Statistical Mechanics of Charged Black Holes in Induced Einstein-Maxwell Gravity

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Abstract

The statistical origin of the entropy of charged black holes in models of induced Einstein-Maxwell gravity is investigated. The constituents inducing the Einstein-Maxwell action are charged and interact with an external gauge potential. This new feature, however, does not change divergences of the statistical-mechanical entropy of the constituents near the horizon. It is demonstrated that the mechanism of generation of the Bekenstein-Hawking entropy in induced gravity is universal and it is basically the same for charged and neutral black holes. The concrete computations are carried out for induced Einstein-Maxwell gravity with a negative cosmological constant in three space-time dimensions.

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1 Introduction

The Bekenstein-Hawking entropy $S^{BH}$ is one of the most intriguing features of black holes. It is generally believed that it is impossible to find its statistical-mechanical explanation in the framework of the classical Einstein gravity. It is more likely that $S^{BH}$ hints to a more fundamental theory of quantum gravity which provides black holes with microscopic degrees of freedom. Such a theory may be quite complicated, like the string theory. Yet one may expect that the mechanism of generation of $S^{BH}$ does not depend on the details and is universal. One of the possibilities to understand this mechanism \cite{1}–\cite{6} is to use the idea of Sakharov’s induced gravity \cite{7}. Sakharov’s basic assumption is that the gravity becomes dynamical as a result of quantum effects of constituent fields. In the models of induced gravity the Bekenstein-Hawking entropy $S^{BH}$ of a black hole has the following statistical-mechanical form

$$S^{BH} = S^{SM} - Q.$$  \hspace{1cm} (1.1)

Here $S^{SM}$ is the statistical-mechanical (entanglement) entropy of the constituents located near the horizon and $Q$ is the Noether charge which appears because of non-minimal couplings. Relation (1.1) has been demonstrated for static \cite{2}–\cite{4} and rotating \cite{6} black holes in four space-time dimensions. These works however considered a pure induced gravity and, hence, the black hole solutions had no charges. In a more realistic situation one may expect that all other long-range fields are induced along with the gravitational one on an equal footing. By extending in this way Sakharov’s assumption one could model a fundamental theory which unifies the gravity with other forces of the Nature and investigate a more general class of black holes.

The aim of the present paper is to consider an induced Einstein-Maxwell gravity, as a simplest model of such a unified theory. In this picture, the graviton $g_{\mu\nu}$ and photon $A_\mu$ are not fundamental, and are collective excitations of the constituent fields. As we will see, the constituents now are to be charged and interact with $A_\mu$ as with an external potential. The action for $A_\mu$ is completely induced by the vacuum polarization and in the low-energy limit it coincides with the Maxwell action. In principle, the additional interaction of the constituents with the black hole electromagnetic field at the horizon can change the statistical entropy $S^{SM}$. We show, however, that this interaction does not affect the divergent part of $S^{SM}$ and in the low-energy limit relation (1.1) for charged black holes preserve its form. Thus, the mechanism of generation of the Bekenstein-Hawking entropy by the constituents seems to be universal and not depending on the type of a black hole.

To simplify the analysis we study induced Einstein-Maxwell gravity in three dimensions, more exactly, the Einstein-Maxwell gravity with a negative cosmological constant. The main interest to this theory is that it admits charged black holes \cite{8},\cite{9} and mimics some properties of the four-dimensional theory. There are also other reasons which motivate our choice. First, in three dimensions, one can easily construct induced gravity
models which are completely free from ultraviolet divergences while in four-dimensions eliminating all the divergences becomes a complicated problem. Second, a special class of solutions in three-dimensional gravity are BTZ black holes [8] whose entropy admits an alternative statistical representation in terms of a conformal field theory [10].

The paper is organized as follows. In Section 2 we construct a model of induced Einstein-Maxwell gravity with negative cosmological constant in three-dimensional space-time. The thermodynamics of charged black holes in this theory is discussed in Section 3. In Section 4 we consider the properties charged fields near a charged black hole. We show that for non-extremal black holes with a weak electric field the gauge interaction near the horizon slightly shifts the mass of a particle. This cause, however, a very little effect on the density of energy levels of the field. By using this result in Section 5 we investigate the statistical-mechanical origin of the Bekenstein-Hawking entropy of charged black holes in three-dimensional induced Einstein-Maxwell gravity. Our concluding remarks in Section 6 concern charged black holes in four-dimensional induced Einstein-Maxwell gravity. Some details regarding computation of the induced action can be found in Appendix.

2 Induced Einstein-Maxwell gravity

The classical Einstein-Maxwell gravity is the theory of the interacting gravitational field, $g_{\mu\nu}$, and the Abelian gauge field $A_\mu$. The corresponding (diffeo- and gauge-invariant) action is

$$I[g, A] = \frac{1}{4} \int d^3x \sqrt{-g} \left[ \frac{1}{4\pi G} (R - 2\Lambda) - F^{\mu\nu} F_{\mu\nu} \right]. \quad (2.1)$$

We consider this theory in three dimensions. In (2.1), $R$ is the scalar curvature defined for the metric $g_{\mu\nu}$, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the Maxwell strength tensor, $G$ and $\Lambda$ are the gravitational (Newton) and cosmological constants, respectively. If the cosmological constant is negative, one of the solutions of (2.1) is anti-de Sitter space-time. For this reason and for the brevity we will call such a theory AdSM-gravity (anti-de Sitter-Maxwell gravity).

In the induced gravity approach action (2.1) is generated as an effective action for a system of quantum constituents. Constituents of our model are charged massive scalar and spinor fields on the space-time with the metric $g_{\mu\nu}$. The fields do not interact to each other but interact to the gauge field $A_\mu$. The theory is described by the quantum action

$$\Gamma[g, A] = \sum_s W_s[g, A] + \sum_d W_d[g, A], \quad (2.2)$$

$$W_s[g, A] = \frac{i}{2} \log \det \left( -D_\mu D^\mu + \xi_s R + m_s^2 \right), \quad (2.3)$$

$$W_d[g, A] = -i \log \det \left( \gamma^\mu D_\mu + m_d \right). \quad (2.4)$$
At least some of scalars are non-minimally coupled with the corresponding constants $\xi_s$. We have $N_s$, scalars with masses $m_s$ and $N_d$ spinors with masses $m_d$. The covariant derivative for the $k$-th constituent with the charge $e_k$ is

$$D_\mu = \nabla_\mu + e_k A_\mu.$$  \hfill (2.5)

Note that in three dimensions $A_\mu$ has the dimensionality of (mass)$^{1/2}$ and, thus, the elementary charges $e_k$ have a nontrivial dimensionality of (mass)$^{1/2}$.

The theory (2.2) in three dimensions has a very important property. It is completely free from the ultraviolet divergences if the parameters of the model subject to the constraints (see Appendix)

$$2N_s - 2N_d = 0 \hspace{1cm} 2 \sum_s m_s^2 - 2 \sum_d m_d^2 = 0 \hspace{1cm} 2N_s + N_d - 12 \sum_s \xi_s = 0. \hspace{1cm} (2.6)$$

In three dimension the spinors have two components and for this reason equations (2.6) coincide with the constraints which provide finiteness of induced 2D gravity [5].

Suppose that all masses have the order of magnitude of some specific scale $M$. Now if the curvature of the space-time and the strength of the gauge field are small, i.e.,

$$|R| \ll M^2 \hspace{1cm} |F^{\mu\nu} F_{\mu\nu}| \ll M^3,$$  \hfill (2.7)

action (2.2) can be approximated by local decomposition in the curvature and the strength tensor. In the leading approximation it coincides with classical action (2.1)

$$\Gamma[g, A] \simeq I[g, A],$$  \hfill (2.8)

where the induced gravitational and cosmological constants are determined by the parameters of the quantum constituents (see Appendix for details)

$$\frac{1}{G} = \frac{1}{3} \left(2 \sum_s (6\xi_s - 1)m_s - \sum_d m_d\right),$$  \hfill (2.9)

$$\frac{\Lambda}{G} = \frac{8}{3} \left(\sum_d m_d^3 - \sum_s m_s^3\right).$$  \hfill (2.10)

It follows from (2.6) and (2.10) that the induced gravity requires that at least some scalar constituents are non-minimally coupled with positive parameters $\xi_s$ in order to provide positivity of $G$. Such models can be constructed for the certain choices of $m_s$, $m_d$ and $\xi_s$. In what follows we will also assume that the induced cosmological constant is negative.

It should be noted, that although the constraints in 2D and 3D induced gravities are the same and the theories are free from all the divergencies there is an important difference between them. The action of induced 2D gravity formulated in form (2.1) does not provide dynamical equations for the metric. That is why one has to introduce massless constituents to get the induced 2D gravity in the Liouville form [5]. The attractive feature of 3D gravity (2.1) is that its properties are similar to the Einstein-Maxwell gravity in four dimensions and it has a number of interesting black hole solutions.
3 Thermodynamics of charged black holes in AdSM gravity

A static charged black hole solution in 3D AdSM-gravity (2.1) was found in [8]. It can be written in the form

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\phi^2 , \quad (3.1) \]

\[ N^2 = \frac{1}{a^2} (r^2 - r_+^2) - \frac{2G}{\pi} q^2 \ln \frac{r}{r_+} , \quad (3.2) \]

where \( a^2 = -1/\Lambda \). The point \( r = r_+ \) corresponds to the black hole horizon, which in three dimensions is a circle. The corresponding gauge potential is

\[ A(r) = -\frac{q}{2\pi} \ln \frac{r}{r_+} dt , \quad (3.3) \]

where \( q \) is the charge of the black hole. We impose condition \( A_t(r_+) = 0 \) which provides regularity of the vector field on the horizon. One can also find a generalization of this solution, a rotating charged black hole, see [9], but we consider only static black holes, for simplicity.

According to the Bekenstein-Hawking formula, the entropy of black hole (3.1)–(3.3) is

\[ S_{BH} = \frac{1}{4G} A = \frac{\pi r_+}{2G} , \quad (3.4) \]

where \( A \) is the length of the black hole horizon. The corresponding Hawking temperature is determined by the surface gravity \( \kappa \) of the horizon

\[ T_H = \frac{\kappa}{2\pi} = \frac{(N^2)'_r}{4\pi} \bigg|_{r=r_+} = \frac{r_+}{2\pi a^2} \left( 1 - \frac{Gq^2a^2}{\pi r_+^2} \right) . \quad (3.5) \]

The black hole becomes extremal \( (T_H = 0) \) at

\[ r_+^2 = \frac{Gq^2a^2}{\pi} . \quad (3.6) \]

By using definitions (3.4) and (3.5) one finds a variational formula

\[ \delta \mathcal{E}_H = T_H \delta S_{BH} + \Phi \delta q , \quad (3.7) \]

\[ \mathcal{E}_H = \frac{r_+^2}{8Ga^2} + \frac{q^2}{4\pi} \ln \frac{R}{r_+} + C \quad , \quad (3.8) \]

\[ \Phi = -A_t(R) \quad , \quad (3.9) \]

where \( R \) and \( C \) are dimensional constants. Note that in an arbitrary gauge, when the condition \( A_t(r_+) \) is not imposed, \( \Phi \) is the difference between electric potentials at the black hole horizon and at some point \( r = R \).
Formula (3.7) has the form of the first law of thermodynamics of a Reissner-Nordström black hole, the parameter $\mathcal{E}_H$ being identified with the energy. If an AdS black hole has no charge one can define its energy by using Abbott-Deser [11] generalization of the ADM mass. This energy can be equivalently written as the integral [12]

$$M_{ADM} = -\frac{1}{8\pi G} \int_{C_R} N^2 K^{-1} \bar{K}$$

(3.10)

computed at the spatial circular boundary $C_R$ at $r = R$. Here $^1K = N(R)/R$ is the extrinsic curvature of $C_R$. Definition (3.10) requires a subtraction of a "reference background" mass. This results in the term in (3.10) depending on the "reference" extrinsic curvature $^1\bar{K}$. In the considered case the natural choice for the "reference background" is the anti-de Sitter space which is another solution of AdSM-gravity

$$ds^2 = -\frac{N^2(R)}{\bar{N}^2(R)} N_0^2(r) dt^2 + \bar{N}^{-2} dr^2 + r^2 d\phi^2$$

(3.11)

$$\bar{N}^2(r) = 1 + \frac{r^2}{a^2}.$$  

(3.12)

The normalization of the Killing vector $\partial_t$ coincides at $r = R$ with normalization of $\partial_t$ for the black hole space-time. Now, if one adopts (3.10) also as the definition of the energy of a charged black hole and chooses $C = -1/(8G)$ in (3.8),

$$\mathcal{E}_H = M_{ADM}.$$  

(3.13)

Two remarks are in order regarding this equation. Firstly, in three dimensions the electric potential diverges logarithmically at infinity. That is why the mass of the black hole cannot be made finite even after subtracting the "reference AdS mass. Secondly, for asymptotically anti-de Sitter spaces the temperature redshifts to zero at infinity. Thus, the standard interpretation of $\mathcal{E}_H$ as an internal thermodynamic energy of a black hole is note quite correct. For the conventional formulation of black hole thermodynamic [13] one should use the quasilocal energy [14]. However, the only difference between this energy and $\mathcal{E}_H$ is in the redshift factor $N(R)$.

It is worth also pointing out another possible definition of the black hole energy which will be important for us later. According to Wald and Iyer [15], one can generally define the black hole energy in terms of the Noether charge and a boundary function,

$$M = \int_{C_R} (Q_{\mu\nu}(t)n^\mu u^\nu + NB)$$

(3.14)

Here $Q_{\mu\nu}$ is the Noether potential associated to the Killing field $\partial_t$, $u^\mu$ and $n^\mu$ are unit normals to $C_R$, inward-pointing and future-directed, respectively. Function $B$ comes out from the boundary term in the action and is introduced to have a well-defined variational procedure. By using formulas of [15] one finds for the Einstein-Maxwell theory (2.1)

$$Q_{\mu\nu}(t) = \frac{1}{8\pi G} t_{\mu\nu} - F_{\mu\nu} A^\lambda,$$  

(3.15)
\[ B = -\frac{1}{8\pi G} K, \]  
(3.16)

where \( K \) is the extrinsic curvature of the spatial boundary of the black hole space-time (at \( r = R \)). One can define the corresponding energy \( M \) for reference AdS space-time (3.11). For these definitions the black hole energy takes the form

\[ M' = M - \bar{M} = M_{\text{ADM}} - q\Phi, \]  
(3.17)

where the last term comes out from the contribution of the gauge field in the Noether potential (3.15). One can also show that energy (3.17) coincides with Hawking-Horowitz [12] definition of the mass as the surface term in the Hamiltonian. According to the general result of Wald and Iyer [15], the first law of black hole thermodynamics looks as follows

\[ \delta M = T_H \delta S_{BH}. \]  
(3.18)

Actually, there is no contradiction between this formula and the first law (3.7). Equation (3.18) holds for boundary conditions which require fixing the value of the potential \( \Phi \) on the boundary. By taking into account (3.17) and the fact that \( \bar{M} \) is fixed one can obtain (3.7) from (3.18).

4 Charged fields near a charged black hole

We begin with the discussion of some features related to interaction of a charged field and electric field of a black hole near the horizon. For simplicity we will be dealing again with static black holes only. However, our basic conclusions will hold for rotating black holes as well. We also restrict the analysis by non-extremal black holes, and then comment on extremal ones. The metric and the gauge potential can be taken in the form

\[ ds^2 = -N^2(x)dt^2 + g_{ik}(x)dx^i dx^k, \]  
(4.1)

\[ A = A_t(x)dt. \]  
(4.2)

The Killing horizon is the surface where \( N^2(x) = 0 \) and it is assumed that on this surface \( A_t = 0 \). We consider a space-time with an arbitrary dimension. Let us investigate the spectrum of single-particle excitations of a scalar field described by the Klein-Gordon equation

\[ (-D^\mu D_\mu + m^2)\phi = 0, \quad D_\mu = \nabla_\mu + ieA_\mu, \]  
(4.3)

where \( e \) is the charge of the field. By substituting the wave-function with the energy \( \omega \)

\[ \phi_\omega(t, x) = e^{-i\omega t} \phi_\omega(x), \]  
(4.4)

one obtains the equation

\[ H^2(\omega)\phi_\omega(x) = \omega^2 \phi_\omega(x), \]  
(4.5)
\[ H^2(\omega) = N^2 [\Delta_x + V(\omega)] \quad , \quad (4.6) \]
\[ \Delta_x \equiv \frac{1}{\sqrt{-g}} \partial_i \sqrt{-g} g^{ik} \partial_k \quad , \quad (4.7) \]
\[ V(\omega) = m^2 - 2\omega e A_t N^{-2} - e^2 A_t^2 N^{-2} \quad . \quad (4.8) \]

Equation (4.5) has the form of the relativistic Schrödinger equation with a specific "Hamiltonian" \( H(\omega) \) where potential term (4.8) depends on the frequency \( \omega \). For antiparticles one obtains the same potential term (4.8) where charge \( e \) should be replaced by \(-e\) (or \( \omega \) by \(-\omega\)).

In general, solution of problem (4.5) may be quite complicated. However, the effect of the gauge interaction near the horizon is easy to understand. In the vicinity of the horizon of a nonextremal black hole

\[ N^2 \simeq \kappa^2 \rho^2 \quad , \quad (4.9) \]
\[ A_t \simeq -\frac{\kappa}{2} E_+ \rho^2 \quad , \quad (4.10) \]
\[ V(\omega) \simeq m^2(\omega) = m^2 + e\omega E_+ \kappa^{-1} \quad , \quad (4.11) \]

where \( \kappa \) is the surface gravity constant and \( \rho \) is the proper distance to the horizon (located at \( \rho = 0 \)).

Therefore, interaction near the horizon with the electric field of the black hole shifts effectively masses of fields.

The parameter \( E_+ \) in (4.10) is the strength of the electric field on the horizon,

\[ E_+ = l^\mu p^\nu F_{\mu\nu} \quad , \quad (4.12) \]

where \( l^\mu \) and \( p^\mu \) are two mutually orthogonal normals to the bifurcation surface, \( l^2 = -p^2 = 1 \), \( p^\mu \) is future directed. For charged black hole (3.2), (3.3)

\[ E_+ = \frac{q}{4\pi r_+} \quad . \quad (4.13) \]

Now, we can take into account that quantum fields which are in a thermal equilibrium with the black hole have to be at the Hawking temperature \( T_H = \kappa/2\pi \). Thus, the main contribution into observable quantities comes from frequencies \( \omega \simeq T_H \) while contribution from \( \omega \gg T_H \) is exponentially small. It means that in order to estimate the effect one can assume that

\[ H^2(\omega) \simeq \kappa^2 \rho^2 \left[ \Delta_x + \tilde{m}^2 \right] \quad , \quad (4.14) \]
\[ \tilde{m}^2 = m^2 + \sigma e E_+ \quad , \quad (4.15) \]

where \( \sigma \simeq O(1) \) is a numerical coefficient. Correspondingly, for antiparticles the charge \( e \) in \( \tilde{m}^2 \) is replaced by \(-e\).
The effective mass $\tilde{m}$ depends on the strength of the electric field. If the electric field is strong, $|E_+| \gg m^2/|e|$, it creates particle–anti-particle pairs and this process results in the instability of the quantum state.

In what follows we assume that the field is weak and there is no pair creation process. As we will see, this condition is satisfied in the induced gravity in the "low energy" limit. In this case, the physical picture is basically the same as for neutral black holes or uncharged fields [16] and the shift of the masses is an irrelevant effect. In particular by using results for uncharged fields one concludes that the spectrum of energies $\omega$ is continuous and does not have the mass gap. The density of the energy levels $dn/d\omega$, of $H(\omega)$ is divergent near the horizon, and this results in the divergence of the entropy of quantum fields. The leading divergence, however, is the same as for uncharged fields.

Now we briefly comment on extremal black holes. In this case the behavior of charged fields is different. Near the horizon of an extremal black hole $N^2 \sim (r - r_+)^2$ and $A_t \sim (r - r_+)$, see, e.g., Eqs. (3.2), (3.3) and (3.6). Thus, the ratio $A_t/N^2$ is singular at $r = r_+$, and the effect cannot be described by shift of the mass (4.11). Instead, behavior of a charged particle near the horizon is similar to moving in a flat space with the potential term $\pm |q\omega A_t|$. However, because this potential vanishes it cannot seriously change the spectrum of single-particle Hamiltonian (4.6) and bring new features into the considered problem. Anyway, the Hawking temperature of extremal black holes is zero and it seems there is no much sense in considering statistical-mechanical entropy of the constituents. We will not be discussing extremal black holes anymore.

5 Black hole entropy in AdSM gravity

We are now ready to discuss statistical mechanical interpretation of the black hole entropy $S^{BH}$ in induced Einstein-Maxwell gravity. We relate $S^{BH}$ to the entanglement entropy $S^{SM}$ of the constituent fields. To regularize the divergences caused by the horizon we use the Pauli-Villars regularization and introduce for the each constituent with the mass $m$ three additional fields, one with the normal statistics and the mass $M_1 = \sqrt{m^2 + 2\mu^2}$ and two with the wrong statistics and the masses $M_2 = \sqrt{m^2 + \mu^2}$. The parameter $\mu$ is the Pauli-Villars cutoff.

The analysis of the divergences in three dimensions is similar to other dimensions, see, e.g., [17],[16]. It can be shown that in three dimensions only the leading divergences are present. By following the method of [17] one immediately finds the regularized values of the densities of the energy levels

$$\left(\frac{dn}{d\omega}\right) = \frac{\tilde{b}(\mu, m)}{2\pi \kappa} A ,$$

for a charged scalar or spinor constituent. Here $A = 2\pi r_+$ is the length of the black hole horizon. Expressions (5.1) include for each field contributions of particles and antiparti-
cles. At large $\mu$

$$\tilde{b}(\mu, m) = 2\gamma \mu - m(\omega) - m(-\omega) \quad , \quad \gamma = 2 - \sqrt{2} \quad ,$$  \hspace{1cm} (5.2)

where $m(\omega)$ and $m(-\omega)$ are effective masses of particles and antiparticles determined in (4.11).

In the "low energy" limit of induced gravity the effect of shifting the masses is negligibly small. According to (A.15), the charges of the constituents $e_k$ should be restricted from above, $|e_k| < \sqrt{m_k}$. Thus, as follows from (2.7),

$$|e_k E_+| \ll m_k^2 ,$$  \hspace{1cm} (5.3)

which guarantees that there is no pair creation by the electric field of the black hole. By using this one can also rewrite (5.2) at $\omega < T_H$ as

$$\tilde{b}(\mu, m) \simeq 2\gamma \mu - 2m + O(\omega^2) \quad .$$  \hspace{1cm} (5.4)

At low energies the terms quadratic in $\omega$ and higher result in small corrections and we neglect them in computations.

The entropy can be found from the free energy, $S = \beta^2 \partial_\beta F[\beta]$, where

$$F[\beta] = \eta \beta^{-1} \int d\omega \frac{dn(\omega)}{d\omega} \ln(1 - \eta e^{-\beta \omega}) \quad ,$$  \hspace{1cm} (5.5)

$\eta = +1$ for bosons and $-1$ for fermions. By using Eqs. (5.5), (5.1) and (5.4) one finds the entropy

$$S_s \simeq \frac{1}{6}(\gamma \mu - m_s)A \quad , \quad S_d \simeq \frac{1}{12}(\gamma \mu - m_d)A \quad ,$$  \hspace{1cm} (5.6)

for scalars and spinors, respectively. The total entropy of the constituents is

$$S^{SM} = \sum_s S_s + \sum_d S_d = \frac{1}{12}\left(\gamma\mu(2N_s + N_d) - 2\sum_s m_s - \sum_d m_d\right)A \quad .$$  \hspace{1cm} (5.7)

According to the induced gravity relation (1.1), the non-minimally coupled constituents give additional contribution to the Bekenstein-Hawking entropy in the form of the Noether charge \cite{2}

$$Q = 2\pi \sum_s \xi_s \int_{\Sigma} \langle (\phi_s) + \phi_s \rangle \quad ,$$  \hspace{1cm} (5.8)

where the field operators are taken on the horizon. The charge $Q$ is ultraviolet-divergent and in the Pauli-Villars regularization

$$Q = \left(\gamma\mu \sum_s \xi_s - \sum_s m_s \xi_s\right)A \quad .$$  \hspace{1cm} (5.9)

Therefore,

$$S^{SM} - Q = \frac{1}{12}\left(\gamma\mu(2N_s + N_d - 12 \sum_s \xi_s) + 2 \sum_s (6\xi_s - 1)m_s - \sum_d m_d\right)A \quad .$$  \hspace{1cm} (5.10)
If the induced gravity constraints (2.6) are satisfied, the divergence of the Noether charge $Q$ compensates the divergence of the entropy $S^{SM}$ and the following identity

\[ S^{SM} - Q = \frac{1}{4G} A = S^{BH}, \tag{5.11} \]

where $G$ is induced gravitational constant (2.9), takes place. Therefore, the induced gravity relation (1.1) holds for charged black holes as well.

The subtraction in entropy formula (5.11) has the same interpretation as for neutral black holes in four-dimensional induced gravity [3], [6]. To see this, it should be noted first that the Bekenstein-Hawking entropy of a charged black hole in induced AdSM gravity is related to the spectrum of the black hole mass $M$ defined by Eq. (3.14). Indeed, consider a small excitation of constituent fields with energy $\mathcal{E}$ over a vacuum ($\mathcal{E} = 0$). Such an excitation results in a change of black hole parameters. The corresponding variational formula was studied in [18] \(^1\)

\[ \delta M = T_H \delta S^{BH} + \mathcal{E}, \tag{5.12} \]

where the energy

\[ \mathcal{E} = \int_{\Sigma_t} T^{\mu \nu} t_\mu d\Sigma_\nu, \tag{5.13} \]

is determined in terms of the stress-energy tensor of the constituents $T^{\mu \nu}$ ($\Sigma_t$ is the hypersurface of constant time $t$, $d\Sigma_t$ is the future-directed vector of the volume element of $\Sigma_t$, and $t_\mu$ are the components of $\partial_t$).

Thus, for a black hole with the fixed area the spectrum of $M$ is related to the spectrum of energies $\mathcal{E}$ of the constituents. On the other hand, the statistical-mechanical entropy $S^{SM}$ of the constituents is determined by the spectrum of their Hamiltonian $\mathcal{H}$ which generates canonical transformations of the system along the Killing field $\partial_t$. The observation crucial for understanding entropy relation (5.11) is that the energy $\mathcal{E}$ and the Hamiltonian $\mathcal{H}$ of the non-minimally coupled constituents differ by a total derivative which picks up a non-vanishing contribution on the inner boundary $\Sigma$ of $\Sigma_t$, i.e., on the horizon. The boundary term is the Noether charge on $\Sigma$

\[ \mathcal{H} - \mathcal{E} = T_H Q, \tag{5.14} \]

where $T_H$ is the Hawking temperature. It is because of Eqs. (5.12), (5.13), (5.14) we expect that the two entropies, $S^{BH}$ and $S^{SM}$, are different and related by (5.11).

\(^1\)It is worth mentioning that Eq. (5.12) was derived in Ref. [18] for uncharged black holes. However, if the mass of the black hole is defined by (3.14), formula (5.12) takes place for charged black holes as well. Equation (5.12) holds in the linear order in perturbations provided that the background black hole metric and the gauge potential satisfy the equations of motion. Also, the variations of matter fields have to vanish on the boundary of the black-hole space-time. This requirement, however, is not important because the constituent fields are trapped inside the potential barrier and can be excited only in a thin layer near the horizon.
Now a remark concerning charged rotating black holes is in order. This sort of black hole solutions in AdSM gravity was found in Ref. [9] and represents a generalization of static solution (3.1)–(3.3). In the corresponding induced gravity the constituent fields co-rotate together with a black hole. In the corotating frame of reference one can define a canonical ensemble of the constituents and compute their entropy $S_{SM}$. Obviously, in the corotating frame the properties of the constituents are as if the black hole were static and, in the leading order, $S_{SM}$ is given by (5.7). (One can verify this by doing a more rigorous analysis, see Ref. [6].) Therefore, in induced AdSM gravity the entropy of rotating charged black holes is still expressed by formula (5.11). Also, the above interpretation of (5.11) can be extended to take into account the rotation, see again Ref. [6].

6 Concluding remarks

In conclusion we comment on black holes in four-dimensional induced Einstein-Maxwell gravity. There is a number of reasons to believe that basic features of the entropy of charged black holes in four dimensions will be similar to properties of the entropy of charged black holes in AdSM gravity.

To see what is happening in four dimensions let us consider first the density of energy levels of an uncharged scalar field computed, say, in Pauli-Villars regularization [16]

$$
\left. \frac{dn(\omega|\mu)}{d\omega} \right|_{\Sigma} = \frac{1}{(4\pi)^2 \kappa} \int_{\Sigma} \left[ 2b + a \left( \frac{\omega^2}{\kappa^2} + 2 \left( \frac{1}{6} - \xi \right) R \right) \right].
$$

(6.15)

Here the integral is taken over the bifurcation surface $\Sigma$ of the horizon. The quantity $R$ is the scalar curvature of the black hole space-time computed on $\Sigma$,

$$
P = 2\mathcal{R} - \mathcal{Q}, \quad \mathcal{Q} = P^{\mu\nu} R_{\mu\nu}, \quad \mathcal{R} = P^{\mu\nu} P^{\lambda\rho} R_{\mu\lambda\nu\rho},
$$

(6.16)

where $P^{\mu\nu} = l^\mu l^\nu - p^\mu p^\nu$ is a projector onto a two-dimensional surface orthogonal to $\Sigma$, and $p^\mu, l^\mu$ are two mutually orthogonal normals of $\Sigma (l^2 = -p^2 = 1)$. The regularization parameter $\mu$ defines the scale of the Pauli-Villars masses, and at large $\mu$

$$
a \simeq \ln \frac{\mu^2}{m^2}, \quad b \simeq \mu^2 \ln \frac{729}{256} - m^2 \ln \frac{\mu^2}{m^2},
$$

(6.17)

where $m$ is the mass of the field (see for details [16]).

As we saw in Section 4, if the field is charged, its interaction with the electric field of the black hole results in the shift of the mass determined by Eq. (4.11). The total density

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2Because the constituents are very heavy and trapped near the horizon, they are automatically inside the ‘null’ cylinder.

3Constructing such a theory with charged constituents would require cancellation of additional ultraviolet divergences, and it would be more complicated than constructing AdSM gravity in three dimensions or induced pure Einstein gravity in four dimensions.
of energy levels of particles and antiparticles will be given by (6.15) where, according with
(4.11) and (6.17), one has to replace constant $a$ and $b$ by

$$a(\omega) = \ln \frac{\mu^2}{m^2(\omega)} + \ln \frac{\mu^2}{m^2(-\omega)},$$

$$b(\omega) = 2\mu^2 \ln \frac{729}{256} - m^2(\omega) \ln \frac{\mu^2}{m^2(\omega)} - m^2(-\omega) \ln \frac{\mu^2}{m^2(-\omega)}.$$  

These expressions are even functions of $\omega$ and, as a consequence, they do not depend on
$\omega$ at large $\mu$. The similar changes in the density of levels take place for charged spinor
fields.

As follows from (6.18), (6.19), the interaction of charged particles with the electric
field does not change the ultraviolet divergence of the density of levels near the horizon.
The electric field strength $E_+$ appears only in finite corrections to $dn/d\omega$. Analogously,
the divergences of the statistical entropy $S_{SM}$ of fields does not depend on the gauge in-
teraction and remain purely geometrical. This property is the same as in case of the
three-dimensional charged black holes.

What properties may one expect in induced Einstein-Maxwell gravity in four dimen-
sions? The Bekenstien-Hawking entropy $S^{BH}$ of a charged black hole is one quarter of
the horizon area $A$ regardless the presence of gauge fields. Also the entropy $S_{SM}$ of the
constituents and the corresponding Noether charge remain proportional to $A$ in the lead-
ing order and do not depend on $E_+$. That is why one can conclude that the entropy $S^{BH}$
for charged and neutral black holes has the universal form (1.1).

This sort of universality is very important and, as we stressed earlier [2], it demon-
strates that the mechanism of generation of the Bekenstein-Hawking entropy is a low-
energy phenomenon which depends neither on the properties of a black hole nor on the
specific structure of an underlying fundamental theory of gravity (on the number of species
of constituents and their parameters, for example).

Our analysis holds for near extremal black holes whose thermodynamical behavior is
known to be similar to properties of a two-dimensional massless quantum gas. It is in-
triguing problem to understand on the level of the induced gravity constituents how this
effective two-dimensional description becomes possible. Another aspect of near-extremal
black holes is that they can be described by an ”effective string theory” (by a 2D super-
symmetric conformal field theory) see, e.g., [19]. Thus, at this point the induced gravity
and string theory derivations of the black hole entropy overlap and one has a chance to
explore whether there is any correspondence between the two pictures.

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Appendix

Here we comment on how Eqs. (2.6), (2.9), (2.10) can be obtained. For simplicity we consider Euclidean theory and use the standard representation

\[ W = -\frac{\eta}{2} \int_\delta^\infty \frac{ds}{s} \text{Tr}(e^{-sL})e^{-m^2s} \]  \hspace{1cm} (A.1)

for the regularized one-loop effective action

\[ W = \eta \frac{1}{2} \log \det(L + m^2) \]  \hspace{1cm} (A.2)

with \( \eta = 1 \) and \( \eta = -1 \) for boson and fermion fields, respectively. Here \( L \) is the wave operator, \( L_s = -D^\mu D_\mu + \xi R \) for scalar constituents, and \( L_d = -D^\mu D_\mu + \frac{1}{4} R \) for spinor ones. The covariant derivative is \( D_\mu = \nabla_\mu + eA_\mu \). The parameter \( \delta \) is the ultraviolet cutoff.

When the mass \( m \) of the field is sufficiently high it is enough to approximate \( W \) by the local expansion over the mass parameter. To this aim one replaces the trace of the heat kernel of \( L \) in (A.1) by the asymptotic expansion over \( s \) with the heat kernel coefficients \( a_n \). In three dimensions the calculation gives

\[ W_\delta = -\frac{\eta}{16\pi^{3/2}} \left[ m^3 a_0 \Gamma(-3/2, m^2\delta) + ma_1 \Gamma(-1/2, m^2\delta) + \pi^{1/2} \frac{1}{m} a_2 + O(m^{-3}) \right] , \hspace{1cm} (A.3) \]

\[ \Gamma(z, x) = \int_x^\infty dt t^{z-1} e^{-t} \]  \hspace{1cm} (A.4)

With the following asymptotics for the incomplete gamma-function

\[ \Gamma(-1/2, x) \simeq 2x^{-1/2} - 2\pi^{1/2} , \hspace{1cm} (A.5) \]
\[ \Gamma(-3/2, x) \simeq \frac{2}{3} x^{-3/2} - 2x^{-1/2} + \frac{4}{3} \pi^{1/2} , \hspace{1cm} (A.6) \]

one gets

\[ W_\delta = W_\delta^{\text{div}} + W^{\text{reg}} , \hspace{1cm} (A.7) \]
\[ W_\delta^{\text{div}} = -\frac{\eta}{16\pi^{3/2}} \left[ \frac{2}{3} \delta^{-3/2} a_0 + 2\delta^{-1/2} (a_1 - m^2 a_0) \right] , \hspace{1cm} (A.8) \]
\[ W^{\text{reg}} = \eta \left[ -\frac{1}{12\pi} m^3 a_0 + \frac{1}{8\pi} ma_1 - \frac{1}{16\pi} \frac{1}{m} a_2 + O(m^{-3}) \right] . \hspace{1cm} (A.9) \]

The heat coefficients \( a_0 \) and \( a_1 \) do not depend on the gauge field \( A_\mu \). For charged scalars

\[ a_0 = 2 \int \sqrt{g} d^3x \ , \hspace{0.5cm} a_1 = 2(\frac{1}{6} - \xi) \int \sqrt{g} d^3x R \]  \hspace{1cm} (A.10)

and for three-dimensional spinors

\[ a_0 = 2 \int \sqrt{g} d^3x \ , \hspace{0.5cm} a_1 = -\frac{1}{6} \int \sqrt{g} d^3x R \]  \hspace{1cm} (A.11)
The next heat-kernel coefficient has the following form

\[ a_2 = \int \sqrt{g} d^3x \left[ \frac{1}{180} (c_1 R^{\mu\nu} R_{\mu\nu} + c_2 R^2) + c_3 F^{\mu\nu} F_{\mu\nu} \right] . \] \hspace{1cm} (A.12)

In this equation we neglect total derivatives and took into account that in three dimensions the Riemann tensor is

\[ R_{\mu\nu\lambda\rho} = g_{\mu\rho} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\rho} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} - \frac{1}{2} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) R . \] \hspace{1cm} (A.13)

The coefficients are: \( c_1 = 6 \), \( c_2 = 6 + 5(1 - 6\xi)^2 \), \( c_3 = -e_s^2/6 \) for a scalar field with the charge \( e_s \) and non-minimal coupling \( \xi \), see, e.g., [20], and \( c_1 = -9 \), \( c_2 = -4 \), \( c_3 = e_d^2/3 \) for a 3D spinor field with the charge \( e_d \).

The induced gravity requires cancelation of the divergences

\[ \sum_s W^{\text{div}}_{\delta,s} + \sum_d W^{\text{div}}_{\delta,d} = 0 \] \hspace{1cm} (A.14)

Equation (A.14) is equivalent to three conditions (2.6) which can be satisfied for a number of models. The total induced action is determined then by the contributions of the regular parts \( W^{\text{reg}} \), Eq. (A.9), only. Let us now impose the additional restriction on the charges of the constituents

\[ \sum_s e_s^2/m_s + 2 \sum_d e_d^2/m_d = 24\pi \] \hspace{1cm} (A.15)

which provides the right coefficient by the Maxwell action. Then the induced Euclidean action in the leading order in curvature and the strength of the gauge field has the form

\[ \Gamma[g, A] = -\frac{1}{4} \int d^3x \sqrt{g} \left[ \frac{1}{4\pi G} (R - 2\Lambda) - F^{\mu\nu} F_{\mu\nu} + a R^{\mu\nu} R_{\mu\nu} + b R^2 \right] . \] \hspace{1cm} (A.16)

with the induced gravitational and cosmological constants defined by Eqs. (2.9), (2.10) and

\[ a = \frac{1}{240\pi} \left( 2 \sum_s \frac{1}{m_s} + 3 \sum_d \frac{1}{m_d} \right) , \] \hspace{1cm} (A.17)

\[ b = \frac{1}{720\pi} \left( \sum_s (6 + 5(1 - 6\xi_s)^2) \frac{1}{m_s} + 4 \sum_d \frac{1}{m_d} \right) . \] \hspace{1cm} (A.18)

Functional (A.16) contains higher curvature terms which can be neglected in low-energy limit. Note that in this limit the terms quadratic in curvature can be also neglected as compared with the ”Einstein-Maxwell” part. Then after the Wick rotation functional (A.16) coincides with the Lorentzian action (2.1).
References


