S-DUALITY FOR LINEARIZED GRAVITY

J. A. Nieto*

Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma
de Sinaloa, 80010 Culiacán Sinaloa, México

Abstract

We develop the analogue of S-duality for linearized gravity in (3+1)-dimensions. Our basic idea is to consider the self-dual (anti-self-dual) curvature tensor for linearized gravity in the context of the Macdowell-Mansouri formalism. We find that the strong-weak coupling duality for linearized gravity is an exact symmetry and implies small-large duality for the cosmological constant.

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*nieto@uas.uasnet.mx
1.- INTRODUCTION

It is well known that linearized Einstein gravitational theory in four dimensions is similar to the source free Maxwell theory. Since S-duality has been successfully applied to Abelian gauge theories and non-Abelian gauge theories [1-10] as well as gravity [11-15], it seems natural to ask whether similar methods may be applied to linearized gravity. However, while in the Abelian case there is an exact strong-weak duality, Yang-Mills and gravity do not possess such a symmetry.

In this work, we show that in spite of gravity, in general, does not possess an exact strong-weak symmetry linearized gravity does. In fact, we derive a S-dual action for linearized gravity which is connected to the original linearized gravitational action by strong-weak duality transformation. Our strategy essentially consists in two steps; first in writing linearized gravity as Abelian gauge theory and second taking recourse of the Macdowell-Mansouri formalism.

Just as $U(1)$ duality has been very important to explore S-duality in larger theories such as $N = 2$ super Yang-Mills theory with gauge group $SU(2)$ [16-17], one should expect that linearized gravitational duality may also be an important tool for investigating S-duality analogue of other theories such as $N = 1$ supergravity in four dimensions.

The plan of this work is as follows: In section II, we show that it is possible to see linearized gravity as an Abelian gauge theory. In section III, we briefly review the Macdowell-Mansouri theory and propose an action for linearized gravity in the context of such a theory. In section IV, we generalize our proposed action to include self-dual and anti-self-dual linearized gravity and we compute the partition function of such a generalized action. Finally, in section V, we make some final comments.
The starting point in linearized gravity is to write the metric of the space-time $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$ as
\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},
\] (1)
where $h_{\mu\nu}$ is a small deviation of the metric $g_{\mu\nu}$ from the Minkowski metric
\[
(\eta_{\mu\nu}) = \text{diag}(-1, 1, 1, 1).
\]
The first-order curvature tensor is
\[
F_{\mu\nu\alpha\beta} = \frac{1}{2} \left( \partial_\mu \partial_\alpha h_{\nu\beta} - \partial_\mu \partial_\beta h_{\nu\alpha} - \partial_\nu \partial_\beta h_{\mu\alpha} + \partial_\nu \partial_\alpha h_{\mu\beta} \right),
\] (2)
which is invariant under the gauge transformation
\[
\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu.
\] (3)
Here, $\xi_\mu$ is an arbitrary vector field and $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$.

It is not difficult to see that $F_{\mu\nu\alpha\beta}$ satisfies the following relations:
\[
F_{\mu\nu\alpha\beta} = -F_{\mu\nu\beta\alpha} = -F_{\nu\mu\alpha\beta} = F_{\alpha\beta\mu\nu},
\]
\[
F_{\mu\nu\alpha\beta} + F_{\mu\beta\nu\alpha} + F_{\mu\alpha\beta\nu} = 0,
\] (4)
\[
\partial_\lambda F_{\mu\nu\alpha\beta} + \partial_\mu F_{\nu\lambda\alpha\beta} + \partial_\nu F_{\lambda\mu\alpha\beta} = 0.
\]
It is interesting to note that the dual $^*F_{\mu\nu\alpha\beta} \equiv \frac{1}{2} \epsilon_{\mu\nu\tau\sigma} F_{\alpha\beta}^{\tau\sigma}$, where $\epsilon_{\mu\nu\alpha\beta}$ is a completely antisymmetric Levi-Civita tensor with $\epsilon_{0123} = -1$ and the indices are raised and lowered by means of $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, does not satisfy the relations (4) unless the vacuum Einstein equations $F_{\mu\nu} = F_{\mu\nu}^{\alpha} = 0$ are satisfied [18].

Let us introduce the ‘gauge’ field
\[ A_{\mu\alpha\beta} = \frac{1}{2} (\partial_\alpha h_{\mu\beta} - \partial_\beta h_{\mu\alpha}). \]  

(5)

Note that \( A_{\mu\alpha\beta} = -A_{\mu\beta\alpha} \). Using (3), we find that \( A_{\mu\alpha\beta} \) transforms as

\[ \delta A_{\mu\alpha\beta} = \partial_\mu \lambda_{\alpha\beta}, \]  

(6)

where \( \lambda_{\alpha\beta} = \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha \).

The expression (5) allows to write the curvature tensor \( F^{\alpha\beta}_{\mu\nu} \) as

\[ F^{\alpha\beta}_{\mu\nu} = \partial_\mu A^{\alpha\beta}_{\nu} - \partial_\nu A^{\alpha\beta}_{\mu}. \]  

(7)

Thus, we have shown that the tensor \( F^{\alpha\beta}_{\mu\nu} \) may be written in the typical form of an abelian Maxwell field strength \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \), where the index \( a \) runs over some abelian group such as \( U(1) \times U(1) \times \ldots \times U(1) \). Note that \( F^{\alpha\beta}_{\mu\nu} \) is invariant under the transformation \( \delta A^{\alpha\beta}_{\mu} = \partial_\mu \lambda^{\alpha\beta} \) which has exactly the same form as the transformation of Abelian gauge fields \( \delta A^a_\mu = \partial_\mu \lambda^a \), where \( \lambda^a \) is an arbitrary function of the coordinates \( x^\mu \).

3.- LINEARIZED GRAVITY a l’á MACDOWELL-MANSOURI

The central idea in the Macdowell-Mansouri theory [19-20] (see also Refs. [21-31]) is to represent the gravitational field as a connection one-form associated to some group that contains the Lorentz group as a subgroup. The typical example is provided by the SO(3,2) anti-de Sitter gauge theory of gravity. In this case the SO(3,2) gravitational gauge field \( \omega^{AB}_\mu = -\omega^{BA}_\mu \) is broken into the SO(3,1) connection \( \omega^{ab}_\mu \) and the \( \omega^a_\mu = e^a_\mu \) vierbein field. Thus, the anti-de Sitter curvature

\[ R^{AB}_{\mu\nu} = \partial_\mu \omega^{AB}_\nu - \partial_\nu \omega^{AB}_\mu + \omega^{AC}_\mu \omega^{B}_\nu C^B - \omega^{AC}_\nu \omega^{B}_\mu C^B \]  

(8)

leads to

\[ R^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \Sigma^{ab}_{\mu\nu} \]  

(9)
and
\[
R_{\mu\nu}^{4a} = \partial_{\mu} e_{\nu}^{a} - \partial_{\nu} e_{\mu}^{a} + \omega_{\mu}^{\alpha c} e_{\nu}^{c} - \omega_{\nu}^{\alpha c} e_{\mu}^{c},
\]
where
\[
R_{\mu\nu}^{ab} = \partial_{\mu} \omega_{\nu}^{ab} - \partial_{\nu} \omega_{\mu}^{ab} + \omega_{\mu}^{ac} \omega_{\nu}^{bc} - \omega_{\nu}^{ac} \omega_{\mu}^{bc},
\]
is the SO(3,1) curvature and
\[
\Sigma_{\mu\nu}^{ab} = e_{\mu}^{a} e_{\nu}^{b} - e_{\nu}^{a} e_{\mu}^{b}.
\]
It turns out that \( T_{\mu\nu}^{a} \equiv R_{\mu\nu}^{4a} \) can be identified with the torsion.

The Macdowell-Mansouri’s action is
\[
S = \frac{1}{4} \int d^{4}x \varepsilon^{\mu \nu \alpha \beta} R_{\mu \nu}^{ab} R_{\alpha \beta}^{cd} \epsilon_{abcd},
\]
where \( \varepsilon^{\mu \nu \alpha \beta} \) is the completely antisymmetric tensor associated to the space-time, with \( \varepsilon^{0123} = 1 \) and \( \varepsilon_{0123} = 1 \), while \( \epsilon_{abcd} \) is also the completely antisymmetric tensor but now associated to the internal group S(3,1), with \( \epsilon_{0123} = -1 \). We assume that the internal metric is given by \( (\eta_{ab}) = (-1, 1, 1, 1) \). Therefore, we have \( \epsilon^{0123} = 1 \). It is well known that, using (9), (11) and (12), the action (13) leads to three terms; the Hilbert Einstein action, the cosmological constant term and the Euler topological invariant (or Gauss-Bonnet term). It is worth mentioning that the action (13) may also be obtained using Lovelock theory (see [32] and references there in).

Let us now apply the Macdowell-Mansouri formalism to linearized gravity as developed in section 2.

First, consider the extended curvature
\[
F_{\mu \nu}^{\alpha \beta} = F_{\mu \nu}^{\alpha \beta} + \Omega_{\mu \nu}^{\alpha \beta},
\]
where
\[
\Omega_{\mu \nu}^{\alpha \beta} = \delta_{\mu}^{\alpha} h_{\nu}^{\beta} - \delta_{\mu}^{\beta} h_{\nu}^{\alpha} - \delta_{\nu}^{\alpha} h_{\mu}^{\beta} + \delta_{\nu}^{\beta} h_{\mu}^{\alpha}.
\]
The central idea is to consider the extended curvature (14) as the analogue of the curvature (9) of the Macdowell-Mansouri formalism. Thus, we may identify \( F_{\alpha\beta}^{\mu\nu} \) and \( \Omega_{\alpha\beta}^{\mu\nu} \) with the linearized versions of \( R_{\mu\nu}^{ab} \) and \( \Sigma_{\mu\nu}^{ab} \), respectively. In fact, if we write \( e_{\mu}^{a} = b_{\mu}^{a} + l_{\mu}^{a} \), where \( b_{\mu}^{a} \) corresponds to the flat metric \( \eta_{\mu\nu} \) and \( l_{\mu}^{a} \) is a small deviation of \( e_{\mu}^{a} \), we find \( h_{\mu\nu} = b_{\mu}^{a} l_{\nu}^{a} \eta_{ab} + b_{\nu}^{a} l_{\mu}^{a} \eta_{ab} \) and therefore using (5) we get \( A_{\mu\alpha\beta} = \omega_{\mu\alpha\beta}^{ab}(l_{\lambda}^{a}) + \partial_{\mu} f_{\alpha\beta} \). Here, \( \omega_{\mu\alpha\beta}^{ab}(l_{\lambda}^{a}) = b_{a\alpha} b_{b\beta} \omega_{ab}^{\mu}(l_{\lambda}^{a}) \) and \( f_{\alpha\beta} = b_{\alpha\beta} - b_{\alpha\beta} l_{\alpha}^{a} \). (It is important to note that \( \omega_{\mu\alpha\beta}^{ab}(l_{\lambda}^{a}) \) can be obtained by setting the torsion (10), for weak gravity, equal to zero.) Thus, up to a gauge transformation \( A_{\mu\alpha\beta} \) is equal to the linearized connection \( \omega_{\mu\alpha\beta}^{ab}(l_{\lambda}^{a}) \). The essential difference between \( \omega_{\mu\alpha\beta}^{ab}(l_{\lambda}^{a}) \) and the full connection \( \omega_{\mu\alpha\beta}^{ab} \) is that the latter is associated to the Lorentz group while the former to an abelian group. This can easily be seen from the transformation \( \delta A_{\mu\alpha\beta} = \partial_{\mu}\lambda_{\alpha\beta} \) of the gauge field \( A_{\mu\alpha\beta} \), since in general a gauge field \( A_{\mu\alpha\beta} \) transforms as \( \delta A = \partial_{\lambda} + [A, \lambda] \).

Let us now propose the action

\[
S = \frac{1}{4} \int d^{4}x \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^{\tau\lambda} F_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho}.
\] \hspace{1cm} (16)

We shall show that this action is reduced to Einstein linearized gravitational action (Fierz-Pauli action) with cosmological constant and a total derivative (‘topological’ term). Substituting (14) into (16) yields

\[
S = \frac{1}{4} \int d^{4}x \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^{\tau\lambda} F_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho} + \frac{1}{2} \int d^{4}x \epsilon^{\mu\nu\alpha\beta} \Omega_{\mu\nu}^{\tau\lambda} F_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho}
\] \hspace{1cm} (17)

\[+ \frac{1}{4} \int d^{4}x \epsilon^{\mu\nu\alpha\beta} \Omega_{\mu\nu}^{\tau\lambda} \Omega_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho}.\]

Using (15) we find that (17) is reduced to

\[
S = \frac{1}{4} \int d^{4}x \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^{\tau\lambda} F_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho} + 2 \int d^{4}x \epsilon^{\mu\nu\alpha\beta} \delta_{\mu}^{\tau} h_{\nu}^{\lambda} F_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho}
\] \hspace{1cm} (18)

\[+ 4 \int d^{4}x \epsilon^{\mu\nu\alpha\beta} \delta_{\mu}^{\tau} h_{\nu}^{\lambda} \delta_{\alpha}^{\sigma} h_{\beta}^{\rho} \epsilon_{\tau\lambda\sigma\rho}.\]

Since \( \epsilon^{\mu\nu\alpha\beta} \delta_{\mu}^{\nu} \epsilon_{\tau\lambda\sigma\rho} = -\delta_{\lambda\rho}^{\alpha\beta} \) and \( \epsilon^{\mu\nu\alpha\beta} \delta_{\mu}^{\nu} \delta_{\alpha}^{\sigma} \epsilon_{\tau\lambda\sigma\rho} = -2 \delta_{\mu}^{\nu} \), where in general \( \delta_{\mu}^{\nu} \) is a generalized delta, we discover that (18) can be written as
$$S = \frac{1}{4} \int d^4 x \epsilon^{\mu \alpha \beta} F_{\mu \nu}^{\tau \lambda} F_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho} + 8 \int d^4 x h^{\mu \nu} (F_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} F)$$

$$-8 \int d^4 x (h^2 - h^{\mu \nu} h_{\mu \nu}).$$

Here, we used the following definitions: $F_{\mu \nu} \equiv \eta^{\alpha \beta} F_{\mu \nu \alpha \beta}$, $F \equiv \eta^{\mu \nu} \eta^{\alpha \beta} F_{\mu \nu \alpha \beta}$ and $h = \eta^{\mu \nu} h_{\mu \nu}$.

We recognize the second term and the third term in (19) as the Einstein action for linearized gravity with cosmological constant, while the first term is a total derivative (an Euler topological invariant or Gauss-Bonnet term). Note that the usual cosmological factor $\Lambda$ in the third term can be derived simply by changing $\Omega \rightarrow a^2 \Omega$, where $a$ is a constant, and rescaling the total action $S \rightarrow \frac{1}{4} \Lambda^{-1} S$, with $\Lambda = a^2$.

### 4.- S-DUALITY FOR LINEARIZED GRAVITY

In order to develop a S-dual linearized gravitational action we generalize the action (16) as follows;

$$S = \frac{1}{4} (\tau^+) \int d^4 x \epsilon^{\mu \alpha \beta} + \mathcal{F}_{\mu \nu}^{\tau \lambda} F_{\alpha \beta}^{\sigma \rho} - \frac{1}{4} (\tau^-) \int d^4 x \epsilon^{\mu \alpha \beta} - \mathcal{F}_{\mu \nu}^{\tau \lambda} F_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho},$$

where $\tau^+$ and $\tau^-$ are two different constant parameters and $\pm \mathcal{F}_{\mu \nu}^{\alpha \beta}$ is given by

$$\pm \mathcal{F}_{\mu \nu}^{\alpha \beta} = (\frac{1}{2}) \pm M_{\alpha \beta}^{\gamma \lambda} \mathcal{F}_{\mu \nu}^{\gamma \lambda},$$

where

$$\pm M_{\alpha \beta}^{\gamma \lambda} = \frac{1}{2} (\delta_{\alpha \beta}^{\gamma \lambda} \mp i \epsilon_{\alpha \beta}^{\gamma \lambda}).$$

It turns out that $+ \mathcal{F}_{\mu \nu}^{\alpha \beta}$ is self-dual, while $- \mathcal{F}_{\mu \nu}^{\alpha \beta}$ is anti self-dual curvature. Therefore, the action (20) describes self-dual and anti-self-dual linearized gravity.

We may now follow a similar procedure as in the references [5-8] for Abelian Yang-Mills and [11, 14, 15] for gravity. Here, however, in order to show the exactness of S-duality symmetry in linearized gravity, it turns out to be more convenient to follow the procedure...
of reference [1] due to Witten. For this purpose let us first introduce a two form $G$ and let us set

$$H_{\mu\nu}^{\alpha\beta} \equiv F_{\mu\nu}^{\alpha\beta} - G_{\mu\nu}^{\alpha\beta}.$$  

(23)

We assume that $G_{\mu\nu}^{\alpha\beta}$ satisfies the same symmetric properties as $F_{\mu\nu}^{\alpha\beta}$, given in (4), namely

$$G_{\mu\nu\alpha\beta} = -G_{\mu\nu\beta\alpha} = -G_{\nu\mu\alpha\beta} = G_{\alpha\beta\mu\nu},$$

$$G_{\mu\nu\alpha\beta} + G_{\mu\beta\nu\alpha} + G_{\mu\alpha\beta\nu} = 0.$$  

(24)

Now, consider the extended action

$$S_E = \frac{1}{4}(\tau^+) \int d^4x \varepsilon_{\mu\nu\sigma\rho} H_{\mu\nu}^{\tau\lambda} + \mathcal{H}_{\alpha\beta\tau\lambda\sigma\rho} - \frac{1}{4}(\tau^-) \int d^4x \varepsilon_{\mu\nu\sigma\rho} H_{\mu\nu}^{\tau\lambda} - \mathcal{H}_{\alpha\beta\tau\lambda\sigma\rho}$$

$$+ \frac{1}{2} \int d^4x \varepsilon_{\mu\nu\tau\lambda} W_{\mu\nu}^{\alpha\beta} G_{\tau\lambda}^{\sigma\rho} \epsilon_{\alpha\beta\sigma\rho} - \frac{1}{2} \int d^4x \varepsilon_{\mu\nu\tau\lambda} - W_{\mu\nu}^{\alpha\beta} - G_{\tau\lambda}^{\sigma\rho} \epsilon_{\alpha\beta\sigma\rho},$$

(25)

where $W_{\mu\nu\alpha\beta} = \partial_\mu V_{\nu\alpha\beta} - \partial_\nu V_{\mu\alpha\beta}$ is the dual field strength satisfying the Dirac quatization law

$$\int W \in 2\pi \mathbb{Z}.$$  

(26)

It is not difficult to see that, beyond the gauge invariance $A \rightarrow A - d\lambda$, $G \rightarrow G$, the partition function

$$Z = \int d^4G d^- G dAdhdV e^{-S_E}$$

is invariant under

$$A \rightarrow A + B and G \rightarrow G + dB,$$  

(27)

where $B_{\mu}^{\alpha\beta}$ is an arbitrary one form.

Starting from (25) one can proceed in two different ways. For the first possibility, we note that the path integral that involves $V$ is

$$\int DV \exp\left( \frac{1}{2} \int d^4x \varepsilon_{\mu\nu\tau\lambda} W_{\mu\nu}^{\alpha\beta} G_{\tau\lambda}^{\sigma\rho} \epsilon_{\alpha\beta\sigma\rho} - \frac{1}{2} \int d^4x \varepsilon_{\mu\nu\tau\lambda} - W_{\mu\nu}^{\alpha\beta} - G_{\tau\lambda}^{\sigma\rho} \epsilon_{\alpha\beta\sigma\rho} \right).$$  

(29)
Integrating over the dual connection \( V \), we get a delta function setting \( dG = 0 \). Thus, using the gauge invariance (28), we may gauge \( G \) to zero, reducing (25) to the original action (20). Therefore, the actions (25) and (20) are, in fact, classically equivalents.

For the second possibility, we note that the gauge invariance (28) enables one to fix a gauge with \( A = 0 \). (It is important to note that, at this stage, we are considering \( A_{\mu \alpha \beta} \) and \( h_{\mu \nu} \) as independent fields in the sense of Palatini.) The action (25) is then reduced to

\[
S_E = \frac{1}{4}(\tau^+) \int dx^4 \varepsilon^{\mu \nu \alpha \beta} G_{\mu \nu}^{\tau \lambda} + G_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho} - \frac{1}{4}(\tau^-) \int dx^4 \varepsilon^{\mu \nu \alpha \beta} - G_{\mu \nu}^{\tau \lambda} - G_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho}
\]

(30)

where

\[
G_{\mu \nu}^{\tau \lambda} \equiv \Omega_{\mu \nu}^{\tau \lambda} - G_{\mu \nu}^{\tau \lambda}.
\]

(31)

In virtue of (31) the extended action (30) becomes

\[
S_E = \frac{1}{4}(\tau^+) \int dx^4 \varepsilon^{\mu \nu \alpha \beta} + G_{\mu \nu}^{\tau \lambda} + G_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho} - \frac{1}{4}(\tau^-) \int dx^4 \varepsilon^{\mu \nu \alpha \beta} - G_{\mu \nu}^{\tau \lambda} - G_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho}
\]

\[
+ \frac{1}{2}(\tau^+) \int dx^4 \varepsilon^{\mu \nu \alpha \beta} + \Omega_{\mu \nu}^{\tau \lambda} + G_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho} + \frac{1}{2}(\tau^-) \int dx^4 \varepsilon^{\mu \nu \alpha \beta} - \Omega_{\mu \nu}^{\tau \lambda} + G_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho}
\]

(32)

Before integrating \( h_{\mu \nu} \), let us consider the identity

\[
\pm M_{\mu \nu}^{\tau \lambda} \pm M_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho} = \pm 4i \pm M_{\mu \nu}^{\tau \lambda} \eta_{\alpha \tau} \eta_{\beta \lambda}.
\]

(33)

which can be obtained from the definition (22). Here, \( \pm M_{\mu \nu}^{\tau \lambda} \eta_{\alpha \tau} \eta_{\beta \lambda} \). Using this identity and the definition (15) for \( \Omega_{\mu \nu}^{\tau \lambda} \) we obtain

\[
- \frac{1}{2}(\tau^+) \int dx^4 \varepsilon^{\mu \nu \alpha \beta} + \Omega_{\mu \nu}^{\tau \lambda} + G_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho} + \frac{1}{2}(\tau^-) \int dx^4 \varepsilon^{\mu \nu \alpha \beta} - \Omega_{\mu \nu}^{\tau \lambda} + G_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho}
\]

\[
= \frac{1}{2} \int dx^4 \varepsilon^{\mu \nu \alpha \beta} \Omega_{\mu \nu}^{\tau \lambda} B_{\alpha \beta}^{\sigma \rho} \epsilon_{\tau \lambda \sigma \rho}
\]

(34)
and
\[
\frac{1}{4}(\tau^+) \int d^4x \epsilon^{\mu\nu\alpha\beta} + \Omega^\tau_{\mu\nu} + \Omega_{\alpha\beta}^\sigma \epsilon_{\tau\lambda\sigma\rho} - \frac{1}{4}(\tau^-) \int d^4x \epsilon^{\mu\nu\alpha\beta} - \Omega^\tau_{\mu\nu} - \Omega_{\alpha\beta}^\sigma \epsilon_{\tau\lambda\sigma\rho}
\]
\[
= \frac{1}{4} \int d^4x \epsilon^{\mu\nu\alpha\beta} \Omega^\tau_{\mu\nu} \Omega_{\alpha\beta}^\sigma \epsilon_{\tau\lambda\sigma\rho},
\]
where
\[
B^\sigma_{\alpha\beta} = -\{ (\tau^+) + G^\sigma_{\alpha\beta} - (\tau^-) - G^\sigma_{\alpha\beta} \} \]  
(36)
and
\[
b = \frac{1}{2}(\tau^+ - \tau^-).
\]  
(37)

Using (34)-(37) we learn that the action (32) can be written as
\[
S_E = \frac{1}{4}(\tau^+) \int d^4x \epsilon^{\mu\nu\alpha\beta} + G^\tau_{\mu\nu} + G_{\alpha\beta}^\sigma \epsilon_{\tau\lambda\sigma\rho} - \frac{1}{4}(\tau^-) \int d^4x \epsilon^{\mu\nu\alpha\beta} - G^\tau_{\mu\nu} - G_{\alpha\beta}^\sigma \epsilon_{\tau\lambda\sigma\rho}
\]
\[
+ \frac{1}{2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \Omega^\tau_{\mu\nu} B^{\sigma\rho}_{\alpha\beta\epsilon_{\tau\lambda\sigma\rho}} + \frac{1}{2} \int d^4x \epsilon^{\mu\nu\alpha\beta} \Omega^\tau_{\mu\nu} \Omega_{\alpha\beta}^\sigma \epsilon_{\tau\lambda\sigma\rho} + \frac{1}{2} \int d^4x \epsilon^{\mu\nu\tau\lambda} - W^{\alpha\beta}_{\mu\nu} G^\sigma_{\tau\lambda\epsilon_{\alpha\beta\sigma\rho}} - \frac{1}{2} \int d^4x \epsilon^{\mu\nu\tau\lambda} - W^{\alpha\beta}_{\mu\nu} G^\sigma_{\tau\lambda\epsilon_{\alpha\beta\sigma\rho}}.
\]  
(38)

We note that the third and fourth term in (38) have exactly the same form as the second and third term of (17). Therefore, using the definition (15) for $\Omega^\tau_{\mu\nu}$ and making similar calculations for obtaining (19) we get
\[
S_E = \frac{1}{4}(\tau^+) \int d^4x \epsilon^{\mu\nu\alpha\beta} + G^\tau_{\mu\nu} + G_{\alpha\beta}^\sigma \epsilon_{\tau\lambda\sigma\rho} - \frac{1}{4}(\tau^-) \int d^4x \epsilon^{\mu\nu\alpha\beta} - G^\tau_{\mu\nu} - G_{\alpha\beta}^\sigma \epsilon_{\tau\lambda\sigma\rho}
\]
\[
+ 8 \int d^4x h^{\mu\nu}(B_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} B) - 8b \int d^4x (h^2 - h^{\mu\nu} h_{\mu\nu})
\]  
(39)
\[
+ \frac{1}{2} \int d^4x \epsilon^{\mu\nu\tau\lambda} + W^{\alpha\beta}_{\mu\nu} G^\sigma_{\tau\lambda\epsilon_{\alpha\beta\sigma\rho}} - \frac{1}{2} \int d^4x \epsilon^{\mu\nu\tau\lambda} + W^{\alpha\beta}_{\mu\nu} G^\sigma_{\tau\lambda\epsilon_{\alpha\beta\sigma\rho}},
\]
where $B_{\mu\nu} \equiv \eta^{\alpha\beta} B_{\mu\alpha\beta}$, and $B \equiv \eta^{\mu\nu} B_{\mu\nu}$. Note that $b$ plays the role of a cosmological constant.

We can now integrate out $h_{\mu\lambda}$ in the reduced partition function
\[ Z = \int d^4Gd^-GdhdVe^{-S_E}, \quad (40) \]

with \( S_E \) given by (39). We first get the relation

\[ B_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}B = 2b(\eta_{\mu\nu}h - h_{\mu\nu}). \quad (41) \]

Using this expression the action \( S_E \) given in (39) becomes

\[ S_E = \frac{1}{4}(\tau^+) \int d^4x\varepsilon^{\mu\nu\alpha\beta}G_{\mu\nu}^{\tau^\lambda\lambda} + G_{\alpha\beta}^{\sigma^\rho} - \frac{1}{4}(\tau^-) \int d^4x\varepsilon^{\mu\nu\alpha\beta} - G_{\mu\nu}^{\tau^\lambda\lambda} - G_{\alpha\beta}^{\sigma^\rho}\]

\[ + \frac{2}{b} \int d^4x(\frac{1}{3}B^2 - B^{\mu\nu}B_{\mu\nu}) \quad (42) \]

Before we perform the integrals over \( +G_{\mu\nu}^{\tau^\lambda\lambda} \) and \( -G_{\mu\nu}^{\tau^\lambda\lambda} \) we still need to rearrange the third integral of (42). For this purpose let us introduce the definition

\[ B_{\mu\nu} = C_{\mu\nu} - \eta_{\mu\nu}C. \quad (43) \]

Note first that \( B = -3C \). Therefore, we find \( \frac{1}{3}B^2 - B^{\mu\nu}B_{\mu\nu} = C^2 - C^{\mu\nu}C_{\mu\nu} \). We now use the identity

\[ -\frac{1}{32}\varepsilon^{\mu\nu\alpha\beta}S_{\mu\nu}^{\tau^\lambda\lambda}C_{\alpha\beta}^{\sigma^\rho}\varepsilon_{\tau^\lambda\lambda}\sigma^\rho = (C^2 - C^{\mu\nu}C_{\mu\nu}), \quad (44) \]

where

\[ S_{\mu\nu}^{\alpha\beta} = \delta_{\alpha}^{\mu}\delta_{\beta}^{\nu} - \delta_{\alpha}^{\nu}\delta_{\beta}^{\mu} + \delta_{\alpha}^{\mu}\delta_{\beta}^{\nu} \quad (45) \]

and we observe that according to (36) we can write \( S_{\mu\nu}^{\alpha\beta} = -(\tau^+) + S_{\mu\nu}^{\alpha\beta} + (\tau^-) - S_{\mu\nu}^{\alpha\beta} \), where

\[ +S_{\mu\nu}^{\alpha\beta} = \delta_{\mu}^{\alpha}\delta_{\nu}^{\beta} - \delta_{\mu}^{\beta}\delta_{\nu}^{\alpha} - \delta_{\nu}^{\alpha}\delta_{\mu}^{\beta} + \delta_{\nu}^{\beta}\delta_{\mu}^{\alpha} \quad (46) \]

and

\[ -S_{\mu\nu}^{\alpha\beta} = \delta_{\mu}^{\alpha}\delta_{\nu}^{\beta} - \delta_{\mu}^{\beta}\delta_{\nu}^{\alpha} - \delta_{\nu}^{\alpha}\delta_{\mu}^{\beta} + \delta_{\nu}^{\beta}\delta_{\mu}^{\alpha}. \quad (47) \]
Here, $\chi^\beta_\nu = G^\alpha_\nu_\alpha - \frac{1}{3} \delta^\beta_\nu G^\lambda_\lambda_\alpha$ and $C_{\mu\nu} = - (\tau^+)_+ \chi_\mu_\nu + (\tau^-)_- \chi_\mu_\nu$. But, due to the symmetry properties of $G^\alpha_\mu_\nu$ given in (24) we discover that $\chi_\mu_\nu = + \chi_\mu_\nu = - \chi_\mu_\nu$. So, $C_{\mu\nu} = - (\tau^+ - \tau^-) \chi_\mu_\nu$.

Thus, we find that (42) can be written as

$$S_E = \frac{1}{4} (\tau^+) \int d^4 x \epsilon^{\mu\nu\alpha\beta} G^\tau_\mu_\nu G^\lambda_\alpha_\lambda + G^\sigma_\rho_\rho - \frac{1}{4} (\tau^-) \int d^4 x \epsilon^{\mu\nu\alpha\beta} - G^\tau_\mu_\nu - G^\rho_\sigma_\rho \epsilon_\tau_\lambda_\sigma \rho$$

$$- \frac{1}{4} (\tau^+) \int d^4 x \epsilon^{\mu\nu\alpha\beta} S^\tau_\mu_\nu + S^\rho_\sigma_\rho \epsilon_\tau_\lambda_\sigma \rho + \frac{1}{4} (\tau^-) \int d^4 x \epsilon^{\mu\nu\alpha\beta} - S^\tau_\mu_\nu - S^\rho_\sigma_\rho \epsilon_\tau_\lambda_\sigma \rho$$

$$+ \frac{1}{2} \int d^4 x \epsilon^{\mu\nu\alpha\beta} W^\tau_\mu_\nu + G^\rho_\sigma_\rho \epsilon_\tau_\lambda_\sigma \rho - \frac{1}{2} \int d^4 x \epsilon^{\mu\nu\alpha\beta} - W^\tau_\mu_\nu - G^\rho_\sigma_\rho \epsilon_\tau_\lambda_\sigma \rho. \quad (48)$$

Here, $S^\alpha_\mu_\nu$ is defined as

$$S^\alpha_\mu_\nu = \delta^\alpha_\mu \lambda_\beta - \delta^\beta_\mu \lambda_\alpha - \delta^\alpha_\nu \lambda_\mu + \delta^\beta_\nu \lambda_\mu.$$ \hspace{1cm} (49)

We are ready to perform the path integral of (48) with respect to $+G^\tau_\mu_\nu$ (similarly for $-G^\rho_\sigma_\rho$). We get the constraint

$$(\tau^+) + G^\tau_\mu_\nu + W^\tau_\mu_\nu + \frac{1}{4} (\tau^+) \epsilon^{\tau_\lambda_\sigma_\rho} S^\alpha_\mu_\nu \epsilon_\tau_\lambda_\sigma_\rho = 0. \quad (50)$$

It is not difficult to see that $(\tau^+) + G^\tau_\tau_\nu + W^\tau_\tau_\nu = 0$ and $(\tau^+) + G^\tau_\tau_\alpha + W^\tau_\tau_\alpha = 0$. Using these two expressions we find

$$(\tau^+) + G^\tau_\mu_\nu + W^\tau_\mu_\nu - \frac{1}{4} \epsilon^{\tau_\lambda_\sigma_\rho} L^\alpha_\mu_\nu \epsilon_\tau_\lambda_\sigma_\rho = 0, \hspace{1cm} (51)$$

where

$$L^\alpha_\mu_\nu = \delta^\alpha_\mu \zeta_\nu^\beta - \delta^\beta_\mu \zeta_\nu^\alpha - \delta^\alpha_\nu \zeta_\mu^\beta + \delta^\beta_\nu \zeta_\mu^\alpha,$$ \hspace{1cm} (52)

with $\pm \zeta_\nu^\alpha = W^{\pm \alpha}_\nu - \frac{1}{3} \delta_\nu^\alpha W^{\lambda}_\lambda$. Now, solving (51) for $+G^\tau_\mu_\nu$ and substituting the result into (48) yields

$$S_E = \frac{1}{4} (-\frac{1}{\tau^+}) \int d^4 x \epsilon^{\mu\nu\alpha\beta} + W^\tau_\mu_\nu + G^\sigma_\rho_\rho \epsilon_\tau_\lambda_\sigma \rho - \frac{1}{4} (-\frac{1}{\tau^-}) \int d^4 x \epsilon^{\mu\nu\alpha\beta} - W^\tau_\mu_\nu - G^\sigma_\rho_\rho \epsilon_\tau_\lambda_\sigma \rho$$

$$- \frac{1}{4} (-\frac{1}{\tau^+}) \int d^4 x \epsilon^{\mu\nu\alpha\beta} + L^\tau_\mu_\nu + L^\tau_\nu_\nu \epsilon_\tau_\lambda_\sigma \rho + \frac{1}{4} (-\frac{1}{\tau^-}) \int d^4 x \epsilon^{\mu\nu\alpha\beta} - L^\tau_\mu_\nu - L^\tau_\nu_\nu \epsilon_\tau_\lambda_\sigma \rho, \hspace{1cm} (53)$$
where we apply similar procedure for \(-G_{\alpha\beta}^{\sigma\rho}\). But, this action can be obtained from the following action:

\[
S_D = \frac{1}{4} \left( -\frac{1}{\tau} \right) \int d^4 x \epsilon^{\mu\nu\alpha\beta} \left[ Q_{\mu\nu}^{\tau\lambda} + Q_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho} \right] - \frac{1}{4} \left( -\frac{1}{\tau} \right) \int d^4 x \epsilon^{\mu\nu\alpha\beta} \left( Q_{\mu\nu}^{\tau\lambda} - Q_{\alpha\beta}^{\sigma\rho} \epsilon_{\tau\lambda\sigma\rho} \right),
\]  

(54)

where

\[
\pm Q_{\mu\nu}^{\tau\lambda} = \pm W_{\mu\nu}^{\tau\lambda} + \pm K_{\mu\nu}^{\tau\lambda},
\]

(55)

with

\[
K_{\mu\nu}^{\alpha\beta\gamma} = \delta^{\alpha}_{\mu} \gamma_{\nu}^\beta - \delta^{\beta}_{\mu} \gamma_{\nu}^\alpha - \delta^{\alpha}_{\nu} \gamma_{\mu}^\beta + \delta^{\beta}_{\nu} \gamma_{\mu}^\alpha.
\]

(56)

Here, \(\gamma_{\mu\nu}\) is an auxiliary field. In fact, by integrating out (54) with respect to \(\gamma_{\mu\nu}\) we get (53). The action (54) is, of course, the dual action. Note that the action (54) has the dual gauge invariance \(V \rightarrow V - d\alpha\) and is of the general type (20) but with \(\tau \rightarrow -\frac{1}{\tau}\), where \(\tau\) can be either \(\tau^+\) or \(\tau^-\). Therefore, we have shown that the coupling transforms as \(\tau \rightarrow -\frac{1}{\tau}\) when one changes from the action (20) to the action (54). It is interesting to note that \(\gamma_{\mu\nu}\) plays the role of dual perturbation metric.

5.-FINAL COMMENTS

In this article, we have shown that it is possible to construct a S-dual action for linearized gravity. Our two main ideas were to rewrite linearized gravity as an Abelian gauge theory and to take recourse of Macdowell-Mansouri formalism. The analysis of S-duality for a linearized gravity then follows as in the case of Abelian gauge theories. An important step was the realization that the transformation (28) provides with enough symmetry to set \(A = 0\). After some long computations we finally prove that the partition function for linearized gravity has the S-dual symmetry

\[
Z_{A,h}(\tau) = Z_{V,\gamma}(-\frac{1}{\tau}),
\]

(57)
where $Z_{A_h}(\tau)$ is the partition function associated to the action (20), while $Z_{V,\gamma}(-\frac{1}{\tau})$ is the partition function associated to the action (54).

It seems that the present work can be extended without essential complications to the case of linearized supergravity. As it is known, the weak field limit of supergravity in four dimensions reduces to the sum of the Fierz-Pauli and Rarita-Schwinger actions, which are the unique actions for spin 2 and 3/2 without ghost (see [31] and references there in). This means that we can write the partition function for linearized supergravity as a product of two partition functions: one corresponding to linearized gravity and the other corresponding to Rarita-Schwinger field. Taking recourse of self-dual and antiself-dual supergravity and considering separately self-dual-antiself-dual linearized gravity and self-dual-antiself-dual Rarita-Schwinger, one can find the linearized supergravity-S-dual symmetry. For linearized gravity one may proceed as in previous sections, while for Rarita-Schwinger field one may use the method developed in reference [11]. In this way, one should expect to get the S-dual linearized supergravity symmetry $Z_{A_h,\psi}(\tau) = Z_{V,\gamma,\varphi}(-\frac{1}{\tau})$, where $\psi$ is the Rarita-Schwinger field and $\varphi$ its dual. At present, we are investigating in detail this possibility and we expect to report our results in the near future.

It is known that the S-duality gauge invariance is a matter of great interest in connection with M-theory [33-39] and supersymmetry [1,16,17] it may be interesting to see whether the present work can be useful in those directions.

Finally, it is worth mentioning the physical meaning, as well as the possible relevance, of the strong coupling limit of linearized gravity. For this purpose, we first need to clarify the physical meaning of the coupling constants $\tau^+$ and $\tau^-$. Write the action (16) as

$$S = \frac{1}{16\Lambda} \int d^4x \epsilon^{\mu\nu\alpha\beta} F_{\tau\lambda}^{\tau\lambda} \epsilon^{\sigma\rho}.$$

As was mentioned at the end of the section 3, $\Lambda$ can be identified with the cosmological constant. Now, using (22) it is not difficult to see that the action (20) can be reduced to

$$S = \frac{1}{8}(\tau^+ - \tau^-) \int d^4x \epsilon^{\mu\nu\alpha\beta} F_{\tau\lambda}^{\tau\lambda} \epsilon^{\sigma\rho} + \frac{i}{8}(\tau^+ + \tau^-) \int d^4x \epsilon^{\mu\nu\alpha\beta} F_{\tau\lambda}^{\tau\lambda} \epsilon^{\sigma\rho}. \quad (59)$$
Thus, if we consider the expressions
\[
\tau^+ = \frac{\Theta}{2\pi} + \frac{1}{4\Lambda}, \\
\tau^- = \frac{\Theta}{2\pi} - \frac{1}{4\Lambda}
\] (60)
we find that the action (59) can be written as
\[
S = \frac{1}{16\Lambda} \int d^4x \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} \tilde{F}_{\alpha\beta} \epsilon_{\tau\lambda\sigma\rho} + \frac{i\Theta}{8\pi} \int d^4x \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} \tilde{F}_{\alpha\beta} \delta_{\tau\lambda\sigma\rho}.
\] (61)
If we now recall the case of gauge theory where \(\tau^+\) and \(\tau^-\) are defined as
\[
\tau^+ = \frac{\theta}{2\pi} + \frac{4\pi}{g^2}, \\
\tau^- = \frac{\theta}{2\pi} - \frac{4\pi}{g^2},
\] (62)
we see that the cosmological constant \(\Lambda\) is playing the role of the gauge coupling constant \(g^2\) and that \(\Theta\) is playing the role of a \(\theta\) constant. Just as in gauge theory the duality transformation \(\tau \rightarrow -\frac{1}{\tau}\), where \(\tau\) can be either \(\tau^+\) or \(\tau^-\), can be considered, in its simple form, as a duality of the gauge coupling constant \(g^2 \rightarrow \frac{1}{g^2}\), in the case of linearized gravity (according to (60)) the duality \(\tau \rightarrow -\frac{1}{\tau}\) can be considered as a duality of the cosmological constant \(\Lambda \rightarrow \frac{1}{\Lambda}\).

This result seems to suggest a new mechanism to make small the cosmological constant, at least in linearized gravity (although similar conclusions must be possible in more general cases as in references [11]-[15]). In particular, the mechanism discussed in this work may be of physical interest in Kaluza-Klein theories where in the process called spontaneous compactification, normally, results a very large cosmological constant.

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