A Bayesian estimate of the skewness of the Cosmic Microwave Background

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ABSTRACT

We propose a formalism for estimating the skewness and angular power spectrum of a general Cosmic Microwave Background data set. We use the Edgeworth Expansion to define a non-Gaussian likelihood function that takes into account the anisotropic nature of the noise and the incompleteness of the sky coverage. The formalism is then applied to estimate the skewness of the publicly available 4 year Cosmic Background Explorer (COBE) Differential Microwave Radiometer data. We find that the data is consistent with a Gaussian skewness, and with isotropy. Inclusion of non Gaussian degrees of freedom has essentially no effect on estimates of the power spectrum, if each $C_\ell$ is regarded as a separate parameter or if the angular power spectrum is parametrized in terms of an amplitude ($Q$) and spectral index ($n$). Fixing the value of the angular power spectrum at its maximum likelihood estimate, the best fit skewness is $S = 6.5 \pm 6.0 \times 10^4 (\mu K)^3$; marginalizing over $Q$ the estimate of the skewness is $S = 6.5 \pm 8.4 \times 10^4 (\mu K)^3$ and marginalizing over $n$ one has $S = 6.5 \pm 8.5 \times 10^4 (\mu K)^3$.

Subject headings: Cosmology: theory – observation – cosmic microwave background: cosmic microwave background
1. Introduction

The main assumption in current analysis of Cosmic Microwave Background (CMB) data is that the perturbations responsible for structure formation in the universe are Gaussian. The theoretical prejudice is that the origin of these fluctuations was a period of Inflation: the vacuum fluctuations of the inflaton field on small scales (which satisfy Gaussian statistics) were linearly amplified and stretched to super horizon scales through the superluminal expansion of the universe. Although there are proposals for non-Gaussian fluctuations within the framework of inflation (Srednicki 1993, Gangui et al 1994, Peebles 1997, Linde & Mukhanov 1997), the paradigm is sufficiently compelling that the assumption of Gaussianity has been unquestioned when constructing methods of data analysis.

Recently, a number of groups have detected non-Gaussianity in the publicly available 4 year COBE DMR data. Ferreira, Magueijo & Górski (1998) found that the COBE normalized bispectrum was inconsistent with that of a Gaussian random field and were unable to attribute it to systematic effects or foreground contaminants. Pando, Valls-Gabaud & Fang (1998), using a statistic defined in terms of the wavelet transform of the COBE data, found evidence for non-Gaussianity localized in the northern hemisphere and Novikov, Feldman and Shandarin (1998) identified deviations from Gaussianity using topological measures; Bromley & Tegmark have confirmed these results. Banday, Zaroubi & Górski (1999) have suggested that the culprit for this non-Gaussianity, is a systematic effect in the time ordered data used to construct the publicly available COBE maps.

Possibly the most important result to come out of the analysis of the 4 year COBE data is an estimate of the angular power spectrum of the CMB; it can be used to normalize theories of large scale structure which when then compared to measures of clustering on other scales can be assessed as to their ability to describe our universe. In all estimates of the angular spectrum, the fluctuations in the CMB are assumed to be Gaussian. Given that the publicly available COBE 4 year data is indeed non-Gaussian, there is a possibility that all estimates of the angular power spectrum are biased in some way. In this letter we present a new estimate of the angular
power spectrum using an approximate likelihood function which includes non-Gaussian degrees of freedom. In section 2 we introduce the Edgeworth expansion; in section 3 we apply it to the 4 year COBE DMR data and in section 4 we discuss our findings.

2. The Formalism

In most of what follows we will use the notation of Górski (1994). A map of the CMB as rendered by the COBE DMR instrument, $\Delta_p$, can be expanded in spherical harmonics

$$\Delta_p = \sum_{\ell=0}^{\ell_{\text{max}}} \sum_{m=-\ell}^{\ell} [(a_{\ell m}^{CMB} + a_{\ell m}^{G}) w_{\ell}^{DMR} + a_{\ell m}^{N}] w_{\ell}^{pix} Y_{\ell m}(n_p)$$

where $Y_{\ell m}$ are real, orthonormal spherical harmonics, $w_{\ell}^{DMR}$ are the filter coefficients of the DMR beam pattern (Wright et al. 1994), and $w_{\ell}^{pix}$ are the filter coefficients used to model smoothing due to pixelization; $a_{\ell m}^{CMB}$, $a_{\ell m}^{Gal}$ and $a_{\ell m}^{N}$ are the harmonic coefficients of the CMB anisotropy, Galactic emission and instrument noise (from now on we will use the single index notation for the spherical harmonic labels, i.e. $i = \ell^2 + \ell + 1 + m$; $n_p$ is the unit vector which points at pixel $p$. A major aim of the DMR analysis is to obtain a good estimate of $a_{\ell m}^{MB}$, separating out the effects due to the Galactic contamination and receiver noise. This can be done by modeling the Galactic emission, using information at different frequencies Bennet et al (1992), and identifying which pixels in $\Delta_p$ are contaminated; one then excises these pixels from the map.

Introducing a Galactic cut in the data renders the $Y_{\ell m}$s non-orthonormal. Górski (1994) proposed a solution to this problem: he identified a new basis, $\psi_i(n_p)$ which is orthonormal on the cut sphere. The $a_i$ and the $Y_i(n_p)$ can be related to the harmonic coefficients, $c_i$ and functions in the new basis through $c_i = L^T a_i$ and $\psi_i(n_p) = L^{-1} Y_i(n_p)$. We have introduced vector notation for quantities with index $i$; $L$ is an triangular matrix so that $c_i$ is a combination of $a_j$ with $j \geq i$. It is straightforward to see that $\Delta_p = c^T \psi(n_p)$.

If the sky is assumed to be Gaussian, then the probability distribution of Fourier amplitudes
\[ G(c) = \frac{\partial c}{(2\pi)^{N_c/2}} e^{-\frac{1}{2}(c^T(C_{CMB} + C_N)^{-1}c)} \]  

(2)

Here \( C_{CMB} = \langle c_{CMB} c_{CMB}^T \rangle = L^T (a_{CMB} a_{CMB}^T) L \), \( (a_{CMB} a_{CMB}^T) = \text{diag}(a_i^2) \) and the noise covariance matrix is \( C_N = \Omega_{\text{pix}} \sum_{p \in \{\text{cutsky}\}} \sigma_p^2 \psi(n_p) \psi^T(n_p) \) (where \( \sigma_p \) is the RMS noise in pixel \( p \) and \( \Omega_{\text{pix}} \) is the solid angle of the pixel). The Bayesian approach is, given the \( c \) to find the \( C_\ell = \langle a_{\ell,m} a_{\ell,m}^T \rangle \), which minimize the Likelihood function, \( \mathcal{L}(C_\ell | C) = G(c | C_\ell) \) (where we have assumed uniform priors for both \( c \) and \( C_\ell \)).

The Edgeworth expansion permits a systematic extension of the Likelihood function to include non-Gaussian degrees of freedom (Kendall & Stuart 1977, Chambers 1967, McCullagh 1984, Juszkiewicz et al. 1995, Amendola 1996); in this letter we shall assume that the non-Gaussianity manifests itself as different levels of skewness, \( S_\alpha \), in sets of pixels, \( P_\alpha \). The Edgeworth expansion for the skewness is

\[ F(c, S_\alpha | c) = G(c | C_\ell)[1 + \sum_\alpha \frac{1}{6} S_\alpha \sum_p \xi^3_p + \sum_\alpha \sum_\beta \frac{1}{72} S_\beta S_\alpha \sum_{p \in P_\beta} \sum_{p' \in P_\alpha} \xi^6_{pp'}] \]  

(3)

where defining \( W = \psi(n_p)(C_{CMB} + C_N)^{-1} \psi^T(n_p') \) and \( \tilde{\Delta}_p = W_{pp'} \Delta_{pp'} \) one has:

\[
\begin{align*}
\xi^2_p &= \tilde{\Delta}_p - W_{pp} \\
\xi^3_p &= \tilde{\Delta}^3_p - 3W_{pp}\tilde{\Delta}_p \\
\xi^6_{pp'} &= \xi^3_p \xi^3_{p'} - 9\xi^2_p W_{pp'} \xi^2_{p'} + 18\tilde{\Delta}_p W_{pp'} \tilde{\Delta}_{pp'} - 6W_{pp'}^2
\end{align*}
\]  

(4)

The Edgeworth expansion contains a number of undesirable properties. Seen as a function of the data, for fixed parameters, a series truncation is in general not a density (not positive definite) and nor unimodal (not a single maximum). These properties follow from the general form of the hermite polynomials. For fixed data, as a function of the parameters, any truncation still suffers from the same, and other equally serious, problems. We work with a prescription which allows us to bypass this inconvenience. We rewrite the likelihood as

\[
F(x) = G(x) \left[ \prod_{\alpha} \prod_{p \in P_\alpha} (1 + \frac{S_\alpha}{6} \xi^3_p + \frac{S_\alpha^2}{72} \xi^6_p) \right] \times \left[ 1 + \sum_{\alpha} \sum_{\beta} \frac{S_\alpha S_\beta}{72} \sum_{p \in P_\alpha \neq p' \in P_\beta} \xi^6_{pp'} - \xi^3_p \xi^3_{p'} \right]
\]  

(5)
which is equivalent to (3) up to second order. The last factor vanishes should there not be any
correlations between pixels. We then expand the logarithm of the likelihood in a power series:

\[
\log F = \log G + \sum_\alpha \frac{S_\alpha}{6} \sum_{p \in P_\alpha} \xi_p^3 + \sum_\alpha \frac{S_\alpha^2}{72} \sum_{p \in P_\alpha} \left[ \xi_p^6 - (\xi_p^3)^2 \right] \\
+ \sum_\alpha \sum_\beta \frac{S_\alpha S_\beta}{72} \sum_{p \in P_\alpha, p \neq p'} \left( \xi_{pp'}^6 - \xi_p^3 \xi_{p'}^3 \right)
\]

(6)

One can check that exponentiation of (6) differs from (3) in terms higher than second order.
However exponentiation of (6) leads to a density. In fact the likelihood as a function of \( s \) is
now a Gaussian distribution. Our expansion is formally the same as the Edgeworth expansion;
however any truncation of a series of the form (6) is very different from a similar truncation of
the Edgeworth series, and is better behaved. This trick is described in [6.19] of Kendall & Stuart
1977. One may check that if the data is nearly Gaussian (in the sense that the sample cumulants
are small), and if there is no noise, then our estimator is the trivial estimator \( \sum_p (\Delta_p^3)/N \). On the
contrary (3) fails to display a maximum in most circumstances.

The goal is then to minimize \(-\log F\), given by Equations (6) and (4), in terms of \( C_\ell \) and \( S_\alpha \).

A few comments are in order. Firstly we have derived the most general Edgeworth expansion in
terms of skewness; a priori it would be natural to attribute one value of the skewness to all the
pixels in the cut sky (so there is only one \( \alpha \) and \( P_\alpha \) is the set of all pixels in the map). However,
a more conservative attitude is to relax statistical isotropy and assume that different parts of the
map will have different levels of skewness. In Section (3) we will estimate the skewness of the
full sky and the skewness of the northern and southern hemispheres. Secondly, the Edgeworth
expansion is valid in the regime of weak non-Gaussianity. Given that there is evidence of strong
non-Gaussianity in the COBE one might expect that we are beyond the regime of applicability of
the formalism. However one must bear in mind that the non-Gaussianity which has been identified
has been “localized” in harmonic space; statistical tests in pixel space have been consistent with
the hypothesis the 4 year COBE DMR data is a realization of a Gaussian random field.
3. Results

In this section we apply the formalism we have developed to the 4 year COBE DMR data. We will be using the inverse noise variance weighted, coadded maps of the 53A, 53B, 90A and 90B COBE-DMR channels at resolution 6, pixelized in the galactic frame. We use the extended galactic cut of Banday et al 1997, and Bennet et al 1996 to remove most of the emission from the plane of the Galaxy. We project out the components of the map corresponding to $\psi_i$ with $i = 1, 4$ which effectively removes the monopole and dipole in the process and keep 961 coefficients.

The first questions we are interested in addressing is if the assumption of zero skewness leads to a bias in the estimate of the angular power spectrum. We have found that the individual estimates of the $C_\ell$ are negligibly affected by introducing skewness; the introducion of the Edgeworth expansion introduces a roughly constant renormalization of the likelihood near the peak and consequently the $C_\ell$ estimates and its errorbars are affect by only few percent. It is convenient to parametrize the angular power spectrum in terms of a normalization, $Q = \sqrt{\frac{5}{4\pi}} a_2$, and a spectral index, $n$ such that

$$C_\ell = a_2^2 \frac{\Gamma(\ell + \frac{n-1}{2})\Gamma(\frac{9-n}{2})}{\Gamma(\ell + \frac{5-n}{2})\Gamma(\frac{3+n}{2})}$$

In Figure 1 we plot the likelihood contours for $Q$ and $n$, marginalizing over the skewness. Any modification to the Gaussian estimate is sufficiently small that the Gaussian likelihood contours and our estimates are indistinguishable. This is a promising result: weakening the assumption of Gaussianity does not change in any way existing estimates of the angular power spectrum.

We now focus on the value and distribution of the skewness. Fixing the various $C_\ell$ at the maximum we find that likelhood as a function of the the skewness takes the form given in Figure 2 (solid line). By construction it is Gaussian with a peak and 1-σ error bars, $S = 6.5 \pm 6.0 \times 10^4 (\mu K)^3$. The gaussian point $(S = 0)$ lies well within the estimated probability distribution i.e. our most stringent estimate of skewness is consistent with Gaussianity. One can relax the assumption of statistical isotropy, and consider the skewness in the South and North hemispheres as separate parameters (dashed and dotted lines respectively in Figure 2). We observe that most of the
statistical significance of our result comes from pixels in the Southern hemisphere. There is no obvious reason why this is so: the mean noise is essentially equivalent in the northern and southern hemisphere as is its range of values within each hemisphere.

An important point should be clarified with regards to our estimate: the value we obtain is larger than what one would expect by applying the naive estimator \( S = \frac{1}{N_{\text{pix}}} \sum_{p} \Delta_{p}^{3} \) as was used in Smoot et al 1994. The latter estimate is only strictly valid if \( \Delta_{p} \) are uncorrelated, not the case of the level 6 pixelization of the COBE data. In fact a rough estimate of the number of effective (or uncorrelated) pixels indicates that the naive estimate should be a factor of 6-8 times smaller than what we get, which is indeed what we find.

One can attempt to identify correlations between the estimates of \( Q \) or \( n \) and \( S \), by analysing the joint likelihood \((Q,S)\) and \((n,S)\) at the peak of the likelihood or marginalizing over \( n \) and \( Q \) respectively. The results are presented in Figure 3 where we plot the 68% and 95% contour levels. Clearly there is very little correlation between the pairs. Marginalizing over \( n \) (Figure 3 a) or \( Q \) (Figure 3 c) greatly increases the spread of the likelihood in \( Q \) and \( n \) respectively but has essentially the same effect on the estimated errors of the skewness; In the former case the maximum likelihood estimate is \( S = 6.5 \pm 8.4 \times 10^{4} (\mu K)^{3} \) and marginalizing over \( n \) one has \( S = 6.5 \pm 8.5 \times 10^{4} (\mu K)^{3} \). Marginalizing over both \( Q \) and \( n \) we obtain or most conservative estimate of the skewness, \( S = 6.5 \pm 8.7 \times 10^{4} (\mu K)^{3} \).

4. Discussion

We set up a Baysian framework with which to study systematically non Gaussianity. This is the natural next step in power spectrum estimation from CMB data sets with more general statistics and should be incorporated in future algorithms. We applied this framework to the estimate of the skewness of publicly available COBE-DMR 4 year maps. We found that the data is consistent with isotropy and Gaussianity as far as the skewness is concerned. In agreement with this, the corrections on power spectrum estimation due to introducing the skewness degree of
freedom are very small. If we parameterize the power spectrum in terms of all the $C_\ell$ the effects are negligible and one can safely use current Gaussian estimates of the angular power spectrum as correct. We then focused on the various possible estimates of the skewness and found our most conservative estimate to be $S = -6.5 \pm 8.7 \times 10^4 \mu K^3$.

We should perhaps comment on the differences between frequentist and Bayesian studies of non-Gaussianity. A frequentist approach (such as the one used in Kogut et al. (1996)) compares the value of a statistic (say skewness) as measured in noisy data with the distribution of what should have been measured if the underlying signal were Gaussian. The value of the measured statistic is not an estimate for its value for our sky. The Bayesian estimate on the contrary, is an estimate of what the signal skewness actually is, given the noise properties of the instrument, and what was observed. The Bayesian estimate for the skewness and its frequentist value need not, in fact, should not be the same. Nonetheless it is reassuring that both answers agree in that the signal’s skewness is Gaussian.

This is the first attempt at estimating the skewness from the publicly available COBE-DMR 4 year maps; Kogut et al. (1996) estimated the pseudocollapsed and equilateral three point function and found these to be also consistent with Gaussianity. Our result reinforces the fact that the non-Gaussianity present in COBE maps does not manifest itself in pixel space; the non-Gaussian signal is mostly localized in frequency space and only statistics that are sufficiently localized in $\ell$ space will be sensitive to it. The natural next step is to extend the Edgeworth formalism to the bispectrum; the complexity of the problem makes it currently numerically intractable for general sky coverage and noise covariance matrices. We are currently exploring various regimes where such a formalism may be feasible.

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Fig. 1.— The likelihood as a function of the amplitude $Q$ and spectral index $n$, marginalized over the skewness, $S$; the 68%, 95% and 99.7% contours are plotted. The result is indistinguishable from the $S = 0$ likelihood.
Fig. 2.— The Likelihood as a function of skewness, $S$ with the angular power spectrum fixed at its maximum likelihood estimates; the dashed line (dotted) line corresponds to using only the northern (southern) hemisphere. The solid line corresponds to using both.
Fig. 3.— The likelihoods as a function of amplitude, $Q$ and skewness, $S$, a) marginalized over the spectral index, $n$ and b) with $n$ fixed at its maximum likelihood estimate. The likelihoods as a function of $n$ and $S$, c) marginalized over $Q$ and d) with $Q$ fixed at its maximum likelihood estimate.