The Weak-Coupling Limit of 3D Simplicial Quantum Gravity

P. Bialas\textsuperscript{a,b}, B. Petersson\textsuperscript{a} and G. Thorleifsson\textsuperscript{a}

\textsuperscript{a} Fakultät für Physik, Universität Bielefeld, 33501 Bielefeld, Germany
\textsuperscript{b} Institute of Comp. Science, Jagellonian University, 30072 Krakow, Poland

We investigate the weak-coupling limit, \( \kappa \to \infty \), of 3D simplicial gravity using Monte Carlo simulations and a Strong Coupling Expansion. With a suitable modification of the measure we observe a transition from a branched polymer to a crinkled phase. However, the intrinsic geometry of the latter appears similar to that of non-generic branched polymer, probably excluding the existence of a sensible continuum limit in this phase.

1. INTRODUCTION

\( D \)-dimensional simplicial quantum gravity is a discretization of Euclidean quantum gravity with the integration over space-time metrics replaced by a sum over all possible \( D \)-dimensional triangulations constructed by gluing together equilateral simplexes. It is defined by the partition function

\[
Z(\mu, \kappa) = \sum_{N_D} e^{-\mu N_D} Z(\kappa, N_D) = \sum_{N_D, N_0} e^{-\mu N_D + \kappa N_0} W_D(N_0, N_D),
\]

where \( N_D = \# D \)-simplexes, \( N_0 = \# \) vertices, and \( C_T \) is the symmetry factor of a labeled triangulation chosen from a suitable ensemble \( T \) (eg combinatorial). \( \mu \) and \( \kappa \) are the discrete cosmological and Newton’s coupling constants.

In \( D = 3 \) and 4 this model has two phases:

- \( \kappa < \kappa_c \) : a (intrinsically) crumpled phase
- \( \kappa > \kappa_c \) : a branched polymer phase

separated (regrettably) by a discontinuous phase transition [1].

As a discontinuous phase transition excludes a sensible continuum limit, there have been several attempts to modify the model Eq. (1) in the hope of finding a non-trivial phase structure. This includes: adding a measure term [2]

\[
W_D(N_0, N_D, \beta) = \sum_{T \in \mathcal{T}(N_0, N_D)} \frac{1}{C_T} \prod_{i=0}^{N_0} q_i^\beta,
\]

where \( q_i \) is the order of the vertex \( i \) (number of simplexes containing \( i \)), and coupling matter fields to the geometry [3].

Such modifications do indeed lead to a more complicated phase diagram (Fig. 1), and suitable modified the model exhibits a new crinkled phase. But does this new phase structure imply a more interesting non-trivial critical behavior? To investigate this we have studied the weak-coupling limit, \( \kappa \to \infty \), of the model Eq. (1) for \( D = 3 \).

2. THE EXTREMAL ENSEMBLE

In the weak-coupling limit the partition function Eq. (1), in \( D = 3 \) and 4, is expected to be dominated by an Extremal Ensemble (EE) of triangulations. For this ensemble, defined as tri-
angulations with the maximal ratio \( N_0/N_D \), the partition function simplifies:

\[
Z(\mu) = \sum_{N_D} e^{-\mu N_D} W_D(N_D) \tag{4}
\]

\[
W_D(N_D) = \sum_{T \in T(N_0^\text{max}, N_D)} \frac{1}{C_T}
\]

where

\[
N_0^\text{max} = \begin{cases} 
\left\lfloor \frac{N_1 + 10}{3} \right\rfloor & D = 3, \\
\left\lfloor \frac{N_1 + 18}{4} \right\rfloor & D = 4. 
\end{cases} \tag{5}
\]

Here \([x]\) denotes the floor function — the biggest integer not greater than \(x\). This in turn defines several distinct series for the EE:

\[
\begin{align*}
3D &: S^0 (N_0, 3N_0 - 10), \quad S^1 (N_0, 3N_0 - 9), \\
    & S^2 (N_0, 3N_0 - 8).
\end{align*}
\]

\[
4D : S^0 (N_0, 4N_0 - 18), \quad S^1 (N_0, 4N_0 - 17).
\]

Assuming the asymptotic behavior \( W_D(N_D) \sim \exp(-\mu_c N_D) N_D^{-\gamma} \), which defines the string susceptibility exponent \( \gamma \), we observe (from a SCE) that for the different series \( S^k \), \( \gamma^k = k + \frac{1}{2} \). This difference in the exponent \( \gamma \) can be understood as the 'higher' series \((k = 1, 2, \ldots)\) can be constructed by introducing \( k \) “defects” (marked points) into triangulations belonging to the minimal series \( S^0 \). Moreover, we observe that the minimal series appears to have very small finite-size effects.

The minimal series \( S^0 \) can be explicitly enumerated as it corresponds to \( D \)–dimensional combinatorial stacked spheres (CSS), i.e. to the surface of a \((D + 1)\)–dimensional simplicial cluster. The number of \((D + 1)\)–dimensional simplicial clusters build out of \( n (D + 1) \) simplexes, rooted at a marked outer face, is given by (where \( n = N_0 - D - 1 \)) \[4\]

\[
\epsilon_{D+1,n} = \sum_{n_1 + \cdots + n_{D+1} = n - 1} \epsilon_{D+1,n_1} \cdots \epsilon_{D+1,n_{D+1}} = \frac{1}{nD + 1} \binom{(D+1)n}{n} \tag{6}
\]

\[
\Rightarrow W_D(N_D) = \frac{D + 2}{N_D} \epsilon_{D+1,n}^{\frac{n-2}{D+1}}. \tag{7}
\]

Expanding this gives

\[
W_4(N_4) = \frac{10}{\sqrt{2\pi} N_4^{3/2}} \frac{256}{27} \left( \frac{N_4^{3/2}}{3125} \right)^{N_4/2} \left( 1 + \frac{83}{48} \frac{1}{N_4} \cdots \right)
\]

\[
W_3(N_3) = \frac{6\sqrt{5}}{\sqrt{2\pi} N_3^{3/2}} \frac{3125}{256} \left( \frac{N_3^{3/2}}{3125} \right)^{N_3/2} \left( 1 + \frac{33}{20} \frac{1}{N_4} \cdots \right)
\]

with \( \gamma = 1/2 \) as expected for branched polymers.

\section{A Modified Measure}

We have investigated the 3D EE including a measure term, using both MC simulations and a SCE \[5\]. We find a continuous phase transition to a crinkled phase at \( \beta = -1 \) (Fig. 2). This is evident in the fluctuations both in the measure term — the “specific heat” \( C_V \) — and in the maximal vertex order \( p_0 \). Scaling analysis of the peak value of specific heat gives: \( C_{V,\text{max}} \approx a + b N_3^{-0.34(4)} \).

To explore the fractal properties of the geometry in the crinkled phase we have measured...
the variations in $\gamma$ with $\beta$ using several different methods (Fig. 3). As in $D = 4$, we find that $\gamma$ becomes negative at $\beta_c$ and decreases with $\beta$. Similarly we find a spectral dimension that increases from $d_s = 4/3$ for $\beta > \beta_c$, to $d_s \approx 2$ as $\beta \to \infty$.

Estimates of the intrinsic fractal dimension $d_H$ differ, on the other hand, substantially depending on how it is defined — on the direct graph (from a vertex-vertex distribution) or on the dual graph (simplex-simplex distribution). The former yields $d_H \to \infty$, the latter $d_H \approx 2$. In addition, we observe that the crinkled phase appears dominated by a gas of sub-singular vertices.

Combined this evidence suggests that the crinkled phase probably corresponds to some kind of non-generic branched polymers phase which makes it unlikely that any sensible continuum limit exist in this phase. This of course does not exclude the possibility that a second order phase transition point exists somewhere on the phase boundary, for example at the end of the first order transition line (Fig. 1).

4. DEGENERATE TRIANGULATIONS

The EE can also be defined with the ensemble of degenerate triangulations introduced in Ref. [6]. In this case degenerate stacked spheres (DSS) are constructed by slicing open a face and inserting a vertex. Different from CSS, Eq. (5), DSS are defined by the maximal ratio:

$$\frac{N_0}{N_D} = \frac{1}{2} + \frac{D}{N_D}.$$  

This ensemble can also be enumerated explicitly. Modifying the measure leads to identical phase structure as is observed for CSS. This is shown in Fig. 4 where we plot the variations in $\gamma$ with $\beta$ for the two ensembles. That the two, very different, ensembles agree on the fractal structure is reassuring and reflects the universal properties of the crinkled phase.

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REFERENCES