \sin^2 \theta_W \cos^2 \theta_W m_Z^2} m_t^2,$$

(326)

where $m_t$ is the mass of the t quark.

The only unknown parameter in the relation (325) is $m_t$. From existing data it follows that

$$m_t = 164 \pm 16 \pm 20 \text{ GeV}.$$  

This is a very important prediction of the Standard Model of electroweak interaction.
Bibliography

The material discussed here can be found in books on quantum field theory and on elementary particle physics. See, for example:


