On the metric operator for quantum cylindrical waves

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ABSTRACT

Every (1 polarization) cylindrical wave solution of vacuum general relativity is completely determined by a corresponding axisymmetric solution to the free scalar wave equation on an auxiliary 2+1 dimensional flat spacetime. The physical metric at radius $R$ is determined by the energy, $\gamma(R)$, of the scalar field in a box (in the flat spacetime) of radius $R$. In a recent work, among other important results, Ashtekar and Pierri have introduced a strategy to study the quantum geometry in this system, through a regularized quantum counterpart of $\gamma(R)$. We show that this regularized object is a densely defined symmetric operator, thereby correcting an error in their proof of this result. We argue that it admits a self adjoint extension and show that the operator, unlike its classical counterpart, is not positive.
1. Introduction

Quantization of cylindrically symmetric gravity was initiated by Kuchar[1]. The system has a ‘z’ (translational) and a ‘φ’ (rotational) Killing field. From the general theory of 1 Killing reductions (see for eg. [2, 3]), an analysis of the space of orbits of the ‘z’ directional Killing field yields an equivalent description in terms of axisymmetric 2+1 gravity coupled to a massless axisymmetric scalar field. This latter description has been studied by Allen [4] and recently, by Ashtekar and Pierri [5].

As a consequence of the axial symmetry, the scalar field satisfies the free wave equation on an auxilliary flat 2+1 spacetime. The physical spacetime metric is then completely determined by the scalar field and the quantization of the system is based on the Fock space associated with the free axially symmetric scalar field[1, 4, 5].

Due to the axial symmetry, the only dynamical 3d metric components are the time-time and radial-radial ones, namely \( g_{TT} \) and \( g_{RR} \), where \( T, R \) are the Einstein Rosen coordinates[1].\(^1\) It turns out that \( g_{RR} = -g_{TT} = e^{\gamma(R,T)} \), where \( \gamma(R,T) \) is the energy of the scalar field in a disc of radius \( R \) at time \( T \).

Among other issues explored and elucidated by Ashtekar and Pierri in [5], such as a careful treatment of the gauge fixing procedure, extraction of the true degrees of freedom and discussion of the issue of time and the Hamiltonian in quantum theory, was the introduction of a strategy to discuss quantum geometry.

They found that since \( \gamma(R,T) \) is the energy of the field in a disc with a sharp boundary at \( R \), it is neither a differentiable function on the classical phase space nor is its quantum counterpart a well-defined operator on the Fock space[5]. They introduced a regulated version, \( \gamma(f_R) \), of \( \gamma(R,T) \) which ‘softens’ the sharp boundary by spatially integrating the scalar field energy density against a smearing function, \( f_R \), which is a smoothening of the step function (the step function falls abruptly to zero at \( R \)).

A key step in their discussion is the proof that the regularized operator, \( \hat{\gamma}(f_R) \), is densely defined. As we show in the appendix, it turns out that the sketch of the

\(^1\)Note that in [1], \( g_{TT}, g_{RR} \) refer to the 4d metric components in contrast to their 3d meaning here. The 4d and 3d components are related by a multiplicative, scalar field dependent factor [3].
argument given in [5] to establish this result is incorrect. In section 3, we rectify this situation by providing a detailed proof of this result. We also argue in favour of existence of self adjoint extensions of this (symmetric) operator.

In section 4 we show that : \( \hat{\gamma}(f_R) : \) admits negative expectation values. This is in contrast to its manifestly positive classical counterpart, but not unexpected since it is well known (see eg. [6]) that stress energy quadratic forms do admit negative expectation values.

Section 5 contains a discussion of our results and some comments.

Before we start on our results, we briefly summarize the relevant contents of [1, 4, 5] in Section 2. We shall use units in which the gravitational constant in 2+1 dimensions, \( G \), Planck’s constant, \( h \), and the speed of light, \( c \), are unity.

**Notation:** The quantum counterpart of a classical quantity, \( X \), is denoted by \( \hat{X} \). The time derivative of \( X \) is denoted by \( \dot{X} \), its radial derivative by \( X' \) and its complex conjugate by \( X^* \). The adjoint of \( \hat{X} \) is \( \hat{X}^\dagger \) and its normal ordered version is : \( \hat{X} \):.

## 2. Review of classical and quantum cylindrical waves

The cylindrical wave system is equivalent to axisymmetric 2+1 gravity coupled to a massless axisymmetric scalar field. It is this equivalent description which is analysed in [5]. We summarize the main results.

The 2+1 dimensional metric is

\[
ds^2 = e^{\gamma(R,T)}(-dT)^2 + (dR)^2 + R^2(d\phi)^2
\]

where \( \phi \) is the angular coordinate associated with the rotational Killing field, \( (R,T) \) are the Einstein Rosen coordinates [1] and

\[
\gamma(R, T) = \frac{1}{2} \int_0^R drr(\dot{\psi}(r, T)^2 + \psi'(r, T)^2)
\]

where \( \psi(R, T) \) is the axisymmetric solution to the flat 2+1 d wave equation,

\[
-\frac{\partial^2 \psi}{\partial T^2} + \frac{\partial^2 \psi}{\partial R^2} + \frac{1}{R} \frac{\partial \psi}{\partial R} = 0.
\]
The mode expansion of $\psi$ is
\[ \psi(R, T) = \int_{0}^{\infty} dk J_0(kR)(A(k)e^{-ikT} + A^*(k)e^{ikT}) \] (4)

The only nontrivial Poisson bracket between the mode coefficients is
\[ \{A(k), A^*(l)\} = -i\delta(l, k). \] (5)

Upon quantization, (4) goes over to the operator valued distribution
\[ \hat{\psi}(R, T) = \int_{0}^{\infty} dk J_0(kR)(\hat{A}(k)e^{-ikT} + \hat{A}^\dagger(k)e^{ikT}) \] (6)
where $\hat{A}(k), \hat{A}^\dagger(k)$ are the annihilation and creation operators for mode $k$. They are represented in the standard way on the Fock space. The only nontrivial commutation relation between these operators is
\[ [\hat{A}(k), \hat{A}^\dagger(l)] = \delta(k, l). \] (7)

In this work, we shall concentrate on the behaviour of operators at $T = 0$. Let $\gamma(R) := \gamma(R, T = 0)$. Then, from (2) and (6),
\[ :\gamma(R) := \frac{1}{2} \int_{0}^{\infty} drr\theta(R - r) :\left((\hat{\psi}(r, 0))^2 + (\hat{\psi}'(r, 0))^2\right) :. \] (8)
where $\theta(x) = 1$ if $x > 0$ and vanishes elsewhere. Direct computation [5] shows that $\| :\gamma(R) : |0 > \|$ diverges ($|0 >$ is the Fock vacuum and $\| \| \|$ denotes the Fock space norm) and so $:\gamma(R) :$ is not a well-defined operator on the Fock space. The divergence comes about due to the discontinuity of the $\theta$ function. The regulated version of this object, introduced in [5] is
\[ :\gamma(f_R) := \frac{1}{2} \int_{0}^{\infty} drrf_r(R)(\hat{\psi}(r, 0))^2 + (\hat{\psi}'(r, 0))^2) :. \] (9)
where $f_r(r)$ is a smooth function \(^2\) which equals 1 for $r \leq R - \epsilon$, smoothly decreases to zero for $R - \epsilon < r < R + \epsilon$ and remains zero for $r \geq R + \epsilon$. $\epsilon$ is a parameter of dimensions of length.

\(^2\)For technical reasons, we require $f_r(r)$ to be a $C^\infty$ function. Note that the particular example of $f_r(r)$ in [5] is not $C^\infty$ at $r = R - \epsilon$ and is not suitable for our purposes.
At \( T=0 \), the quantum line element becomes (see (1))

\[
\hat{d}s^2 = e^{\hat{\gamma}(f_R)}\left( -(dT)^2 + (dR)^2 \right) + R^2(d\phi)^2. \tag{10}
\]

Substituting (6) into (9), we obtain

\[
: \hat{\gamma}(f_R) : = \frac{1}{2} \int_0^\infty dk_1 \int_0^\infty dk_2 (2F_+(f_R, k_1, k_2)\hat{A}^\dagger(k_1)\hat{A}(k_2) + F_-(f_R, k_1, k_2)(\hat{A}^\dagger(k_1)\hat{A}(k_2) + \hat{A}(k_1)\hat{A}(k_2))) \tag{11}
\]

with

\[
F_\pm(f_R, k_1, k_2) = \pm k_1k_2 \int_0^\infty dr r f_R(r)(J_0(k_1r)J_0(k_2r) \pm J_1(k_1r)J_1(k_2r)). \tag{12}
\]

The rest of this work is devoted to an analysis of the properties of (11).

3. \( :\hat{\gamma}(f_R) : \) is densely defined

In this section we show that \( :\hat{\gamma}(f_R) : \) is a (symmetric) densely defined operator. We first demonstrate that it maps the vacuum into the Fock space. Next we show that it maps states of the form \( \int dk_1..dk_n g(k_1, .., k_n)\hat{A}^\dagger(k_1)\hat{A}^\dagger(k_n)|0> \), where \( g(k_1..k_n) \) is a smooth function of rapid decrease in \( (k_1, .., k_n) \) space, into the Fock space. The above states, for all \( n \), along with \( |0> \) span a dense subspace of the Fock space. Finally, we show that the operator is symmetric on this domain and argue for existence of its self adjoint extensions.

\( :\hat{\gamma}(f_R) : |0> \) is in the Fock space iff \( || :\hat{\gamma}(f_R) : |0> || \) is finite. But from (11)

\[
|| :\hat{\gamma}(f_R) : |0> ||^2 = 2 \int dk_1dk_2|F_-(f_R, k_1, k_2)|^2. \tag{13}
\]

We now demonstrate that \( \int dk_1dk_2|F_-(f_R, k_1, k_2)|^2 \) is finite. In fact, we shall prove the result for a general class of smearing functions, \( f(r) \), of which \( f_R(r) \) is a particular case. Let \( f(r) \) be \( C^\infty \) on \( (0, \infty) \) and let \( f(r) \) and all its derivatives vanish faster than any power of \( r^{-1} \) as \( r \to \infty \). Further, let all derivatives of \( f(r) \) vanish faster than
any power of $r$ as $r \to 0$. Denote the set of such functions by $S$.\(^3\)

**Lemma 1:**

\[
F_- := -k_1 k_2 \int_0^\infty dr f(r) (J_0(k_1 r) J_0(k_2 r) - J_1(k_1 r) J_1(k_2 r))
\]  

(14)

has good enough ultraviolet behaviour in $k_1, k_2$ that

\[
\int_0^\infty dk_1 \int_0^\infty dk_2 |F_-(f, k_1, k_2)|^2
\]

(15)

is finite.

**Proof:** We shall use the Hankel transform, $G(k)$, of $rf(r)$ with respect to $J_1(kr)$. From the appendix, $G(k) \to 0$ faster than any power of $k$ as $k \to \infty$, where (see for eg. page 453 of [7])

\[
G(k) = \int_0^\infty r f(r) J_1(kr) r dr
\]

(16)

\[
rf(r) = \int_0^\infty G(k) J_1(kr) k dk.
\]

(17)

Using (17) in (14), we obtain

\[
-F_- = k_1 k_2 \int_0^\infty dk G(k) \int_0^\infty df J_1(kr) (J_0(k_1 r) J_0(k_2 r) - J_1(k_1 r) J_1(k_2 r)).
\]

(18)

From page 411 of [7],

\[
\int_0^\infty dr J_1(kr) J_0(k_1 r) J_0(k_2 r) = \frac{\theta}{k \pi} \quad \text{if } k_1, k_2, k \text{ form the sides of a triangle, with } \theta \text{ the angle between sides of length } k_1 \text{ and } k_2
\]

\[
= \frac{1}{k} \quad \text{if } k > k_1 + k_2
\]

\[
= 0 \quad \text{otherwise}
\]

(19)

\(^3\)Note that $f \in S$ is a rotationally symmetric function of $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 = r^2$. Its behaviour as $r \to 0$ ensures that it is a $C^\infty$ function on $\mathbb{R}^2$ at the origin. It is obviously $C^\infty$ elsewhere. Thus $f \in S$ is a smooth function of rapid decrease on $\mathbb{R}^2$.  

5
and

\[ \int_0^\infty dr J_1(kr)J_1(k_1 r)J_1(k_2 r) = \frac{\sin \theta}{k \pi} \quad \text{if } k_1, k_2, k \text{ form the sides of a triangle,} \]
\[ \text{with } \theta \text{ the angle between sides of length } k_1 \text{ and } k_2 \]
\[ = 0 \quad \text{otherwise.} \quad (20) \]

Using (19) and (20) in (18), we get
\[ -F = k_1 k_2 \int_{|k_1 - k_2| < k < k_1 + k_2} dk G(k) \left( \frac{\theta - \sin \theta}{\pi} \right) + k_1 k_2 \int_{k > k_1 + k_2} G(k) dk. \quad (21) \]

We examine the behaviour of \( F \) in the UV regime, i.e. as \( k_1 \to \infty \) or \( k_2 \to \infty \) or both \( k_1, k_2 \to \infty \). From the Appendix A1, the last term gives rise to convergent contributions to (15) because it falls off faster than any power of \( k_1 + k_2 \). Hence, it suffices to examine the UV behaviour of

\[ I := k_1 k_2 \int_{|k_1 - k_2| < k < k_1 + k_2} dk G(k) \left( \frac{\theta - \sin \theta}{\pi} \right) \quad (22) \]

We first consider the case \( k_1 > k_2, \ k_1 \to \infty. \Rightarrow k_1 - k_2 < k < k_1 + k_2 \). From the definition of \( \theta \) (see (19)),

\[ k^2 = (k_1 - k_2)^2 + 4k_1 k_2 \sin^2 \frac{\theta}{2} \]
\[ kdk = k_1 k_2 \sin \theta d\theta. \quad (23) \]

\[ \Rightarrow I = \frac{(k_1 k_2)^2}{\pi} \int_0^\pi d\theta G(k) \sin \theta \frac{(\theta - \sin \theta)}{k}. \quad (24) \]

Let \( I_1 := \frac{(k_1 k_2)^2}{\pi} \int_{\theta_1}^\pi d\theta G(k) \sin \theta \frac{(\theta - \sin \theta)}{k} \) for some fixed small value of \( \theta_1 < 1 \) independent of \( k_1, k_2 \).

For \( \theta > \theta_1 \) and large enough \( k_1 \), from (23)

\[ (i) \quad 0 < k_2 < k_1^\frac{1}{3} \quad \Rightarrow (k_1 - k_2)^2 > k_1 \Rightarrow k > k_1^\frac{1}{3} \]

\[ (ii) \quad k_1^\frac{1}{3} < k_2 < k_1 \quad \Rightarrow k > 2k_1^\frac{2}{3} \sin \frac{\theta_1}{2}. \quad (25) \]

Thus for \( \theta > \theta_1 \), with fixed \( \theta_1 \) and large enough \( k_1, k > k_1^\frac{1}{3} \). Then from A1, \( G(k) \to 0 \) as \( k_1 \to \infty \) rapidly and so does \( I_1 \). Since \( I_1 \) gives convergent contributions to (15), it suffices to examine \( I_2 := \frac{(k_1 k_2)^2}{\pi} \int_{\theta_1}^\pi d\theta G(k) \sin \theta \frac{(\theta - \sin \theta)}{k} \).
Let \( p = k_1 - k_2, \ dk_1 dk_2 = dk_1 dp \) with \( k_1 \to \infty \). Then consider \( I_2 \) when 

(a) \( \frac{1}{10} p < k_1: \) \footnote{The specific choices of the values of the exponents as \( \frac{1}{10} \) here and of \( \frac{2}{9} \) in (26) are made only for concreteness.} From (23), \( k > p \). Thus \( k > k_1^{1/10} \) and from A1, \( G(k) \) renders the contribution in this range to (15) convergent.

(b) \( 0 < p < \frac{1}{10} k_1: \Rightarrow k_1 > k_2 > k_1 - \frac{k_1}{k_2} \) and \( k_2 \to \infty \). Put \( I_2 = I_{2(i)} + I_{2(ii)} \) with

\[
I_{2(i)} := \frac{(k_1 k_2)^2}{\pi} \int_{(k_1 k_2)^{-\frac{2}{9}}}^{\theta_1} d\theta G(k) \sin \theta \frac{(\theta - \sin \theta)}{k}.
\]

For \( (k_1 k_2)^{-\frac{2}{9}} < \theta < \theta_1 \), there exists some \( C > 0, C \) independent of \( k_1, k_2 \) such that for large enough \( k_1 \)

\[
k^2 > k_1 k_2 \sin^2 \frac{\theta}{2} > C k_1 k_2 (k_1 k_2)^{-\frac{2}{9}}.
\]

\( \Rightarrow k > k_1^{\frac{1}{3}} \) and again, from A1, \( G(k) \) renders this contribution convergent.

Finally consider

\[
I_{2(ii)} := \frac{(k_1 k_2)^2}{\pi} \int_0^{(k_1 k_2)^{-\frac{2}{9}}} d\theta G(k) \sin \theta \frac{(\theta - \sin \theta)}{k}.
\]

Note that \( |G(k)| \) is a bounded function. \( \Rightarrow \) There exists \( C_1 > 0 \) such that

\[
|I_{2(ii)}| < C_1 (k_1 k_2)^2 \int_0^{(k_1 k_2)^{-\frac{2}{9}}} d\theta \frac{\sin \theta (\theta - \sin \theta)}{\sqrt{k_1 k_2 \theta}} < C_1 (k_1 k_2)^2 \left( \frac{\theta^4}{\sqrt{k_1 k_2}} \right)_{\theta=(k_1 k_2)^{-\frac{2}{9}}} \leq C_1 (k_1 k_2)^{-\frac{2}{9}} \]

\( \Rightarrow |I_{2(ii)}| < C_1 (k_1 k_2)^{-\frac{2}{9}} \)

\[
\Rightarrow \int_0^{k_1^{1/10}} |I_{2(ii)}|^2 dp < C_1^2 \int_0^{k_1^{1/10}} dp (k_1 (k_1 - p))^{-\frac{2}{9}} < 2 C_1^2 k_1^{\frac{9}{50}}
\]

\[
\Rightarrow \int_{k_1 \to \infty} dk_1 \int_0^{k_1^{1/10}} |I_{2(ii)}|^2 dp < \frac{2 C_1^2}{k_1^{\frac{9}{50}}} \to 0.
\]
Thus $I_{2(ii)}$ also gives a convergent contribution. Hence $\int_{k_2<k_1} |F_-|^2 dk_1 dk_2$ is finite. Since $F_-$ is symmetric under interchange of $k_1$ and $k_2$, (15) is finite.

We have also checked that no divergent terms appear in the expression for $||:\gamma(f):|g,n>||$ where

$$|g,n> := \int dk_1..dk_n g(k_1,..,k_n)\hat{A}^\dagger(k_1)..\hat{A}^\dagger(k_n)|0>,$$  \hspace{1cm} (33)

with $g(k_1,..,k_n)$ a smooth function of rapid decrease in $(k_1,..,k_n)$.

We sketch the main steps of the relevant calculation. The key point is that all terms in $||:\gamma(f):|g,n>||$ involving the potentially divergent expression $\int_0^\infty F_+(f,k_1,k_2)^2 dk_1 dk_2$ cancel. The remaining terms depend on $F_+$ through either the expression

$$\int_0^\infty ds dp dq \int dk_2..dk_n F_+(f,p,s)F_+(f,s,q)g^*(p,k_2,..,k_n)g(q,k_2,..,k_n)$$ \hspace{1cm} (34)

or the expression

$$\int_0^\infty ds dt dp dq \int dk_3..dk_n F_+(f,s,t)F_+(f,p,q)g^*(s,p,k_3,..,k_n)g(t,q,k_3,..,k_n)$$ \hspace{1cm} (35)

Since $g(k_1..k_n)$ vanishes at infinity rapidly in each of its arguments, we can use Lemma 2 below to show that these terms are finite.

Lemma 2 : Let $G_{\pm}(k_1) := \int_0^\infty dk_2 F_{\pm}(f,k_1,k_2)g(k_2)$ with $g(k_2)$ a smooth function of rapid decrease as $k_2 \to \infty$. Then $G_{\pm}(k_1)$ falls off rapidly as $k_1 \to \infty$.

Proof : From (16)- (20),

$$\pm F_{\pm} = k_1 k_2 \int_{|k_1-k_2|<k<k_1+k_2} dkG(k)\frac{(\theta \pm \sin \theta)}{\pi} + k_1 k_2 \int_{k>k_1+k_2} G(k)dk.$$ \hspace{1cm} (36)

Let $0 < k_2 < k_1 - k_1^{\frac{1}{2}}$, $k_1 \to \infty$.:

From (23) and Appendix A1, the first term falls off rapidly with large $k_1$. From A1, so does the second term. Hence

$$G^{(1)}_{\pm}(k_1) := \int_0^{k_1-k_1^{\frac{1}{2}}} dk_2 F_{\pm}(f,k_1,k_2)g(k_2)$$ \hspace{1cm} (37)
also vanishes rapidly as $k_1 \to \infty$.

Let $k_2 > k_1 - k_1^{1/2}$, $k_1, k_2 \to \infty$.

Since $J_0(x), J_1(x)$ are bounded functions of $x$, there exists $C_1$ such that

$$|J_0(k_1 r)J_0(k_2 r) \pm J_1(k_1 r)J_1(k_1 r)| < C_1$$

$$\Rightarrow |F_{\pm}(f, k_1, k_2)| < C_f k_1 k_2,$$

$C_f := C_1 \int_0^\infty df r(r).$ (38)

From (38) and the rapid fall off of $g(k_2)$ with large $k_2$, it is clear that

$$G_{\pm}^{(2)}(k_1) := \int_{k_1 - k_1^{1/2}}^{\infty} dk_2 F_{\pm}(f, k_1, k_2) g(k_2)$$

falls off rapidly as $k_1 \to \infty$.

Since $G_{\pm}(k_1) = G_{\pm}^{(1)}(k_1) + G_{\pm}^{(2)}(k_1)$, $G_{\pm}(k_1)$ also falls off rapidly as $k_1 \to \infty$. This completes the proof.

Using Lemmas 1 and 2, it can also be verified that all conceivable terms in $||: \tilde{\gamma}(f) : g, n > ||$ involving $F_-$ are also finite.

Thus, the result of these calculations is that $: \tilde{\gamma}(f) :$ maps the dense domain consisting of all $|g, n >$ and $|0 >$ into the Fock space. Further, since $F_{\pm}(f, k_1, k_2)$ are real it is also symmetric and we have shown that $: \tilde{\gamma}(f) :$ is a densely defined symmetric operator.

We now argue in favour of existence of its self adjoint extensions along the lines of [5]. We shall assume that there is no obstruction to an application of the standard treatments [8] of free fields on $\mathbb{R}^n$, to the axisymmetric case. Although we do not define the relevant function space below, we expect that a more careful treatment will convert our argument into a rigorous proof.

Let $\hat{C}$ be the complex conjugation operator in the Schroedinger representation on $L^2(S', d\mu)$ where $S'$ is the appropriate space of distributions and $d\mu$ the standard Gaussian measure associated with the restriction of the Laplacian in 2d to rotational symmetry. Then $\hat{C}\hat{A}(k) = \hat{A}(k)\hat{C}$ and $\hat{C}\hat{A}^\dagger(k) = \hat{A}^\dagger(k)\hat{C}$. Also $\hat{C}$ leaves the domain of $: \tilde{\gamma}(f) :$ invariant. This coupled with the reality of $F_\pm$ implies that $\hat{C}$ commutes

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6The likely candidate for $S'$ is the topological dual to the space $S$ introduced earlier in this section.
with $\hat{\gamma}(f)$: Then the theorem of Von Neumann [9] cited in [5] shows that the operator admits self adjoint extensions.

4. Non-positivity of $\hat{\gamma}(f_R)$:

We demonstrate the existence of states on which $\hat{\gamma}(f_R)$ has negative expectation value. The candidate states are motivated by those considered in [6]. Let

$$\langle \phi | := \frac{1}{1 + 2\lambda^2} |0 > + \lambda \int_0^\infty dp_1 dp_2 g(p_1, p_2) A^\dagger(p_1) A^\dagger(p_1)$$

(40)

with

$$\int_0^\infty dp_1 dp_2 g(p_1, p_2) g^*(p_1, p_2) = 1 \quad g(p_1, p_2) = g(p_2, p_1)$$

(41)

Then a straightforward computation yields

$$\langle \phi | \hat{\gamma}(f_R) | \phi > = \frac{4\lambda}{1 + 2\lambda^2} \int_0^\infty dk_1 \int_0^\infty dk_2$$

$$\left(2\lambda F_+(f_R, k_1, k_2) \int_0^\infty dl g(l, k_2) g^*(k_1, l) + F_-(f_R, k_1, k_2) \text{Re}(g(k_1, k_2)) \right)$$

(42)

where ‘Re’ refers to ‘real part of’.

We shall choose $g(k_1, k_2)$ to be a step function. Thus, it is not contained in the dense domain defined in the previous section. However, the choice of step function is just for pedagogy - as can easily be verified, the conclusions are unaltered even if we replace the step function by a suitable smooth function of compact support which drops to zero sufficiently fast. Choose

$$g(k_1, k_2) = \frac{1}{2\delta} \quad |k_1 - k_0| < \delta, |k_2 - k_0| < \delta$$

$$= 0 \quad \text{otherwise}$$

(43)

where $k_0$ is a parameter which will be fixed later. Then for sufficiently small $\delta$ to leading order in $\delta$,

$$\langle \phi | \hat{\gamma}(f_R) | \phi > = \frac{8\lambda\delta}{1 + 2\lambda^2} \left(2\lambda F_+(f_R, k_0, k_0) + F_-(f_R, k_0, k_0) \right)$$

(44)
Now choose $k_0$ such that $k_0(R + \epsilon) << 1$. This enables us to use the small argument expansions of the Bessel functions appearing in (12) and to leading order in $k_0(R + \epsilon)$ we obtain,

$$F_{\pm}(f_R, k_0, k_0) \sim \pm (k_0(R + \epsilon))^2 M, \quad M = \frac{1}{(R + \epsilon)^2} \int_{0}^{R+\epsilon} drr f_R(r).$$

Thus, to leading order in $\delta$ and $k_0(R + \epsilon)$,

$$<\phi| \hat{\gamma}(f_R) |\phi> = \frac{8\lambda\delta (k_0(R + \epsilon))^2}{1 + 2\lambda^2} (2\lambda - 1) M. \quad (46)$$

For $0 < \lambda < \frac{1}{2}$, $\hat{\gamma}(f_R)$ has negative expectation values. Hence, in contrast to the positivity of its classical counterpart, $\hat{\gamma}(f_R)$ is not a positive operator.

### 5. Open issues and discussion

By virtue of its solvability, the cylindrical wave midisuperspace is a very useful toy model for issues in full quantum gravity. In contrast to the cosmological minisuperspace models, we can learn about the nonlinear field theoretic aspects of quantum gravity from this system. To do this, it is important to understand the quantum metric operator.

We have shown that the regulated metric operator, $\hat{\gamma}(f_R)$, defined in [5] is indeed a densely defined symmetric operator. In contrast to its classical positivity, $\hat{\gamma}(f_R)$ is not a positive quantum operator.

Our results can also be viewed from the standpoint of flat space quantum field theory. Indeed, apart from its significance for the quantum geometry of the cylindrical wave system, $\hat{\gamma}(f_R)$ can be thought of as a spatially smeared Hamiltonian operator for (axisymmetric) free quantum field theory on a flat 2+1 spacetime. In the absence of axisymmetry, such spatially smeared Hamiltonians are well-defined quantum operators in 1+1 dimensions but not in 2+1 (see appendix) or higher dimensions [10, 11]. Thus the axisymmetric 2+1 case is a ‘borderline’ case in which all nonzero angular momentum modes of the field are switched off. Due to the absence of vacuum fluctuation contributions from these modes, the smeared Hamiltonian turns out to be a well-defined operator. Indeed, it is the presence of these fluctuations which
invalidates the idea of [5] to apply non axisymmetric 2+1 flat spacetime quantum field theory results to the axisymmetric case.

The next step in understanding $\hat{\gamma}(f_R)$ would be to see if the operator is unbounded below or not. In this regard the results of Helfer [12] suggest that since $\hat{\gamma}(f_R)$ is a well-defined operator, it should be bounded from below. If so, it would be of interest to calculate a lower bound for this operator, maybe by trying to generalise the techniques of [13].

In case there is a lower bound, it is possible to say more about the properties of the operator. For example, by Theorem XII.5.2 of [15], $\hat{\gamma}(f_R)$ would then admit a self adjoint extension with the same lower bound. Moreover, qualitative information regarding the spectrum may be obtained using the min-max principle of [16].

Ultimately we would like to compare the spectrum of metric based operators (such as the determinant of the metric and the area operator) in this Fock representation of cylindrical waves with the spectrum of analogous operators in the (non Fock) spin network representation of quantum gravity [17, 18]. In this regard note that for cylindrical waves, the (regulated) area operator associated with an annulus between radii $R_1$ and $R_2$ is

$$\hat{A}(R_1, R_2) = \frac{1}{2\pi} \int_{R_1}^{R_2} \sqrt{\hat{g}_{RR}} RdR = \frac{1}{2\pi} \int_{R_1}^{R_2} e^{\frac{\gamma(f_R)}{2}} RdR,$$

(47)

where $\hat{g}_{RR} = e^{\gamma(f_R)}$. The $\hat{g}_{RR}$ operators at different radii do not necessarily commute [5] and it is difficult to infer anything about the spectrum of the area operator from that of $\hat{g}_{RR}$.

We leave these and other issues (such as alternative regularizations of the metric/area operators) for possible future work.

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Appendix

A1

Let \( f(r) \in S \) (see beginning of section 3 for definition of \( S \)). We show below that
\[
G(k) = \int_0^\infty r f(r) J_1(kr) r dr \tag{48}
\]
vanishes faster than any power of \( k^{-1} \) as \( k \to \infty \).

Note that (see footnote 3, section 3) as a rotationally symmetric function on \( \mathbb{R}^2 \), \( f(r) \) is Schwartz. Hence its 2d Fourier transform, \( F(k) \), \(^7\) is also Schwartz (see for eg. \[19\]).

\[
F(k) = \frac{1}{2\pi} \int_0^\infty dr \int_0^{2\pi} d\theta f(r) e^{ikr \cos \theta}. \tag{49}
\]

From \[20\] \( J_0(kr) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{ikr \cos \theta} \). Hence
\[
F(k) = \int_0^\infty dr r f(r) J_0(kr). \tag{50}
\]

Thus, the Hankel transform of \( f(r) \) with respect to \( J_0(kr) \) is the same as its 2d Fourier transform. Hence the Hankel transform \( F(k) \) is a function of rapid decrease. Using properties of the Hankel transform and the Bessel functions\(^7\) we get
\[
f(r) = \int_0^\infty dk k J_0(kr) F(k) = \frac{1}{r} \int_0^\infty dk F(k) \frac{d}{dk}(J_1(kr)k) \tag{51}
\]
A by parts integration using the asymptotic properties of \( F(k) \) gives
\[
\Rightarrow rf(r) = - \int_0^\infty dk k J_1(kr) \frac{d}{dk} F(k) \tag{52}
\]
But \( G(k) = -\frac{d}{dk} F(k) \) is a function of rapid decrease because \( F(k) \) is a function of rapid decrease.

Q. E. D.

A2

In this appendix we identify the error in the proof of \[5\], which attempts to show that \( ||\hat{\gamma}_{\hat{f}_R}||_0 > || \) is finite. \(^8\) We assume familiarity with the relevant part of \[5\].

\(^7\)The Fourier transform of a rotationally symmetric function evaluated at the vector \( \vec{k} \) depends only on the magnitude \( |\vec{k}| \) of \( \vec{k} \) as can be seen from (49).

\(^8\)We use unpublished results of C. Torre below and we thank him for allowing us to reproduce his results here.
The claim in [5] is that

\[ I := \int d^2\vec{k}_1 d^2\vec{k}_2 |G_-(f_R, \vec{k}_1, \vec{k}_2)|^2 \]  

(53)
is finite with

\[ G_-(f_R, \vec{k}_1, \vec{k}_2) = -\frac{(k_1 k_2 + \vec{k}_1 \cdot \vec{k}_2)}{\sqrt{k_1 k_2}} f(\vec{k}_1 + \vec{k}_2) \]  

(54)

where \( f(\vec{X}) \) is the 2d Fourier transform of \( f_R(r) \) evaluated at wave vector \( \vec{X} \) and the magnitude of \( \vec{X} \) is denoted by \( |\vec{X}| = X \). We shall show that, in fact, \( I \) diverges.

Note that (53) converges iff it converges absolutely. Therefore, in what follows, we are permitted to change variables and orders of integration [21].

Put \( \vec{k}_1 + \vec{k}_2 = \vec{q} \). Let \( \theta \) be the angle between \( \vec{k}_1 \) and \( \vec{q} \) and let \( \theta_1 \) be the angular coordinate of \( \vec{k}_1 \). Then, with \( 0 < \theta_1 < 2\pi \) and \( \theta_1 < \theta < \theta_1 + 2\pi \),

\[ d^2\vec{k}_1 d^2\vec{k}_2 = k_1 dk_1 d\theta_1 q dq d\theta. \]  

(55)

Then,

\[ |G_-(f_R, \vec{k}_1, \vec{k}_2)|^2 = |f(\vec{q})|^2 |\Phi_{\vec{q}}(\vec{k}_1)|^2 \]  

(56)
with

\[ \Phi_{\vec{q}}(\vec{k}_1) = \frac{(k_1 |\vec{q} - \vec{k}_1| + \vec{k}_1 \cdot (\vec{q} - \vec{k}_1))}{\sqrt{k_1 |\vec{q} - \vec{k}_1|}}. \]  

(57)

We now expand the numerator of \( \Phi_{\vec{q}}(\vec{k}_1) \) in powers of \( \frac{q}{k_1} \) to get the leading behavior as \( q/k_1 \to 0 \):

\[ \Phi_{\vec{q}}(\vec{k}_1) \sim \frac{q^2 \sin^2 \theta}{2k_1}. \]  

(58)

From (55), (56) and (58), \( I \) diverges logarithmically with \( k_1 \) as \( k_1 \to \infty \). (Note that when \( R = \infty \), \( f(\vec{q}) \) has support only at \( q = 0 \) and hence the integral vanishes. This is as expected because in this limit \( f_{R=\infty} = 1 \) and \( \hat{\gamma}(f_{\infty}) \) is just the total energy operator for the scalar field. But for any finite \( R \), the integral \( I \) diverges.)
References


