Abstract

Silent universes are studied using a “3+1” decomposition of the field equations in order to make progress in proving a recent conjecture that the only silent universes of Petrov type I are spatially homogeneous Bianchi I models. The infinite set of constraints are written in a geometrically clear form as an infinite set of Codacci tensors on the initial hypersurface. In particular, we show that the initial data set for silent universes is “non-contorted” and therefore (Beig and Szabados [1]) isometrically embeddable in a conformally flat spacetime. We prove, by making use of algebraic computing programs, that the conjecture holds in the simpler case when the spacetime is vacuum. This result points to confirming the validity of the conjecture in the general case. Moreover, it provides an invariant characterization of the Kasner metric directly in terms of the Weyl tensor. A physical interpretation of this uniqueness result is briefly discussed.

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1 Introduction

In the last few years there has been intense activity in the study of silent universes, which are dust spacetimes such that the fluid velocity vector is irrotational and the magnetic part of the Weyl tensor with respect to this vector vanishes. The concept was first put forward by S.Matarrese, O.Pantano and D.Sáez [2] in an astrophysical and cosmological context in order to describe structure formation in the universe. Moreover, silent universes were interesting also from an exact solutions point of view because this class contains the Szekeres family of spacetimes [3], which has been extensively analyzed in the literature (see [4] for a comprehensive review). The Szekeres class exhausts the silent universes of Petrov type D. Since silent universes are necessarily of Petrov type 0, D or I (Barnes and Rowlingson [5]) and the conformally flat case is well understood, the interesting new physics was in the Petrov type I subclass and a number of relevant new results were expected. This fact triggered active research on silent universes [6]-[11]. In particular, it was soon realized that the Einstein equations and the Bianchi identities, when written in a suitable orthonormal tetrad, decouple into a set of ordinary differential equations (which describe the time evolution of the spacetime) and a set of constraint equations (which only involve derivatives along spatial directions). Most of the studies focussed in analyzing the evolution equations, mainly by making use of a dynamical system formulation of the Einstein field equations (see [12] for a detailed account of the method). The existence problem of silent universes of Petrov type I was apparently settled in [10] where it was claimed that solving the constraint equations on an initial hypersurface is necessary and sufficient to obtain a silent universe (or, equivalently, that the time evolution of the constraints is automatically satisfied if the constraints hold initially). Unfortunately, this claim turned out to be untrue [13]. So, the problem of how large the class of Petrov type I silent universes is had to be reanalyzed. In two independent works [14]-[15] this issue was addressed. The analysis of the time-evolution of the constraints was performed in [14] using a coordinate approach and in [15] using a tetrad approach. In both cases it was found that the successive time derivatives of the constraints become larger and larger expressions which are algebraically independent from each other. This led these authors to conjecture that silent universes of Petrov type I are extremely scarce and that the whole class reduces to some Bianchi models. Proving this conjecture turns out to be a very difficult problem because the successive constraints become huge just after a few time derivatives and they become unmanageable even for algebraic computing programs. Our aim in this paper is twofold. First, we perform a study of silent universes using the ADM splitting [16] with respect to the hypersurface \( \Omega \) orthogonal to the fluid velocity. The main advantage of this method is that the full set of constraints can be written in a geometrically neat way as an infinite sequence of symmetric tensors satisfying the Codacci equation (i.e. a sequence of so-called Codacci tensors). Moreover, we show that the initial data set \((\Omega, h_{ab}, K_{ab})\) satisfies the so-called non-contorted condition. R.Beig
and L.B. Szabados [1] have recently proven that, locally, non-contorted initial data sets can be isometrically embedded in a conformally flat spacetime. Thus, each hypersurface orthogonal to the fluid velocity vector in a silent universe can be locally viewed as a hypersurface in a conformally flat spacetime. We have not been able to exploit this fact fully yet, but we believe that an appropriate use of this result could prove essential to settle the question on the non-existence of spatially inhomogeneous silent universes of Petrov type I. However, this question is not analyzed here any further. Rather, we concentrate in the second objective of the paper. The great complexity of analyzing the set of constraints in the general case simplifies in the vacuum case (essentially because there are five unknown functions instead of six). This allows us to prove, by making use of algebraic computing, that a vacuum silent universe of Petrov type I must be locally isometric to the Kasner spacetime [17] (which is a one-parametric family of spatially homogeneous Bianchi I vacuum spacetimes) thus showing that the conjecture holds in vacuum. Although the concept of silent universe was originally put forward for dust spacetimes, the definition makes obviously sense for vacuum spacetimes as well. The only difference is that in vacuum there is no privileged timelike congruence, unlike in the dust case, and the definition must be modified accordingly (see Definition 1). Since Petrov type D vacuum silent universes are just the vacuum subclass of the Szekeres family, and vacuum conformally flat silent universes are, of course, locally Minkowski, our result provides a complete classification of vacuum silent universes. This result is interesting at least in three respects. First, it suggests the validity of the conjecture in the general case. Furthermore, it provides an invariant characterization of the Kasner spacetimes without involving isometries (the Kasner family is well-known to be the most general spatially homogeneous Bianchi type I vacuum metric). Finally, this result can also be interpreted physically as follows. The uniqueness result we prove states that non-trivial vacuum silent spacetimes must be of Petrov type D (or 0) (we consider Kasner as trivial due to its high degree of symmetry). Gravitational fields of Petrov type D have often been considered analogous to Coulombian electromagnetic fields in flat space. This analogy is, obviously, loose because the Petrov type of a spacetime is a local property, while characterizing an electromagnetic field as Coulombian involves conditions on the decay rate near spatial infinity, which is a global property. Nevertheless, accepting this analogy and considering Petrov type D gravitational fields as “locally Coulombian”, our uniqueness result could be interpreted physically as establishing that vacuum gravitational fields which are purely electric with respect to an inertial observer must be of Coulombian type (except for the Kasner metric, which is a special case). A similar result also holds in electromagnetism in flat space, where it is true that electromagnetic fields which are purely electric with respect to an inertial observer must be Coulombian. However, this analogy should be taken cautiously because the terms “inertial” and “Coulombian” have a local meaning in gravity (corresponding to “geodesic irrotational observer” and “Petrov type D”, respectively) while they are global in flat space (meaning “orthogonal to hyperplanes” and “with decay rate as
Nevertheless, we believe that this analogy provides some physical explanation for the uniqueness result we have obtained.

The paper is organized as follows. In section 2 we describe briefly the ADM formalism for an arbitrary energy-momentum tensor. This fixes our notation and conventions. In section 3 we adapt these equations to the silent universe case. In particular, we show that the evolution equations can be written as a system of ordinary differential equations when suitable variables are chosen. Then, we concentrate in the study of the constraint equations. We prove that the whole set of constraints take the form of an infinite sequence of tensors \((S_n)^a_b\) which satisfy the so-called Codacci equation. The tensors \((S_n)^a_b\) depend algebraically on the second fundamental form \(K_{ab}\) and the Ricci tensor of \(h_{ab}\). Then, we show that the Codacci equation for \((S_0)^a_b\) and \((S_1)^a_b\) imply that the initial data set is non-contorted and quote the result by Beig and Szabados mentioned above. In Section 4 we study the Codacci equation by rewriting it as a Pfaffian system for one-forms. This allows us to study the integrability conditions of this equation easily and show that \((S_n)^a_b\) are symmetric and commute with each other. Using these results, the evolution equations are rewritten in terms of the eigenvalues of \((S_0)^a_b\) and \((S_1)^a_b\). In section 5 we restrict ourselves to the vacuum case and we prove our main theorem, namely that an arbitrary vacuum silent universe of Petrov type I is locally isometric to Kasner. The computations necessary to show this result have been performed using Reduce. However, the proof is not straightforward and needs some discussion. Hence we have included the source code of the program, as well as the necessary explanations in Appendix A. Reading this Appendix in detail requires some knowledge of Reduce but we believe that the general idea of the proof can be understood also by the non-expert just by following the comments we have included.

2 Preliminaries and basic results

In this section we review briefly the ADM formalism [16] and write down its basic equations. This will fix our notation and conventions.

The study of silent universes was motivated, among other reasons, by the fact that the Einstein field equations split into a set of ordinary differential equations (the evolution equations) and a set of constraint equations. In order to see why this happens, let us start by writing down the “3+1” equations for an arbitrary energy-momentum tensor. Consider a smooth irrotational timelike congruence of curves on a spacetime \((\mathcal{M}, g)\) with unit tangent vector \(\vec{u}\). Then, there exists an embedded hypersurface \(\Omega\) which is orthogonal to \(\vec{u}\), i.e. \(\Omega\) is a three-dimensional manifold and \(\varphi_0 : \Omega \to \mathcal{M}\) is an embedding such that \(\varphi_0^*(u) = 0\) \((u)\) is obtained by lowering the index to \(\vec{u}\) and

\[ r^{-2} \text{ at infinity} \).
the \( \star \) denotes the pullback, as usual). Let us choose an arbitrary point \( p \in \Omega \) and an open, connected neighborhood \( U_p \subset \Omega \) of \( p \) with compact closure. Let \( t \) denote the proper-time parameter of the congruence satisfying \( t = 0 \) on \( \varphi_0(U_p) \). Then, there exists a positive real number \( t_0 \) such that the integral curves of \( \vec{u} \) crossing \( \varphi_0(U_p) \) exist for \( t \in I_0 = (-t_0, t_0) \subset \) and do not intersect \( \varphi_0(U_p) \) again (also for \( |t| < t_0 \)). Let \( \Phi_t \) denote the one-parameter local group of transformations generated by \( \vec{u} \). The map \( \varphi_t = \Phi_t \circ \varphi_0|_{U_p} \) is an embedding \( \varphi_t : U_p \to M \) and \( \varphi_t(U_p) \) is a spacelike hypersurface in \( M \). The set \( \mathcal{M}_p \equiv \bigcup_{|t|<t_0} \varphi_t(U_p) \) is diffeomorphic to \( U_p \times I_0 \) and \( (\mathcal{M}_p, g|_{\mathcal{M}_p}) \) is globally hyperbolic. Given an \( r \)-index covariant tensor field \( T \) on \( \mathcal{M}_p \), we define a one-parameter family of covariant tensor fields on \( U_p \) by \( T_{j_1 \ldots j_r}(t) \equiv \varphi_t^*(T)_{j_1 \ldots j_r}, \ t \in I_0 \) (tensors on \( U_p \) carry Latin indices and tensors on \( M \) carry Greek indices). The definition of Lie derivative yields \( \varphi_t^*(\mathcal{L}_\vec{u} T)_{j_1 \ldots j_r} = \partial_t T_{j_1 \ldots j_r}(t) \). Define as usual the acceleration and the deformation tensor by \( a_\mu = u^\nu \nabla_\nu u_\mu \) and \( \Sigma_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta \nabla_\alpha u_\beta \), where \( h_\mu{}^\nu = \delta_\mu{}^\nu + u_\mu u^\nu \) is the projector. \( U_p \) can be endowed with a family of symmetric tensors \( h_{ab}(t) \) and \( K_{ab}(t), \ t \in I_0 \) by

\[
    h_{ab}(t) = \varphi_t^*(g)_{ab}, \quad K_{ab}(t) = \varphi_t^*(\Sigma)_{ab}.
\]

For fixed \( t \), \( h_{ab}(t) \) is a positive definite metric on \( U_p \). We denote the corresponding Levi-Civita covariant derivative by \( D^t \) and the Riemann and Ricci tensors by \( R_{abcd}(t) \) and \( R_{ab}(t) \). For objects at \( t = 0 \) we will drop the argument, i.e. \( h_{ab}(0) \), \( K_{ab}(0) \) and \( D^0 \) will be written simply as \( h_{ab} \), \( K_{ab} \) and \( D \) respectively (and similarly for other tensors on \( U_p \)). The Gauss and Codacci identities relating the Riemann tensors of \( (\mathcal{M}_p, g|_{\mathcal{M}_p}) \) and \( (U_p, h_{ab}(t)) \) are [19]

\[
    \varphi_t^*(\mathcal{R})_{abcd} = R_{abcd}(t) + K_{ac}(t)K_{bd}(t) - K_{ad}(t)K_{bc}(t),
\]

\[
    \varphi_t^*(\mathcal{R}(\vec{u}))_{abc} = D^t K_{ab}(t) - D^t K_{ac}(t),
\]

where \( \mathcal{R} \) is the Riemann tensor of \( (\mathcal{M}_p, g|_{\mathcal{M}_p}) \) with all the indices lowered, and \( \mathcal{R}(\vec{u}) \) is the tensor \( \mathcal{R}(\vec{u})_{\alpha\beta\gamma} = u^\nu \mathcal{R}_{\delta\alpha\beta\gamma} \). Since the Ricci and the Riemann tensors are equivalent in three dimensions, we can contract (1) with \( h^{ac}(t) \) without loss of information. The result is best written by introducing the electric and magnetic parts of the Weyl tensor in \( (\mathcal{M}_p, g|_{\mathcal{M}_p}) \) as \( E_{\alpha\beta} = u^\mu u^\nu C_{\mu\alpha\nu\beta} \), \( H_{\alpha\beta} = \frac{1}{2} u^\mu u^\nu \eta_{\mu\sigma\tau\alpha} C_{\sigma\tau\nu\beta} \), \( (\eta_{\alpha\beta\gamma\delta} \) is the volume form of the spacetime). Using now the Einstein field equations for an arbitrary energy-momentum tensor, we can decompose the Ricci tensor of \( (\mathcal{M}_p, g|_{\mathcal{M}_p}) \) as

\[
    \mathcal{R}_{\alpha\beta} = \Pi_{\alpha\beta} + (p + p) u_\alpha u_\beta + \frac{1}{2} (p - p) g_{\alpha\beta} - u_\alpha q_\beta - u_\beta q_\alpha.
\]

where \( \Pi_{\alpha\beta} \) is symmetric and trace-free and \( \Pi_{\alpha\beta}, \ q_\alpha \) are orthogonal to \( \vec{u} \). Let us define \( E_{ab}(t), \ H_{ab}(t), \ P_{ab}(t), \ q_\alpha(t), \ a_\alpha(t), \ \rho(t) \) and \( p(t) \) as the pullbacks with respect to \( \varphi_t \) of \( E_{\alpha\beta}, \ H_{\alpha\beta}, \ \Pi_{\alpha\beta}, \ q_\alpha, \ a_\beta, \ \rho \) and \( p \) respectively. The contracted Gauss identity reads

\[
    E_{ab}(t) + \frac{1}{2} P_{ab}(t) + \frac{2}{3} \rho(t) h_{ab}(t) = R_{ab}(t) + K(t) K_{ab}(t) - K_{ac}(t) K^c_b(t) \equiv S_{ab}(t) + S(t) h_{ab}(t), \quad \tag{3}
\]
where \( K(t) \equiv h^{ab}(t)K_{ab}(t) \), \( S(t) \equiv h^{ab}(t)S_{ab}(t) \), Latin indices are raised with \( h_{ab}(t) \) and the symmetric tensor \( S_{ab}(t) \) is defined through the second identity. The Codacci identity (2) can be written as

\[
\eta^{a}_{abc} \eta^{c}_{def} + 1/2 \left( h_{ac}(t) g_{b} - h_{ab}(t) g_{c}(t) \right) = D_{c}K_{ba}(t) - D_{b}K_{ca}(t) \equiv Z_{cba}(t). \tag{4}
\]

where \( Z_{cba}(t) \) is defined by the second identity and \( \eta^{a}_{abc} \) stands for the volume form of \((U_{t}, h_{ab}(t))\). Applying \( \varphi_{i}^{*} \) to the trivial identity \( \mathcal{U}_{a} g_{a\beta} = \nabla_{a}u_{\beta} + \nabla_{\beta}u_{a} \) yields

\[
\frac{\partial h_{ab}(t)}{\partial t} = 2K_{ab}(t). \tag{5}
\]

Finally, applying \( \varphi_{i}^{*} \) to \( \mathcal{U}_{a} \Sigma_{\mu\nu} = \Sigma_{\alpha\beta} \Sigma_{\beta}^{\alpha} + a_{\alpha}a_{\beta} + h_{\mu}^{\alpha}h_{\nu}^{\beta} \nabla_{\alpha}a_{\beta} - h_{\mu}^{\alpha}u^{\beta}h_{\nu}^{\gamma}u^{\delta} R_{\alpha\beta\gamma\delta} \) (which is a well-known, and in any case easily verifiable, identity) we get, after raising one index and using (3),

\[
\frac{\partial K_{c}(t)}{\partial t} = -K_{d}(t)K_{c}^{d}(t) + a^{b}(t)a_{c}(t) + D_{d}a^{b}(t) - S_{c}(t) + \Pi_{c}(t) - \frac{p(t)}{2} \delta_{c}^{b}. \tag{6}
\]

The geometrical identities (3),(4) are the so-called constraint equations and (5)-(6) are the evolution equations.

### 3 ADM formalism for silent universes

The evolution equations (5)-(6) are, in general, partial differential equations. The main property of silent universes is that, using an appropriate set of variables, the evolution equations become ordinary differential equations. To show this, we first write down the evolution equation for \( R_{b}(t) \). Let \( \Gamma_{a}^{b}(t) \) denote the Christoffel symbols of \( h_{ab}(t) \). An easy consequence of (5) is \( \partial_{t} \Gamma_{bd}(t) = D_{d} \Gamma_{b}^{c}(t) + Z_{c}^{a} \ h_{b}(t) \). The identity \( \partial_{t}R_{b}^{a}(t) = D_{d} \left[ \partial_{t} \Gamma_{bd}(t) \right] - D_{d} \left[ \partial_{t} \Gamma_{bd}^{c}(t) \right] \) becomes, after using the Ricci identity,

\[
\frac{\partial R_{b}^{a}(t)}{\partial t} = \left[ D_{c}Z_{b}^{ac} - D_{b}Z_{b}^{ac} + R_{c}^{a}K_{b} - KR_{b}^{a} - \frac{1}{2}RK_{b}^{a} + \delta_{b}^{a} \left( \frac{1}{2}RK - \text{Tr}(RK) \right) \right] \bigg|_{t} \tag{7}
\]

where \( \text{Tr}(RK)(t) = R^{b}_{b}(t)K^{a}_{a}(t) \) and “|\(_{t}” means that all the objects enclosed (including the covariant derivative) are to be taken at the value \( t \). The crucial fact in (7) is that spatial derivatives act only on the tensor \( Z_{abc}(t) \). Hence, by imposing conditions on this object it is possible to obtain a system of ordinary differential equations for \( K_{b}^{a}(t) \) and \( R_{b}^{a}(t) \). The simplest possibility is to set \( Z_{abc}(t) \equiv 0 \), which, from equation (4), is equivalent to \( q_{a}(t) \equiv 0 \) and \( H_{ab}(t) \equiv 0 \). Moreover, (6) shows that some evolution equations for \( a_{c}(t) \), \( p(t) \) and \( \Pi_{ab}(t) \) are needed in order to obtain a closed system of differential equations. Again, the simplest possibility is that they vanish identically. This motivates the standard definition of silent universe.
**Definition 1** A spacetime \((\mathcal{M}, g)\) is called a silent universe if it admits a unit timelike vector field \(\vec{u}\) with is irrotational and geodesic, the magnetic part of the Weyl tensor with respect to \(\vec{u}\) vanishes and the Ricci tensor takes the form \(R_{\alpha\beta} = \rho (u_\alpha u_\beta + \frac{1}{2} g_{\alpha\beta})\), where \(\rho\) is an arbitrary smooth function (possibly zero).

The condition that \(\vec{u}\) is geodesic is superfluous in the dust case \((\rho \neq 0)\) because of the contracted Bianchi identities, but it must be required additionally in the vacuum case.

The term “silent” stems precisely from the fact that the evolution equations become ordinary differential equations and hence, no influence from neighbouring points (apart from the one encoded in the initial data) arises during evolution.

From now on, the spacetime \((\mathcal{M}, g)\) will denote a silent universe. Then, the Codacci identity becomes

\[
Z_{a}^{c}(t) = D_{a}^{b} K^{c}_{b}(t) - D_{b}^{c} K^{a}_{b}(t) = 0
\]  

and the evolution equation for \(K^{a}_{b}(t)\) reads

\[
\frac{\partial K^{a}_{b}(t)}{\partial t} = - \left( K^{d}_{a} K_{c}^{d} + S_{c}^{d} \right)_{t}.
\]

Regarding the remaining evolution equation, it turns out to be more convenient to use \(S_{b}^{a}(t)\) instead of \(R_{b}^{a}(t)\). A straightforward, if somewhat long, calculation using (7) and (9) yields

\[
\frac{\partial S_{b}^{a}(t)}{\partial t} = \left[ 2 S_{c}^{a} K_{b}^{c} + K_{a}^{c} S_{b}^{c} - 2 S K_{b}^{a} - 2 K S_{b}^{a} + \delta_{a}^{b} (S K - \text{Tr}(S K)) \right]_{t},
\]

where \(\text{Tr}(S K)(t) = S_{b}^{a}(t)K_{a}^{b}(t)\). In order to derive this equation the following algebraic identity (which is valid for any \(3 \times 3\) matrix \(A\)) has been used

\[
A^{3} = \text{tr}(A) A^{2} + \frac{\text{Tr}(A^{2}) - \text{Tr}^{2}(A)}{2} A + \text{I}_{3} \left[ \frac{1}{3} \text{Tr}(A^{3}) - \frac{1}{2} \text{Tr}(A) \text{Tr}(A^{2}) + \frac{1}{6} \text{Tr}^{3}(A) \right].
\]

Thus, the evolution equations for silent universes are indeed very simple (the definition was designed for this purpose). In return, the set of constraints become highly non-trivial. Indeed, the set of constraints \(Z_{a b c}(t) = 0\) provide, after time differentiation, new constraints which must be satisfied identically. Let us describe this in detail. For the initial data problem, a triple \((U_{p}, h_{a b}, K_{a b})\) satisfying the Codacci constraint \(D_{[a} K_{b]} = 0\) must be given. Then, the tensor \(S_{a b}\) is defined via \(S_{a b} + Sh_{a b} = R_{a b} + K K_{a b} - K_{a c} K_{b}^{c}\). Next, \(K_{a}^{b}(t)\) and \(S_{b}^{a}(t)\) are obtained as the unique solutions of the ordinary differential system (9)-(10) with initial data \(K_{a}^{b}(0) = K_{a}^{b}\) and \(S_{b}^{a}(0) = S_{b}^{a}\). Afterwards \(h_{a b}(t)\) can be found as the unique solution of (5) (the right hand-side is already known) satisfying \(h_{a b} = h_{a b}(0)\). It is easy to show that the tensor \(S_{b}^{a}(t)\) constructed a posteriori from \(h_{a b}(t)\) and \(K_{a b}(t)\) coincides with the solution of (10) we started from. It remains the
check as to whether the Codacci equation for $K^b(t)$ is fulfilled for all $t$. A necessary condition is that all the time derivatives of $Z_{abc}(t)$ vanish at $t = 0$ (whether this is also sufficient would require extra, non-trivial work). In [14] and [15] the time derivatives were discussed in a coordinate and tetrad setting respectively, and they were found to become increasingly large and intractable (the explicit expressions for the first few constraints appear in [20]) but almost nothing about the geometric structure of the successive constraints was revealed. The ADM formalism turns out to be more useful for this purpose. To that aim, let us find an expression for the commutation between successive time derivatives of $D_a A^b_c(t)$ following sequence of one-parameter families of tensors on $U_p$. A simple calculation yields

$$\frac{\partial}{\partial t} \left( D^t_a N^b_c(t) \right) = \left[ D_a \left( \frac{\partial}{\partial t} N^b_c \right) + N^d_c D_a K^b_d - N^b_d D_a K^d_c \right]_t. \quad (11)$$

**Lemma 1** Let $(\mathcal{M}, g)$ be a silent universe and $(U_p, h_{ab}(t), K_{ab}(t))$ constructed as in Sect. 2. Let $A^b_c(t)$ be a family of tensors on $U_p$ satisfying $D^t_a A^b_c(t) = 0 \forall t$. Then the family of tensors

$$B^b_c(t) = \partial_t A^b_c(t) + K^b_d(t) A^d_c(t)$$

also satisfies $D^t_a B^b_c(t) = 0 \forall t$.

**Proof.** Taking the time derivative of $D^t_a A^b_c(t) = 0$ and using the commutation formula (11) we find

$$0 = \frac{\partial}{\partial t} \left( D^t_a A^b_c(t) \right) \bigg|_t = \left[ D^t_a \left( \frac{\partial}{\partial t} A^b_c \right) + A^t_d \left( D^t_a \right) (K^b_d) \right] \bigg|_t = D^t_a \left( A^d_c K^b_d + \frac{\partial}{\partial t} A^b_c \right) \bigg|_t$$

where $D^t_a K^b_c(t) = 0$ was used in the second equality and $D^t_a A^b_c(t) = 0$ in the third one.

This lemma, together with the constraint equations (8) suggests defining the following sequence of one-parameter families of tensors on $U_p$

$$(S_0)^a_b(t) = -K^a_b(t), \quad (S_n)^a_b(t) = \frac{\partial}{\partial t} \left[ (S_{n-1})^a_b(t) \right] + K^b_d(t) (S_{n-1})^d_c(t), \quad n \in \mathbb{Z}. \quad (12)$$

Lemma 1 shows that $(S_n)^a_b(t)$ satisfies $D^t_a (S_n)^c_b(t) = 0$, $\forall n \in \mathbb{Z} \cup \{0\}, \forall t$. Hence, the successive time derivatives of $Z_{abc}(t)$ at $t = 0$ can be written as $D^t_a (S_n)^c_b(t) = 0$, $\forall n \in \mathbb{Z} \cup \{0\}$ and these are the full set of constraints that the initial data $(U_p, h_{ab}, K_{ab})$ must satisfy.

Notice that the evolution equation (9) implies $(S_1)^a_b(t) = S^a_b(t)$. Furthermore, if follows from (9) and (10) that each $(S_n)^a_b(t)$ depends algebraically on $K^a_b(t)$ and $S^c_b(t)$. Moreover, it follows easily that this dependence is polynomial and that if we decide
that $K^a_b(t)$ carries a degree equal to one and $S^a_b(t)$ carries a degree equal to two, then $(S_n)^a_b(t)$ is homogeneous of degree $n + 1$ (i.e. it consists of a sum of terms of degree $n + 1$). Unfortunately, obtaining an explicit general formula for $(S_n)^a_b(t)$ in terms of $K^a_b(t)$ and $S^a_b(t)$ seems to be a difficult task. The first two constraint equations read simply

$$D_{[a} K^c_{b]} = 0, \quad D_{[a} S^c_{b]} = 0.$$  

In particular, this shows that the initial data set $(U_p, h_{ab}, K_{ab})$ is non-contorted (see [1] for a definition). In that paper, the authors prove that any non-contorted initial data set can, locally, be isometrically embedded in a conformally flat spacetime. Hence, each hypersurface orthogonal to the fluid flow in a silent universe can be locally embedded in a conformally flat spacetime. This already gives substantial geometric information about the initial data set (and only the first two constraints have been used!). We believe that this fact can provide the key for proving the conjecture that silent universes of Petrov type I must be spatially homogeneous (in terms of the initial data set, being spatially homogeneous is equivalent to $(U_p, h_{ab})$ being flat and $K_{ab}$ being covariantly constant). However we have not been able to exploit this fact fully yet and the question remains under investigation.

Summarizing, in this section we have obtained the evolution equations for the silent universes as ordinary differential equations in terms of the tensors $K^a_b(t)$ and $S^a_b(t)$ and we have written the full set of constraints in the form $D_{[a} (S_n)^c_{b]}(t) = 0 \quad \forall t$, for a collection of tensors $(S_n)^a_b(t)$ which depend algebraically on $K^a_b(t)$ and $S^a_b(t)$. Since all these tensors satisfy the same equation, the next section is devoted to study it in some detail.

### 4 Consequences of the Codacci equation

Let us start with a standard definition

**Definition 2** Let $(V, \gamma)$ be a (pseudo-)Riemannian manifold and $D$ the Levi-Civita covariant derivative. A symmetric tensor field $Q_{ab}$ is called a Codacci tensor iff it satisfies the Codacci equation

$$D_{[a} Q^c_{b]} = 0.$$  

(13)

Codacci tensors have received considerable attention in the mathematics literature (see [21] for an account) mainly because of the crucial rôle they play in isometric embeddings of Riemannian manifolds into Euclidean manifolds. In our context they are interesting because the full set of constraints for silent universes take the form of an infinite set of Codacci tensors $(S_n)^a_b(t)$ in the initial data set $(U_p, h_{ab}, K_{ab})$ (the symmetry of $(S_n)^a_b$ will be shown later). Let us now consider the integrability conditions of the
Codacci equation (13). To that aim, we will rewrite this equation as a differential system for one-forms (this way of writing the Codacci equation is new, as far as we know). Let $Q_{ab}$ be a Codacci tensor on $(\mathcal{V}, \gamma)$. Choose an arbitrary cobasis $\{\theta^a\}$ on this Riemannian manifold and denote by $\omega^a_b$ the connection one-forms in this basis. The torsion-free condition is $d\theta^a + \omega^a_b \wedge \theta^b = 0$. Denoting by $Q^a_b$ the components of the Codacci tensor $Q$ (with one index raised) in the basis $\{\theta^a\}$ and its dual $\{\bar{e}^a\}$, we can define three one-forms $Q^a_b$ by $Q^a_b \equiv Q^a_{cb} \theta^b$. A simple calculation gives

$$dQ^a + \omega^a_b \wedge Q^b = \left( D[Q^a]_b \right) \theta^c \wedge \theta^b = 0.$$ (14)

Thus, the Codacci equation for $Q_{ab}$ is equivalent to $dQ^a + \omega^a_b \wedge Q^b = 0$. To study the integrability conditions of this system we only need to take its exterior derivative, which gives

$$\Omega^a_b \wedge Q^b = 0 \quad (15)$$

where $\Omega^a_b$ are the curvature two-forms $\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$. Let us now assume that $\mathcal{V}$ is three-dimensional (this is the relevant case for silent universes) and define the so-called Ricci one-forms by $P^a_b = R^a_{bc} \theta^c$ ($R^a_{bc}$ are the components of the Ricci tensor in the triad $\{\theta^a\}$). The following identity holds in three dimensions

$$\Omega^a_b = P^a - P_b \wedge \theta^a - \frac{1}{2} R \theta^a \wedge \theta_b,$$

where indices are raised and lowered with $h_{ab} = h(e^a, e^b)$ (this identity just states the well-known equivalence between the Riemann and Ricci tensors in three dimensions). The fact that $Q_{ab}$ is symmetric can be written as $Q^a \wedge \theta_a = 0$ so that the integrability conditions (15) become $P_b \wedge Q^b = 0$ or, in index notation,

$$R^a_{bc} Q^b_c - Q^a_b R^b_c = 0.$$ (16)

This equation states that $R^a_{bc}$ and $Q^a_b$ commute when viewed as linear maps on the tangent space $T_q \mathcal{V}$, $q \in \mathcal{V}$. Let us now put $(\mathcal{V}, \gamma) = (U_p, h_{ab}(t))$. Applying (16) to the Codacci tensor $K^a_b$ we obtain, after using the definition of $S^a_b$ (3),

$$S^a_b K^b_c - K^a_b S^b_c = 0.$$ (17)

Since all $(S_n)^a_b$ are polynomials in $K^a_b$ and $S^a_b$, we obtain $(S_n)^a_b(S_m)_c - (S_m)^a_b(S_n)_c = 0$, $\forall n, m \in \mathbb{Z} \setminus \{0\}$. In particular, each tensor $(S_n)^a_b$ commutes with $K^a_b$. The defining recursion (12) then implies that $(S_n)_a^b$ are symmetric and hence Codacci tensors.

Obviously, the same arguments hold for an arbitrary value of $t \in I_0$. Thus $(S_n)^a_b(t)$ are Codacci tensors in $(U_p, h_{ab}(t))$ and they commute with each other (for fixed $t$).
Therefore, there exists an orthonormal basis of vectors \( \{ \hat{e}_a(t) \} \) in \((U_p, h_{ab}(t))\) such that \((S_n)_{ab}^p(t), \forall n\) diagonalize simultaneously. Denoting by \( \{ \hat{\theta}^a(t) \} \) the dual cobasis, we have

\[
K(t) = \sum_{c=1}^{3} \lambda_0^c(t) \hat{\theta}^c(t) \otimes \hat{e}_c(t), \quad S(t) = \sum_{c=1}^{3} \lambda_1^c(t) \hat{\theta}^c(t) \otimes \hat{e}_c(t),
\]

where \( K(t), S(t) \) denote the tensors with indices \( K_{0}^{ab}(t) \) and \( S_{0}^{ab}(t) \), and \( \lambda_0^c(t) \) and \( \lambda_1^c(t) \) are their respective eigenvalues. Writing equation (10) in this frame gives

\[
(\lambda_0^a - \lambda_0^b) \hat{\theta}^b \left( \frac{\partial}{\partial t} \hat{e}_a(t) \right) \bigg|_t = \delta_0^b \left[ -\frac{\partial \lambda_0^b}{\partial t} + 3\lambda_0^b \lambda_0^c - 2S \lambda_0^c - 2K \lambda_0^e + SK - \text{Tr}(SK) \right] \bigg|_t
\]

where the Einstein summation convention has been suspended. Until here no restriction on the Petrov type of the silent universe has been imposed. However, silent universes of Petrov type D and 0 are well-understood and attention can be restricted to Petrov type I. Of course, the Petrov type of a spacetime need not remain constant everywhere but it is well-known that the set of points with Petrov type I is open (see e.g. [22]) and therefore a submanifold, so we can assume Petrov type I everywhere without loss of generality (because only local properties are considered in this paper). For silent universes, Petrov type I is equivalent (see e.g. [5]) to \( E_{ab}(t) \) having three different eigenvalues (of course only two of them are linearly independent because of the treecfree condition of \( E_{ab}(t) \)). Eq. (3) shows that this is also equivalent to \( S_{ab}(t) \) having three different eigenvalues. Thus, (18) for \( a \neq b \) gives

\[
\hat{\theta}^b \left( \frac{\partial}{\partial t} \hat{e}_a(t) \right) = 0, \quad b \neq a,
\]

which allows us to rewrite equation (9) as

\[
\frac{\partial \lambda_0^a(t)}{\partial t} = - \left( \lambda_0^a(t) + [\lambda_0^b(t)]^2 \right).
\]

We now quote without proof a lemma due to Barnes and Rowlingson [5] which we will use in the vacuum case. The proof can be easily rewritten in our formalism by using the Codacci equation in the form (14) and the relations (18), (19) and (20).

**Lemma 2** (Barnes and Rowlingson [5]) Let \((\mathcal{M}, g)\) be a silent universe of Petrov type I and \((U_p, h_{ab}(t), K_{ab}(t))\) constructed as in Sect. 2. Fix an arbitrary point \( q \in U_p \). Then,

1. the subset of \( I_0 \) where \( K \mid_q(t) \) has three different eigenvalues is dense in \( I_0 \).

2. Fix an arbitrary value \( t \in I_0 \). Each one-form in the cobasis \( \{ \hat{\theta}^a(t) \} \) introduced above is integrable.

3. There exists a coordinate system \( \{ x, y, z \} \) in \( U_p \) in which \( K(t) \) and \( S(t) \) are diagonal and such that the metric \( h_{ab}(t) \) takes the form \( ds^2(t) = A(t, x, y, z)dx^2 + B(t, x, y, z)dy^2 + C(t, x, y, z)dz^2 \).
5 Vacuum silent universes of Petrov type I

Vacuum silent universes are characterized by $\rho = 0$ which, from equation (3), is equivalent to $S(t) = 0$. Hence, the evolution equation (18) takes the simpler form

$$\frac{\partial \lambda^a_i(t)}{\partial t} = \left[3\lambda^a_1\lambda^a_0 - 2\lambda^a_0 K - \text{Tr}(SK)\right]_t.$$  \hspace{1cm} (21)

We need the following lemma

**Lemma 3** Let $(\mathcal{M}, g)$ be a vacuum silent universe of Petrov type I and construct $(U_p, h_{ab}(t), K_{ab}(t))$ as in Sect. 2. Fix an arbitrary point $q \in U_p$. Let $\mu_q(t)$ denote one of the eigenvalues $\lambda^a_A|q(t)$, $A = 0, 1$ defined in (17). Then, the set $\{t \in I_0; \mu_q(t) \neq 0\}$ is dense in $I_0$.

**Proof.** Let first $\mu_q(t)$ be one of the eigenvalues of $S$. We can choose $\mu_q(t) = \lambda^1_1|q(t)$ without loss of generality. Suppose that the lemma does not hold, i.e. that $\mu_q(t)$ vanishes for $t \in I_1$ where $I_1$ is open and non-empty. Then, equation (21) implies $\text{Tr}(SK)|q(t) = 0 \forall t \in I_1$, which, after using $S(t) = 0$, implies $\lambda^0_1(t) - \lambda^0_3(t)|q = 0 \forall t \in I_1$. This is impossible from the first conclusion of Lemma 2. Let now $\mu_q(t)$ be one of the eigenvalues of $K(t)$, say $\lambda^0_1|q(t)$. The claim of the lemma follows easily because if $\mu_q(t)$ vanished on a non-empty open set $I_1$, then (20) would imply $\lambda^1_1|q(t) = 0$ on $I_1$, which we have just shown to be impossible.

This lemma combined with Lemma 2 shows that there exists an open dense subset $W \subset U_q \times I_0$ where the eigenvalues $\lambda^a_0(t)$ are non-zero and mutually distinct and that the same holds for $\lambda^a_1(t)$. Then, the following parametrization exists on $W$

$$\lambda^1_0(t) = w|t, \quad \lambda^0_2(t) = w(1 + u)|t, \quad \lambda^3_0(t) = w(1 + v)|t,$$

$$\lambda^1_1(t) = w^2s(k + 1)|t, \quad \lambda^2_1(t) = w^2s(k - 1)|t, \quad \lambda^3_1(t) = -2w^2sk|t,$$

where $w, u, v, s$ and $k$ are nowhere vanishing scalar functions on $W$. Furthermore, the combinations $u - v, v + 1, u + 1, 3k - 1, 3k + 1, k - 1$ and $k + 1$ are also nowhere zero on $W$ (these statements just translate the fact that $S(t)$ and $K(t)$ have three different and non-vanishing eigenvalues on $W$). This type of parametrization is convenient for the algebra computing calculations we describe in Appendix A. The evolution equations (20) and (21) take the following form on $W$

$$\partial_tw = -w^2(sk + s + 1), \quad \partial_tu = w(usk + us + 2s - u^2 - u),$$
$$\partial_tv = w\left(ksv + 3ks + (v + 1)(s - v)\right), \quad \partial_tk = w\left(-ku + 2kv - \frac{1}{2}u - \frac{3}{2}k^2u\right),$$
$$\partial_ts = ws\left(2sk - 1 + 2s - \frac{u}{2} - 2v - \frac{3}{2}ku\right).$$  \hspace{1cm} (22)

We are now in a position to prove our main theorem.
Theorem 1 Let \((\mathcal{M}, g)\) be a vacuum silent universe of Petrov type I. Then the spacetime is locally isometric to a Kasner spacetime.

Before proving this result, let us recall that the Kasner spacetime [17] is defined as the manifold \(+ \times ^3\) endowed with the metric
\[
ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \quad t > 0, \quad -\infty < x, y, z < +\infty
\] (23)
where \(p_1, p_2, p_3 \in \) and satisfy the relations \(p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1\). Hence each element of the family is parametrized by one real parameter. This spacetime contains a three-dimensional abelian isometry group acting transitively on the spacelike hypersurfaces \(t = \) const. Is is thus a Bianchi type I vacuum spacetime. Furthermore, it is well-known (see e.g. [18]) that any spacetime admitting an abelian three-dimensional Lie algebra of Killing vectors which span spacelike hypersurfaces must be locally isometric to Kasner (a global isometry requires further conditions on the topology of the spacetime). Hence, the Kasner family can be locally characterized by the existence of these Killing vectors. Theorem 1 provides another characterization for the Kasner family directly in terms of the Weyl tensor and without involving isometries, namely, a vacuum spacetime which is of Petrov type I and silent must be locally isometric to Kasner. Obviously a global isometry cannot be expected because the Petrov type I and the silent conditions are purely local and they place no restriction on the global topology of the spacetime (which could be, for instance, \(\times T^3\), where \(T^3\) is the three-dimensional torus, endowed with the metric (23)).

The proof of Theorem 1 involves combining several constraint equations in order to show that they lead to incompatibilities unless the spacetime is very simple, namely a homogeneous Bianchi model. Since the expressions involved soon become very large, the proof would be impossible (using this direct method) without employing algebraic computing. However, the proof can be followed in exactly the same way as an ordinary proof can. The only requirement is some knowledge of Reduce, which is the algebraic computing program we have used. Even for those who are not familiar with Reduce, we believe that the general idea of the proof can be followed with relatively little effort. Hence, the Reduce program and the necessary explanations on the logic of the proof have been included in Appendix A (the meaning of the commands is not explained, for the interested reader we recommend the introduction to Reduce by M.A.H.MacCallum and F.Wright [23]).

Proof of the Theorem.

Let us consider a point \(p \in \mathcal{M}\) and construct \((U_p, h_{ab}(t), K_{ab}(t)), t \in I_0\) as described in Sect 2. Using Lemma 2, the metric \(h_{ab}(t)\) in \(U_p\) can be written as
\[
ds^2(t) = A(t, x, y, z) dx^2 + B(t, x, y, z) dy^2 + C(t, x, y, z) dz^2.
\] (24)
In Appendix A we proof that, under the assumptions of the theorem, the functions \(A, B\) and \(C\) depend only on \(t\). Then, the spacetime metric in \(\mathcal{M}_p, g|_{\mathcal{M}_p}\) can be
reconstructed from $h_{ab}(t)$ in the usual way to give $ds^2|_{\mathcal{M}_p} = -dt^2 + A(t)dx^2 + B(t)dy^2 + C(t)dz^2$. Thus, the spacetime admits three commuting Killing vectors which span spacelike hypersurfaces. Hence, the metric must must take the form (23) for some values of $p_1$, $p_2$ and $p_3$. Conversely, a simple calculation shows that the Kasner spacetime is a silent universe of Petrov type I. This concludes the proof of the theorem. \hfill \Box

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Appendix

This Appendix contains the bulk of the proof of Theorem 1. The aim is to prove that the one-parametric family of metrics (24) on $U_p$ (for vacuum silent spacetimes of Petrov type I) can be written in the form $ds^2(t) = A(t)dx^2 + B(t)dy^2 + C(t)dz^2$. In order to prove this, we use the algebraic computing program Reduce. In this Appendix we include the code lines, which are written using the sans serif font, and we explain the logic of the proof (this is written in plain text). A few key output results\footnote{In Reduce, a code line ending with \$ does not produce any written output (of course the operation is performed). If the line ends with \";\" the result is written in the screen (or the standard output).}, which are required to follow the proof, are also included. These will be written in italic characters.

\begin{verbatim}
load-package groebner$ on gcd$
depend w,t,z$
depend u,t,z$
depend v,t,z$
depend s,t,z$
depend k,t,z$

Only the dependence on $t$ and $z$ of the functions is prescribed. Of course, all of these functions depend on $y$ and $x$ as well, but we will not invoke any equations containing partial derivatives with respect to $x$ or $y$. This may seem inadequate because we are not exploiting all the available equations. However, the strategy of the proof relies on showing a certain property for the $z$ variable (the vanishing of some associated Christoffel symbols) and then use the property that the $z$ variable is on the same footing as the $x$ and $y$ variables (there is an obvious symmetry between $x < -y < -z$ in the metric (24)) in order to imply the vanishing of some other Christoffel symbols. Hence concentrating only on the $z$ variable is sufficient.

Let us now introduce the evolution equations (22) and the set of Codacci tensors ($S_n$).

\begin{verbatim}
s(0,1):=-w$s(0,2):=-w*(1+u)$s(0,3):=-w*(1+v)$s(1,1):=w^2*s*(1+k)$s(1,2):=w^2*s*(k-1)$s(1,3):=-2*w^2*s*k$
\end{verbatim}
\end{verbatim}
\begin{align*}
\text{dwt} &:= -w^2(s^2 k + s + 1) \\
\text{dut} &:= w(v^2 s k + v^2 s + 3 k^2 s - v^2 - v) \\
\text{dst} &:= w(s^2 (2 s k - 1 + 2 s - u^2 - 2 s v - 3 - 2 k u)) \\
\text{dkt} &:= w(-k u + 2 k v - u^2 + 2 k^2 u) \\
\text{df}(w, t) &:= \text{dwt} \\
\text{df}(u, t) &:= \text{dut} \\
\text{df}(v, t) &:= \text{dst} \\
\text{df}(s, t) &:= \text{dkt} \\
\text{df}(k, t) &:= \text{dkt} \\
d &:= 6 \\
\text{for } l := 2 : d \text{ do for } i := 1 : 3 \text{ do } s(l, i) := \text{df}(s(l-1, i), t) - s(l-1, i) \\
\text{cdz}(l, 1) &:= \text{df}(s(l, 1), z) + \Gamma_{1,1,3}^1 (s(l, 1) - s(l, 3)) \\
\text{cdz}(l, 2) &:= \text{df}(s(l, 2), z) + \Gamma_{2,2,3}^2 (s(l, 2) - s(l, 3)) \\
\text{df}(w, z) &:= \text{rhs first solve(cdz(0, 1), df(w, z))} \\
\text{df}(u, z) &:= \text{rhs first solve(cdz(0, 2), df(u, z))} \\
\text{df}(v, z) &:= \text{rhs first solve(cdz(1, 2), df(v, z))} \\
\text{for } i := 0 : (d-3) \text{ do } r(i) := \text{num cdz}(i+3, 2)/w^{4+i}/s \\
\text{for } i := 0 : (d-4) \text{ do } n(i) := \text{resultant}(r(0), r(i+1), \Gamma_{1,1,3}^1)/2 \\
\text{Our next aim is to analyze the case in which } \Gamma_{2,2,3}^2 \equiv 0 \text{ on a non-empty open subset } \tilde{W} \subset W \text{ and } \Gamma_{1,1,3}^1 \text{ is nowhere zero on } \tilde{W}. \\
\text{gam}(2, 2, 3) &:= 0 \\
\text{pro} &:= r(0)/\text{gam}(1, 1, 3) \\
\text{proz} &:= \text{df(pro, z)}/\text{gam}(1, 1, 3) \\
\text{prozz} &:= \text{df(proz, z)}/\text{gam}(1, 1, 3) \\
n1 &:= -2 v^2 + 12 v (k - 1) - 9 k^2 + 30 k - 13 \\
n2 &:= 9 k - 4 v - 7 \\
factorize\{\text{prozz} - n1*\text{pro} - n2*\text{proz}\}; \{18, -v, v, 3k + 1, 3k + 1, k - 1\} \\
\text{But this is impossible because no element in this list can vanish anywhere on } W. \\
\text{This concludes this case. Next, we consider the situation in which } \text{gam}(2, 2, 3) \neq 0 \text{ everywhere on } W \text{ (from now on we will restrict } W \text{ to an open dense subset thereof whenever necessary and without further notice).} \\
\text{Subcase 0: } c_0 = 0 \text{ identically on } W_1. \\
c0 &:= -27 k^4 u^2 - 72 k^3 v u + 36 k^3 u^2 + 96 k^2 v^2 - 96 k^2 v u
\[+42k^2u^2 - 24k^2v^2u + 12k^2u^2 + u^2\]
\[c0t:=\text{diff}(c0,t)/2/w\]  
\[c01:=\text{resultant}(c0t,pol1,s)\]
\[\text{rg}:=\text{resultant}(c0,c01,v)/(3k+1)^2/(3k-1)^2/k^2/3131031158784/u^14\]
\[\text{rg}+(-117k^4+30k^2-1)^2;\]
\[0\]

So, \(-117k^4 + 30k^2 - 1\)|\(_{W_1}\) \(\equiv 0\). In particular \(k\) is constant and its derivative is identically zero. Then
\[
\text{part}(\text{groebner}\{dkt/w*2,-117k^4+30k^2-1,c0\}),4);
\]
\[u^2\]

which is impossible because \(u\) cannot vanish anywhere on \(W\). So we can assume \(c_0 \neq 0\) everywhere on \(W\) and divide out this factor whenever necessary. This finishes Subcase 0. We continue with the generic case.

\[v:=u*l\]  
\[u1:=\text{resultant}(\text{pol1},\text{pol2},s)/(3k+1)/(3k-1)/(u-v)/v/u^7/c0/324\]
\[u2:=\text{resultant}(\text{pol1},\text{pol3},s)/c0/162/v/u^6/(u-v)\]

By construction, the polynomials \(u_1\) and \(u_2\) depend only on \(k, l\) and \(u\).

\[q1:=\text{sub}(k=2,l=2,u1)\]  
length coeff(q1,u); length coeff(u1,u);
\[5\]  
\[q2:=\text{sub}(k=2,l=2,u2)\]  
length coeff(q2,u); length coeff(u2,u);
\[7\]

The two polynomials \(q_1\) and \(q_2\) depend only on \(u\). The two last output results show that \(q_1\) and \(u_1\) are of the same degree as polynomials in \(u\) and the same happens for \(q_2\) and \(u_2\).

\[
\text{groebner}\{q1,q2\};
\]
\[\{1\}\]

Hence \(q_1\) and \(q_2\) have no common zeros. From this we can conclude that the resultant of \(u_1\) and \(u_2\) with respect to \(u\) is not identically zero. Indeed, if it were zero, this would mean that the polynomials \(u_1\) and \(u_2\) have common solutions for \(u\) at any value of \(k\) and \(l\). In principle, it could happen that the common solution for \(u\) tends to \(\infty\) when we approach \(k = 2\) and \(l = 2\), but this can also be excluded because \(q_1\) and \(u_1\) are polynomials in \(u\) of the same degree (and similarly for \(q_2\) and \(u_2\)). So, there should exist at least one finite value of \(u\) which solves \(q_1 = 0\) and \(q_2 = 0\) simultaneously. But we have proven that \(q_1\) and \(q_2\) have no common zeros. Thus, the resultant of \(u_1\) and \(u_2\) with respect to \(u\) is a non-identically vanishing polynomial in \(l\) and \(k\). The reason why we do not prove this by direct calculation is that the polynomials are already very big and performing the full resultant takes far too long.

So, \(k\) and \(l\) must belong to the set of zeros of a polynomial in \(k\) and \(l\), which we denote by \(\sigma\) (we view \(\sigma\) as as a subset of the \(\{k,l\}\) plane). Next, we prove that
restricting the polynomial $u_1$ to any point on $\sigma$ provides a non-trivial polynomial in $u$ (i.e. non-constant). Similarly, we show that $pol_2$ (which depends on $k, l, u, s$) when restricted to any point on $\sigma$ is a non-trivial polynomial in $s$. These two facts imply that $\{k, l, u, s\}$ must lie on a one-dimensional manifold. So, let us first prove that $u_1$ is a non-trivial polynomial in $u$ at any point on $\sigma$.

\[
\begin{align*}
j_1 &= \text{lterm}(u_1, u)/u^4 \quad j_2 := \text{lterm}(u_2, u)/u^6 \\
j_3 &= \text{sub}(u=0, u_1)/(3k + 1)/(3k - 1) \\
g &= \text{groebner}\{j_3, j_4\} \\
y_1 &= \text{resultant}(j_1, g_2, l)/(k - 1)^3/(k + 1)^5/k^{12}/(3k + 1)^7/(3k - 1)^7/471859200 \\
y_2 &= \text{resultant}(j_2, g_2, l)/3774873600/(3k - 1)^6/(3k + 1)^6/(k + 1)^5/(k - 1)^3/k^{14} \\
\end{align*}
\]

which proves that $u_1$ is nowhere identically zero on $\sigma$. Since a constant non-zero value is also impossible (recall that $u_1 = 0$ is a consequence of the Codacci equations) the claim follows. Regarding $pol_2$, it suffices to notice that the leading term (in the variable $s$) of this polynomial is

\[
\begin{align*}
\text{lterm}(pol_2, s) \text{ on factor} \quad &\text{ws; off factor} \\
-44(3k + 1)^2(3k - 1)^2s^3 \\
\end{align*}
\]

which is nowhere zero on $W$. Hence, $u, k, v, s$ lie on a curve. Therefore, there exists a non-zero vector field $\vec{a} = a_1 \partial_z + a_2 \partial_t$ ($a_1$ and $a_2$ unknown functions on $W$) which annihilates $k, s, u, v$. Consider first the possibility that $a_1$ vanishes on a non-empty open set, i.e. suppose that $k, s, u, v$ do not depend on $t$ on that open set. Then

\[
\begin{align*}
v &:= \text{rhs first solve}(2*dkt/w, v) \\
s &:= \text{rhs first solve}(2*k*dst/w/s, s) \\
\text{third groebner}(\text{num dut/w, num dvt/w, num pol1/u, num pol2, u, k})/(3k + 1)^7/2 \\
4k^2 - 1 &\quad \text{Hence } k \text{ is constant.} \\
\text{first groebner}(\{\text{num dut/w, num dvt/w, 4*k^2-1}\}, \{u, k\}); \\
\{8k + 3u + 8, 4k^2 - 1\} &\quad \text{So, } u \text{ is also constant. In particular, its derivative with} \\
\text{respect to } z \text{ vanishes.} \\
\text{first groebner}(\{df(u, z), r(0), 2*u+8+8*k, 4*k^2-1\}; \text{clear } s) &\quad \text{clear } v \\
\text{gam}(2, 2, 3) \\
\end{align*}
\]

This is impossible in the case we are analyzing. So, we can assume without loss of generality that the vector $\vec{a}$ takes the form $\vec{a} = \partial_z - b \cdot \text{gam}(2, 2, 3)/w\partial_t$ for some unknown function $b$. This provides a new set of equations (we also rewrite $\text{gam}(1,1,3) = f*\text{gam}(2,2,3)$ where $f$ is an unknown function)

\[
\begin{align*}
\text{clear } v &\quad \text{gam}(1,1,3) := \text{gam}(2,2,3)*f \\
\text{pr}(1) &:= 2*(df(k, z)-b*\text{gam}(2,2,3)*df(k, t)/w)/\text{gam}(2,2,3) \\
\text{pr}(2) &:= 2*(df(s, z)-b*\text{gam}(2,2,3)*df(s, t)/w)/\text{gam}(2,2,3)/s \\
\end{align*}
\]
\[ \text{pr}(3) = \frac{df(u,z) - b \cdot \text{gam}(2,2,3) \cdot df(u,t) / w}{\text{gam}(2,2,3)} \]
\[ \text{pr}(4) = \frac{df(v,z) - b \cdot \text{gam}(2,2,3) \cdot df(v,t) / w}{\text{gam}(2,2,3)} \]
\[ \text{pr}(5) = \frac{r(0)}{\text{gam}(2,2,3)} \]

for \( i := 1:4 \) do \( x(i) := \text{resultant}(\text{pr}(1), \text{pr}(i+1), f) \)
\[ z1 := \frac{-\text{resultant}(x(1), x(2), b)}{(k-1)(3k+1)/2} \]
\[ z2 := \frac{(-2 \cdot \text{resultant}(x(1), x(3), b)/(k-1)(3k+1)/2 + (3k+1) \cdot z1)/(3k-1)}{3k-1} \]
\[ z3 := \frac{-\text{resultant}(x(1), x(4), b)/(k-1)(3k+1)/2}{3k-1} \]
\[ f1 := \frac{9k^2u - 24k + 30k + 3u}{2} \]
\[ f2 := \frac{-9k^2u - 6k - 36k + 3u + 12v + 12}{2} \]
\[ z3 := \frac{z3 - f1 \cdot z1 - f2 \cdot z2}{(3k-1)} \]

Before following with the general case, we must study a few particular cases. They are characterized by the vanishing of one of the following expressions on a non-empty open subset \( W_2 \subset W \):

\[ c1 := 2k^2 + 3k + 2k + v + 1 \]
\[ c2 := 2k^2 + 3k + 1 \]
\[ c3 := -3k^2 + 2k^2v + 2k + 2k^2v + 1 \]
\[ c4 := -4k^3 + 3k^3 - 10k^2 + 2k + 2k^2 + 1 \]

**Subcase 1:** \( c_1 \) vanishes identically on \( W_2 \).

\[ c1t := 2df(c1,t)/w \]
\[ v := \text{rhs first solve}(c1,v) \]
\[ c1t := \frac{c1t}{3k+1} \]
\[ u := \text{rhs first solve}(c1t,u) \]
\[ v := \text{rhs first solve}(c1t,v) \]

Thus, \( k = 1/5 \) or \( 2ks + 2s - 1 = 0 \) on some non-empty open set. \( k = 1/5 \) is impossible as the following calculation shows:

\[ k := 1/5 \]

Thus, \( s := 1/(k+1) \)

This is also impossible and therefore \( c_1 \) cannot vanish on \( W_2 \). As usual, we can assume that it is nowhere zero on \( W \).

**Subcase 2:** \( c_2 \) vanishes identically on \( W_2 \).

\[ c2t := df(c2,t)/w + 2s \]
\[ g := \text{groebner} \{ dkt/w, pol1 \} \]
\[ s := 1/(k+1) \]

Thus, \( k \) must by constant and hence \( s, v \) and \( u \) are also constants. In particular they do not depend on \( t \) and we have seen above that this is impossible for \( \text{gam}(2,2,3) \neq 0 \).

**Subcase 3:** \( c_3 \) vanishes identically on \( W_2 \).
\[ c_{3t} := \frac{df(c_3, t)}{w} \]

\[ gr := \text{groebner}(\{c_3, c_{3t}\}, \{s, v\}) \]

\[ v := \text{rhs first solve}(\text{second } gr, v) \]

\[ s := \text{rhs first solve}(\text{first } gr, s) \]

\[ cc := \text{factorize}(\text{num } z_1/(3k+1)) \]

\[ \text{length } cc; 2 \]

\[ \text{second groebner}(\{\text{first } cc, z_2\}, \{u, k\}) \text{ on factor}; \text{ off factor} \]

\[ (3k^2 + 1)(k^2 + 1)(3k + 1)(3k - 1)(k - 1) \]

\[ \text{second groebner}(\{\text{second } cc, pol1, z_2\}, \{u, k\}) \text{ on factor}; \text{ off factor} \]

\[ (3k + 1)(3k - 1)^2(k - 1) \]

\[ \text{clear } u; \text{ clear } v; \text{ clear } s \]

This gives a contradiction to \( c_3 = 0 \) on \( W_2 \), hence \( c_3 \) is non-zero everywhere on \( W \).

**Subcase 4:** \( c_4 \) vanishes identically on \( W_2 \)

\[ c_{4t} := 2*\frac{df(c_4, t)}{w} \]

\[ \text{groebner}(\text{coeff}(c_4, v)); \{1\} \]

This shows that we can solve for \( v \)

\[ v := \text{rhs first solve}(c_4, v) \]

\[ \text{groebner}(\{z_1, z_2, 5k^2-1, c_{4t}\}); \{1\} \]

We can solve \( s \) in \( c_{4t} \)

\[ s := \text{rhs first solve}(c_{4t}, s) \]

\[ \text{length } \text{coeff}(\text{resultant}(\text{num } z_1, \text{num } z_2, u), k); 52 \]

Hence, \( k \) must be constant and its derivative must be identically zero. Then

\[ \text{second groebner}(\{\text{dkt}, z_1, z_2\}, \{u, k\})/\text{num}(v) \text{ on factor}; \text{ off factor} \]

\[ \text{clear } v; \text{ clear } s; (k^2 + 1)(3k - 1) \]

Again a contradiction. This concludes the last particular case.

**Generic Case:** We can now consider the generic situation (we divide by \( c_1, c_2, c_3 \) and \( c_4 \) whenever necessary)

\[ h(1) := \text{resultant}(z_1, z_2, s)/c_1/2 \]

\[ h(2) := \text{resultant}(z_1, z_3, s)/c_1/2 \]

\[ h(3) := \text{resultant}(z_1, \text{pol1}, s) \]

\[ t(1) := \text{resultant}(h(1), h(2), u)/9216/(k^4/(k+1)^2/(3k-1)^2/(3k+1)^2/c_2/c_3^2/(c_4^2/(k-1)/(v+1)/v) \]

\[ t(2) := \text{resultant}(h(1), h(3), u)/442368/(k^6/(3k-1)^4/v^3/(3k+1)^3/c_3^2/c_4^2/(v+1)/(k+1)) \]

As happened above, evaluating the resultant of \( t(1) \) and \( t(2) \) with respect to \( u \) is unworkable and we must use the same type of argument employed above in order to show that this resultant is not identically zero. Consider the particular value \( k = 2 \) in \( t(1) \) and \( t(2) \). The resulting polynomials, called \( tt(1) \) and \( tt(2) \) are shown to be of the same degree as \( t(1) \) and \( t(2) \) respectively (as polynomials in \( u \)). We then show that \( tt(1) \) and \( tt(2) \) have no common zeros. Hence the resultant of \( t(1) \) and \( t(2) \) with respect to \( u \) is not identically zero.

\[ tt(1) := \text{sub}(k=2, t(1)) \]

\[ \text{length } \text{coeff}(tt(1), v); 9 \]

\[ \text{length } \text{coeff}(t(1), v); 9 \]
\tt(2):=\text{sub}(k=2,\tt(2))$

$ length \text{coeff}((\tt(2),v); 14$ \text{length} \text{coeff}(\tt(2),v); 14$
groebner(\tt(1),\tt(2))$;

$\{1\}$

Since the resultant($t(1),t(2),u$) depends only on $k$ and it is non-zero, we can conclude that $k$ must be constant. It only remains to show that $v$, $u$ and $s$ are also constants. This is shown in the next few lines

\text{groebner}(\text{coeff}(t(1),v));

$\{1\}$ \quad So, $v$ must be constant

\text{factorize lterm}(h(1),u);

$\{6, u, u, u, k, k - 1, k + 1, 3k + 1\}$ \quad $u$ must also be constant

\text{factorize lterm}(z3,s);

$\{32, -s, s, k, k + 1, 3k + 1\}$ \quad $s$ is also constant.

But $k, s, u, v$ constants is impossible when $\text{gam}(2,2,3) \neq 0$ (because, in particular, their time derivative should vanish and we have excluded this case before). Thus, it follows that $\text{gam}(2,2,3) \neq 0$ on $W$ is impossible. At the very beginning we also showed that $\text{gam}(2,2,3) = 0$ and $\text{gam}(1,1,3) \neq 0$ on a non-empty open subset is impossible. Hence, $\text{gam}(1,1,3) = \text{gam}(2,2,3) = 0$ everywhere on $W$. Furthermore, there is an obvious symmetry between the coordinates $x, y$ and $z$ (there is nothing special about $z$). It follows necessarily $\text{gam}(1,1,3) = \text{gam}(2,2,3) = \text{gam}(1,1,2) = \text{gam}(3,3,2) = \text{gam}(2,2,1) = \text{gam}(3,3,1) = 0$ everywhere on $W$. Since $W$ is dense on $U_p \times I_0$ (we have restricted $W$ always to smaller open dense subsets thereof) all these Christoffel symbols vanish identically on $U_p \forall t \in I_0$. In terms of the metric (24) at $t = 0$, this means $ds^2|_{t=0} = A(x)dx^2 + B(y)dy^2 + C(z)dz^2$. Performing a trivial coordinate change we obtain $ds^2|_{t=0} = A\tilde{x}d\tilde{x}^2 + B\tilde{y}d\tilde{y}^2 + C\tilde{z}d\tilde{z}^2$. The Codacci equations $D[aK^c_{b}] = D[aS^c_{b}] = 0$ imply that $K^c_{b}(t)$ and $S^c_{b}(t)$ are diagonal constant matrices in the coordinate system $\{\tilde{x}, \tilde{y}, \tilde{z}\}$. Then, the evolution equations (9), (10) imply that $K^c_{b}(t)$ and $S^c_{b}(t)$ are independent of $\tilde{x}, \tilde{y}, \tilde{z}$. Finally, the evolution equation (5) shows that $h_{ab}(t)$ is diagonal and depends only on $t$ (in the coordinate system $\{\tilde{x}, \tilde{y}, \tilde{z}\}$). After dropping the tildes we readily obtain $ds^2(t) = A(t)dx^2 + B(t)dy^2 + C(t)dz^2$, which was the aim of this Appendix.

References


