Scattering of short laser pulses from trapped fermions

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We investigate the scattering of intense short pulses of light off trapped cold fermionic atoms. We discuss the sensitivity of the scattered light to the quantum statistics of the atoms. The temperature dependence of the scattered light spectrum is also calculated. Comparisons are made with a system of classical atoms who obey Maxwell-Boltzmann statistics. We find the total scattering increases as the fermions become cooler but eventually tails off at very low temperatures (far below the Fermi temperature). At these low temperatures the fermionic degeneracy plays an important role in the scattering as it inhibits spontaneous emission into energy levels below the Fermi one. Static atom-atom interactions are neglected in our treatment since s-wave collision is forbidden between fermions in the same state. Light induced dipole interactions are not explicitly included as we assume the single atom dipole Rabi frequency of the external driving field to be very large.

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I. INTRODUCTION

Rapid advances of trapping and cooling of alkali atoms [1] have led to the recent dramatic achievement of Bose-Einstein condensation (BEC) of these trapped atoms [2–5]. Focusing these highly successful trapping and cooling methods onto fermions instead of bosons offers additional rich opportunities for studying degenerate fermionic atomic gases. The description of these magnetically trapped Fermi gases presents additional simplification as the Pauli exclusion principle forbids the unsuppressed low energy s-wave collisions among atoms in the same hyperfine state. Thus these dilute trapped atoms behave very close to the ideal Fermi gas so ubiquitous in physics textbooks for decades. Recent highlights from several leading experimental groups have indicated that we are at the edge of being able to explore quantum degenerate Fermi gas [6,7] inside laboratories.

Once such a degenerate gas is achieved, how do we perform diagnostic measurements on its properties? A standard method in atomic physics is to probe the gas with light scattering. This is the problem to be addressed in our paper. These spectroscopic light scattering methods have already been suggested for BEC in both the weak and strong field scattering regimes [8,9]. These early theoretical investigations had only limited impact on experimental observations so far, partly because the resonant photon-atom interaction becomes a complicated many body problem when a condensate is involved. The recent dramatic demonstration of low group velocity of light propagation inside a condensate [10], the Bragg scattering experiment [11], and the surprising observation of super-radiant collective spontaneous emission from MIT [12] all calls for more detailed applications of quantum field theory of photons interacting with atoms. In the last several years, the light scattering off Fermi degenerate atoms have already been discussed by several groups; some of these investigations have focused on the case of a distribution of ideal atoms obeying Fermi-Dirac statistics [13]; while other considered more exotic state for a Cooper paired fermionic condensate [14]. One notable feature for a Fermi degenerate gas is that the Pauli exclusion principle blocks (inhibits) scattering events for atoms into already occupied states [15].

In a previous paper involving two of us (LY and ML) an optical method for detection of the properties of BEC was proposed [16]. This detection involves the limiting case of scattering short but intense laser pulses from a system of cooled bosonic atoms in a trap. In particular, the case of laser pulses with areas of $2\pi K$ was investigated. (The pulse area to be defined later is proportional to the integral of the slowly varying envelope of the electric field multiplied by the atomic dipole moment). It was shown that such pulses mainly cause cyclic Rabi oscillations for atoms in their excited and ground states. Thus to zero-th order involves no photon scattering (except the stimulated interaction with the driving field). However, atoms can spontaneously emit photons while in their excited states, therefore, the previous zero-th order picture involving coherent evolution of all atoms can be interrupted by spontaneous emissions by individual atoms. Thus scattered light from these emissions can be collected and their properties reflect to a certain degree the properties of the trapped atoms. It was shown that for the case of Bose-Einstein condensation, above the critical temperature, $T_c$, the coherent scattering is very weak and is predominately in the forward direction due to phase matching effects. However, below the critical temperature the number of scattered photons increases dramatically and the coherent scattering occurs within a solid angle determined by the size of the condensate. For sufficiently short $2\pi K$-pulses the system is
preserved even below $T_c$ so that this method can be used as a non-destructive probe of BEC.

This paper presents an extension of our investigations to the analogous case of a system of trapped fermionic atoms. The ideal atoms of a Fermi-Dirac distribution is considered in this paper. The static properties of harmonic trapped ideal fermionic atoms as considered by several previous groups will be essentially used as inputs [17]. The more subtle case involving a weak attractive interactions between atoms could potentially develop into a BCS type Cooper paired condensate [18], and whose pulsed light scattering properties [14] will be further explored in a future publication. In the present study involving ideal non-interacting fermionic atoms, we do not expect to see a dramatic change in the spectrum as one cools the gas since no phase transition occurs in the trapped gases even at zero temperature. However, with a finite number of atoms (say 1 million) and cooled far below the Fermi temperature [6] one does expect that the quantum statistics should play a role in the spectrum.

This paper is organized as follows: first we review the formulation as was presented previously [8] for trapped bosons. In the second quantized form, the only difference now would be the commutation relations between atomic operators. We then calculate the spectra of scattered light taking into account the fermionic nature of the atomic operators. Our new results are displayed numerically in Section IV. A detailed discussion section follows where both the angular and spectral distribution will be presented. We also find the total number of atoms scattered for two typical experimental parameter sets. Finally we conclude with a summary of the essential physics learned from this investigation. In the appendix we show how we can re-write the expression for the spectrum of the coherently scattered light in a form that is suitable for numerical calculation.

As will become obvious by the introduction of many newly derived analytic formulae, the trapped fermion case is numerically far more difficult to calculate than the previous bosonic one. At low temperatures the fermionic system consists of many stacked energy levels [17] whereas the bosonic system nicely condenses to a few of the lowest energy levels. Also the numerical methods for infinite series summations used previously for bosons are only applicable for the high temperature regime of a Fermi gas. New methods are therefore developed to cope with the additional difficulties of the fermionic system.

The first few sections of this paper follows closely to the BEC situation. Initially the treatment is formally the same for both bosons and fermions, the only difference being the commutation relations. However, these differences become very significant later especially with the calculation of the spectrum.

II. THE MODEL

We consider a system consisting of $N$ fermionic atoms confined in a trap interacting with light. Let us write down its Hamiltonian in the Fock representation and in the second quantized form [8]:

$$
H = \sum_{\tilde{n}} \omega^g_{\tilde{m}} g^g_{\tilde{n}} + \sum_{\tilde{m}} (\omega^e_{\tilde{m}} + \omega_0) e^e_{\tilde{m}} e^e_{\tilde{m}}^\dagger + \sum_{\tilde{n},\tilde{m}} \int d\tilde{k} g(y) [\eta_{\tilde{n}\tilde{m}}(\tilde{k}) g^g_{\tilde{n}} a^e_{\tilde{k}\mu} e^e_{\tilde{m}} e^e_{\tilde{m}}^\dagger + h.c.] + \sum_{\mu} \int d\tilde{k} c k a^e_{\tilde{k}\mu} a^e_{\tilde{k}\mu}^\dagger.
$$

(1)

where the Franck-Condon factors are

$$
\eta_{\tilde{n}\tilde{m}}(\tilde{k}) = \langle g,\tilde{n}|e^{-ik\cdot\tilde{R}}|e,\tilde{m}\rangle
$$

(2)

with $\tilde{R}$ denoting the position operator of the atom (its nucleus). We have used atomic units (unless otherwise stated), the rotating wave and the dipole approximation. The atomic annihilation and creation operators for the $n$-th state of the center of mass motion of the atoms in the trap are denoted by $g_\tilde{n}$ and $g^g_{\tilde{n}}$ respectively. Since these operators are associated with the atoms in the ground electronic state, for the case of a spherically symmetric harmonic trap potential, $\tilde{n}$ has three components $(n_x,n_y,n_z)$ and energy $\omega^g_{\tilde{n}} = \omega (n_x + n_y + n_z)$ where $\omega$ is the trap frequency. The size of the trap is related to the size of the ground state of the trap potential, $a = \sqrt{1/2M\omega_0}$ ($h = 1$). The atomic annihilation and creation operators in the excited state $e^e_{\tilde{m}}$ and $e^e_{\tilde{m}}^\dagger$ may experience different trap potential from that of the ground state. However, from our analysis of short pulse scattering, we find that the particular shape of the excited state potential is unimportant since atoms only spend a very short period of time in the excited state [8]. The electronic transition occurs at the frequency $\omega_0$. Since we are treating an $s$-state to a $p$-state transition the excited state operators are vectors $e^e_{\tilde{m}}$ and $e^e_{\tilde{m}}^\dagger$. Annihilation and creation operators for photons of momentum $\tilde{k}$ and linear polarization $e^e_{\tilde{k}\mu} (\mu = 1,2)$ are denoted by $a^e_{\tilde{k}\mu}$ and $a^e_{\tilde{k}\mu}^\dagger$. All atomic operators obey standard fermionic anti-commutation relations. $g(k)$ is a slowly varying coupling which is dependent on $k$. Its relation to the natural linewidth is $\gamma = (8\pi^2 k_0^2/3c)|g(k_0)|^2$, with $k_0 = \omega_0/c$. For notational convenience, we will suppress the indices $g,e$ for the internal states. The convention being that the indices $\tilde{n}$, $\tilde{m}$ represents the center of mass states in the electronic ground potential whereas $\tilde{n}$, $\tilde{m}$ denotes the center of mass states in the excited state potentials.

Note that the strong resonant atomic dipole-dipole interaction resulting from the exchange of transverse photons is included in Eq. (1).

If the system is driven by a coherent laser pulse then we may neglect spontaneous emission effects during the
pulse if the pulse is sufficiently short and intense. We can then safely substitute the electric field operator entering the interaction Hamiltonian in Eq. (1) by a c-number. The pulses we intend to use should have duration \( \tau_L \leq 300 \) ps or shorter, i.e. width \( \gamma_L = 1/\tau_L \approx 3 \times 10^9 - 10^{11} \) Hz. A first estimate shows that \( \gamma_L \gg \gamma \approx 2.5 \) MHz, so that spontaneous emission may be neglected during the interaction time of the pulse with the atoms. Note that the current estimate for spontaneous emission is a far better one than in the bosonic case as there is no Bose enhancement. However, the fermionic nature may come into play at very low temperatures where the effective spontaneous emission rate will be greatly reduced due to suppression by the Fermi sea of ground levels. Therefore it is more valid to assume that the effects of dissipative spontaneous emission and dispersive dipole-dipole interactions are small compared to the coherent driving laser during the interaction between the atoms and the laser pulse. We can then replace the product of the electric field operator and the absolute value of the electronic transition dipole moment by

\[
d\hat{E}(+) = \frac{\Omega}{2} \sum_\mu \int d\tilde{k} \, \hat{g}(\tilde{k}, \mu)e^{i\tilde{k} \cdot \vec{R} - i\omega t}. \tag{3}
\]

The envelope of the laser pulse is defined by \( \hat{g}(\tilde{k}, \mu) \). \( \Omega \) is the peak Rabi frequency of the laser pulse. Using the assumption that the pulse is a plane wave packet moving in the \( \tilde{k} \) direction with a central frequency \( \omega_L \) and a linear polarization \( \vec{e}_L \), we obtain

\[
d\hat{E}(+) = \frac{\Omega}{2} \hat{e}_L \mathcal{T}[\gamma_L(t - \tilde{k} \cdot \vec{R}/\omega_L)]e^{i\tilde{k} \cdot \vec{R} - i\omega_L t}. \tag{4}
\]

The time dependent profile of the pulse, \( \mathcal{T}(\gamma_L t) \), is chosen to be real and we assume a gaussian shape with a peak equal to one at \( t = 0 \).

In order for Eq. (4) to be valid, we need \( \hat{g}(\tilde{k}, \mu) \) to vary in momentum of the order of \( \gamma_L/c \approx 10 - 300 \) m\(^{-1}\). However, the Franck-Condon factors \( \eta_{\tilde{n}m}(\tilde{k}) \) change by \( \delta k \), on the order of \( 1/\sqrt{n} \approx 10^5 \) m\(^{-1}\) for low \( n \). For higher \( n \)'s, \( \delta k \) scales as \( 1/\sqrt{n} \), so it becomes \( \sim 10^3 \) m\(^{-1}\) for the highest energy levels that are still available in the trap.

Thus, we have \( \delta k \gg \gamma_L/c \) for all \( \gamma_L \), so we may validly replace \( \tilde{k} \) by \( \tilde{k}_L \) inside \( \eta_{\tilde{n}m}(\tilde{k}) \). Inserting this substitution into Eq. (3) and Eq. (4), Eq. (1) can be written as

\[\begin{align*}
\mathcal{H} &= \sum_{\tilde{n}} \omega_{\tilde{n}m}^2 g_{\tilde{n}m} + \sum_{\tilde{n}} (\omega_{\tilde{n}m}^2 + \omega_0) \tilde{e}_{\tilde{n}m}^\dagger \tilde{e}_{\tilde{n}m} \\
&+ \frac{\Omega}{2} \mathcal{T}(\gamma_L t) \left[ \exp(i\omega_L t) \sum_{\tilde{n}} g_{\tilde{n}m} \tilde{e}_L \cdot \hat{f}_{\tilde{n}} + \text{h.c.} \right], \tag{5}
\end{align*}\]

where we have re-written the operators in terms of annihilation and creation operators of wave packets of excited states which originate from the \( \tilde{n} \)-th state of the ground state potential,

\[
\hat{f}_{\tilde{n}} = \sum_{\tilde{m}} \eta_{\tilde{n}\tilde{m}}(\tilde{k}_L) \tilde{e}_{\tilde{m}}. \tag{6}
\]

These annihilation operators and their conjugate creation ones also obey the standard fermionic anti-commutation relations, i.e. \( \{ \hat{f}_{\tilde{n}}^\dagger, \hat{f}_{\tilde{n}} \} = \delta_{\tilde{n}\tilde{n}'} \delta_{qq'} \), with \( q, q' = x, y, z \) enumerating the components of the vectors \( \hat{f}_{\tilde{n}}^\dagger \) and \( \hat{f}_{\tilde{n}} \). Their energies \( \omega_{\tilde{n}m} \) also vary very slowly for their corresponding states and therefore for each of the wave packets \( \hat{f}_{\tilde{n}}, \hat{f}_{\tilde{n}}^\dagger \); their energy can be approximated by \( \omega_{\tilde{n}m}^0 + \omega_0 + k_L^2/(2M) \). This assumes that the atomic wavepackets in the excited state potential will not experience much coherent oscillation or diffusion (i.e. are in a sense frozen in shape) within the duration of the laser pulse (\( \tau_L \ll 1/\omega_L \)).

The Heisenberg equations that follow from the Hamiltonian (5) now becomes linear. Thus at resonance, \( \omega_L \approx \omega_0 + k_L^2/(2M) \), and in the rotating frame in which \( g_{\tilde{n}} \to e^{-i\omega_{\tilde{n}m}^0t} g_{\tilde{n}}, \hat{f}_{\tilde{n}} \to e^{-i(\omega_{\tilde{n}m}^0+\omega_0)t} \hat{f}_{\tilde{n}} \), they become

\[
\begin{align*}
g_{\tilde{n}}(t) &= -i \frac{\Omega}{2} \mathcal{T}(\gamma_L t) \tilde{e}_L \cdot \hat{f}_{\tilde{n}}(t), \\
\tilde{e}_L \cdot \hat{f}_{\tilde{n}}(t) &= -i \frac{\Omega}{2} \mathcal{T}(\gamma_L t) g_{\tilde{n}}(t). \tag{7}
\end{align*}
\]

They can be easily solved analytically for any pulse envelope

\[
\begin{align*}
g_{\tilde{n}}(t) &= g_{\tilde{n}}(-\infty) \cos [A(t)], \\
\tilde{e}_L \cdot \hat{f}_{\tilde{n}}(t) &= -ig_{\tilde{n}}(-\infty) \sin [A(t)], \\
\tilde{e}_L \cdot \hat{f}_{\tilde{n}}(t) &= +e^{\gamma_L(-\infty) \cos [A(t)]}, \tag{8}
\end{align*}
\]

with the pulse area

\[
A(t) = \frac{\Omega}{2} \int_{-\infty}^t \mathcal{T}(\gamma_L t') dt'. \tag{11}
\]

Thus we see that each of the \( \tilde{n} \) levels of the ground state oscillator (when populated) creates an independent wave packet \( \hat{f}_{\tilde{n}} \) which is a superposition of the excited state wavefunctions. The population oscillates coherently between the \( \tilde{n} \)-th ground state and the corresponding excited state wave packet. We can view the behavior of the system as a set of independent two-level atoms coherently driven by the laser pulse. By using a pulse whose area is a multiple of \( 2\pi \) the system will be left in the same state after the duration of the pulse. Clearly, as \( |\tilde{n}| \) increases, the approximations become less valid, but they should hold very well for the lowest available \( 10^4 \) states of the ground state potential.

The linear relations \( \tilde{f}_{\tilde{n}} \) and \( \tilde{e}_{\tilde{m}} \) remain the same as before for bosonic atoms \([8]\) and their inverse can be easily derived using the sum rules

\[
\begin{align*}
\sum_{\tilde{n}} \left[ \eta_{\tilde{n}\tilde{m}}(\tilde{k}_L) \right]^* \eta_{\tilde{n}\tilde{m}'}(\tilde{k}_L) &= \delta_{\tilde{m}\tilde{m}'}, \tag{12}
\sum_{\tilde{n}} \left[ \eta_{\tilde{n}\tilde{m}}(\tilde{k}_L) \right]^* \eta_{\tilde{n}\tilde{m}'}(\tilde{k}_L) &= \delta_{\tilde{n}\tilde{n}'}. \tag{13}
\end{align*}
\]
They will allow us to express solutions for $g_{\alpha}(t)$ and $\bar{e}_m(t)$ as given above in Eqs. (9) and (10) (and their conjugates) uniquely in terms of $g_{\alpha}(-\infty), \bar{e}_m(-\infty)$, etc.

Since spontaneous emission rate $\gamma$ from a dipole allowed excite states is non-zero, we do in practice have photons scattered from the atoms. The detailed discussions were given in our previous work [8]. Following the same perturbative treatment as developed there. One can work out the spectra of time dependent light scattering as given below.

III. SPECTRUM OF THE SCATTERED LIGHT

In order to calculate the spectrum of the scattered light we need to first determine the initial conditions for Eqs. (9) and (10) at $t = -\infty$. We assume that initially the ground state energy levels were populated in accordance to a Fermi-Dirac distribution (FDD) for non-interacting atoms in the harmonic well [17]. Neglecting the interactions between magnetic trapped fermionic atom (in the same hyperfine state) is a much better approximation for fermions than for bosons. $S$-wave scattering, the predominant collision channel is forbidden for fermions within the same spin state due to the required anti-symmetry of the two atom wavefunction. We can also neglect the much weaker $p$-wave collisions as they are energetically suppressed in the temperature range of interests here.

The mean number of atoms in the $\bar{n}$-th level at $t = -\infty$ is therefore

$$N_{\bar{n}} = \langle \hat{g}_{\bar{n}}^\dagger \hat{g}_{\bar{n}} \rangle = z e^{-\beta \bar{n} \omega_s} / (1 + z e^{-\beta \omega_s}),$$  \hspace{1cm} (14)

where $\beta = 1/kT$, and $z$ is the fugacity. The relation $\sum_{\bar{n}} N_{\bar{n}} = N$ determines $z$ as a function of $\beta$ and $N$. We can now calculate the spectrum of scattered photons by using the Hamiltonian (1). We derive the Heisenberg equation for the photon annihilation operator,

$$\dot{a}_{\bar{k}\mu} = -i c k a_{\bar{k}\mu} - i g(k) \sum_{\bar{n},\bar{m}} \bar{g}_{\bar{n}} \bar{e}_{\bar{m}} \cdot \bar{e}_{\bar{k}\mu} \bar{n}_{\bar{m}}(\bar{k}),$$  \hspace{1cm} (15)

and its Hermitian conjugate for $a_{\bar{k}\mu}^\dagger$. These equations are now solved perturbatively with respect to the atom-photon field coupling $g(k)$. The formal solution of Eqs. (15) is

$$a_{\bar{k}\mu}(t) = e^{-i ck t} a_{\bar{k}\mu}(-\infty) - i g(k) \sum_{\bar{n},\bar{m}} \bar{n}_{\bar{m}}(\bar{k})$$

$$\times \int_{-\infty}^t dt' e^{-i ck (t-t')} \bar{e}_{\bar{k}\mu} \cdot \bar{e}_{\bar{m}}(t') \bar{g}_{\bar{n}}^\dagger (t').$$  \hspace{1cm} (16)

The perturbative solution is then obtained similar to the case of bosons [8] but with care to take into account the anti-commutation relation between fermionic atomic operators.

The total spectrum of scattered photons is defined as

$$C(\bar{k}; \mu) = \lim_{t \to -\infty} \langle \hat{a}_{\bar{k}\mu}^\dagger (t) \hat{a}_{\bar{k}\mu}(t) \rangle,$$  \hspace{1cm} (17)

and can be divided into coherent and incoherent parts,

$$C(\bar{k}, \mu) = C_{\text{coh}}(\bar{k}, \mu) + C_{\text{in}}(\bar{k}, \mu).$$  \hspace{1cm} (18)

The coherent part from $\langle \hat{a}_{\bar{k}\mu}(t) \rangle$, as in the single atom case, is proportional to the square modulus of the Fourier transform of the mean atomic polarization. The incoherent part however is due to the quantum fluctuations of the atomic polarization. Although in usual experiments, only the total spectrum can be measured, the division into coherent and incoherent parts is meaningful since they have significantly different angular characteristics, as we show latter. The total spectrum is

$$C(\bar{k}; \mu) = |g(k)|^2 |\bar{e}_L \cdot \bar{e}_{\bar{k}\mu}|^2$$

$$\times \sum_{\bar{m}, \bar{m}_1, \bar{m}_2} \left[ \eta_{\bar{m}, \bar{m}_1} (\bar{k}) \right] \eta_{\bar{m}_2, \bar{m}_2} (\bar{k})$$

$$\times \sum_{\bar{m}_1, \bar{m}_2} \left[ \eta_{\bar{m}_1, \bar{m}_1} (\bar{k}) \right] \left[ \eta_{\bar{m}_2, \bar{m}_2} (\bar{k}) \right]^*$$

$$\times \int_{-\infty}^\infty dt_1 \int_{-\infty}^\infty dt_2 e^{-i (\bar{c}k - \omega_L) t_1} e^{i(ck - \omega_L) t_2}$$

$$\times \langle \bar{f}_{\bar{m}_1}^\dagger (t_1) \cdot \bar{e}_L \bar{g}_{\bar{m}_1} (t_1) \bar{g}_{\bar{m}_2}^\dagger (t_2) \bar{f}_{\bar{m}_2}^\dagger (t_2) \cdot \bar{e}_L \rangle.$$  \hspace{1cm} (19)

In the perturbative limit, we insert now the solutions of Eqs. (9) and (10) into Eq. (19). At $t = -\infty$, the Heisenberg picture coincides with the Schrödinger picture, so that we omit in the following the explicit time dependence of the operators at $t = -\infty$. Since initially all atoms are in the ground electronic state, we obtain

$$\langle \bar{f}_{\bar{m}_1}^\dagger (t_1) \cdot \bar{e}_L \bar{g}_{\bar{m}_1} (t_1) \bar{g}_{\bar{m}_2}^\dagger (t_2) \bar{f}_{\bar{m}_2}^\dagger (t_2) \cdot \bar{e}_L \rangle$$

$$= \langle \bar{g}_{\bar{m}_1}^\dagger \bar{g}_{\bar{m}_1} \bar{g}_{\bar{m}_2} \bar{g}_{\bar{m}_2}^\dagger \rangle$$

$$\times \sin \left[ A(t_1) \right] \sin \left[ A(t_2) \right] \sin \left[ A(t_2) \right]$$

$$+ \langle \bar{g}_{\bar{m}_1}^\dagger \bar{e}_L \cdot \bar{f}_{\bar{m}_1} \bar{e}_L \cdot \bar{f}_{\bar{m}_2} \bar{g}_{\bar{m}_2}^\dagger \rangle \sin^2 \left[ A(t_1) \right] \sin^2 \left[ A(t_2) \right],$$  \hspace{1cm} (20)

where the expectation values $\langle \cdots \rangle$ with respect to the FDD remains to be evaluated. Thus the only difference between the bosons and fermions so far is the evaluation of the expectation values according to the relevant statistics. The formal expressions are the same.

The single atom spectrum can also be written as a sum of coherent and incoherent parts [8],

$$S(\bar{\omega}) = S_{\text{coh}}(\bar{\omega}) + S_{\text{in}}(\bar{\omega}),$$  \hspace{1cm} (21)

with $\bar{\omega} = (ck - \omega_L)/\gamma_L$, and

$$S_{\text{coh, in}}(x) = \frac{3}{8 \pi^2 \gamma_L} \frac{\gamma c}{k_0} \langle \bar{e}_L \cdot \bar{e}_L \rangle^2 W_{\text{coh, in}}(x).$$  \hspace{1cm} (22)
Making use of the properties of the FDD we obtain the same as for the boson case considered earlier \[8\]

\[ W_{\text{coh}}(x) = \gamma_L^2 \left| \int_{-\infty}^{\infty} e^{-ix\gammaLt'} \cos \left[ A(t') \right] \sin \left[ A(t') \right] dt' \right|^2, \]  

(23)

\[ W_{\text{in}}(x) = \gamma_L^2 \left| \int_{-\infty}^{\infty} e^{-ix\gammaLt'} \sin^2 \left[ A(t') \right] dt' \right|^2. \]  

(24)

For a hyperbolic secant pulse \(1/\cosh(\gammaLt)\) of area \(2\pi\), one has

\[ W_{\text{coh}}(x) = \pi x^2 / \cosh^2(\pi x/2), \]  

(25)

\[ W_{\text{in}}(x) = \pi x^2 / \sinh^2(\pi x/2). \]  

(26)

Since the results are only weakly dependent on a particular pulse shape we shall present results for the hyperbolic secant pulse only.

Using the relation

\[ \sum_{n} [\eta_{\bar{n}}(\bar{k})]^{*} \eta_{\bar{n}}(\bar{k}) = \eta_{\bar{n}}(\bar{k} - \bar{k}), \]  

(27)

we obtain

\[ C(\bar{k}, \mu) = |g(k)|^2 |\bar{e}_{L} \cdot \bar{e}_{\mu}|^2 \]

\[ \times \sum_{\bar{n}_{1}, \bar{n}_{2}, \bar{n}_{1}', \bar{n}_{2}'} [\eta_{\bar{n}_{1}}(\bar{k} - \bar{k})]^{*} \eta_{\bar{n}_{2}}(\bar{k} - \bar{k}) + \eta_{\bar{n}_{1}'}^{*} \eta_{\bar{n}_{2}'} \]

\[ \times \frac{1}{\gamma_L} \left[ \langle g_{\bar{n}_{1}}^{\dagger} g_{\bar{n}_{1}'} g_{\bar{n}_{2}}^{\dagger} g_{\bar{n}_{2}'} \rangle W_{\text{coh}}(\bar{\omega}) + \langle g_{\bar{n}_{1}}^{\dagger} \bar{e}_{L} \cdot \bar{f}_{\bar{n}_{1}}^{\dagger} \bar{e}_{\mu} \cdot \bar{f}_{\bar{n}_{2}}^{\dagger} g_{\bar{n}_{2}'} \rangle W_{\text{in}}(\bar{\omega}) \right]. \]  

(28)

Making use of the properties of the FDD we obtain

\[ \langle g_{\bar{n}_{2}'}^{\dagger} g_{\bar{n}_{1}}^{\dagger} g_{\bar{n}_{1}} g_{\bar{n}_{2}} \rangle = \delta_{\bar{n}_{1}, \bar{n}_{1}'} N_{\bar{n}_{1}} \delta_{\bar{n}_{2}, \bar{n}_{2}'} N_{\bar{n}_{2}} \]

\[ + \delta_{\bar{n}_{1}, \bar{n}_{2}} \delta_{\bar{n}_{1}', \bar{n}_{2}'} \left[ \langle g_{\bar{n}_{1}'} g_{\bar{n}_{1}} \rangle^2 - N_{\bar{n}_{1}'}^2 \right] \]

\[ + \delta_{\bar{n}_{1}, \bar{n}_{2}} \delta_{\bar{n}_{1}', \bar{n}_{2}'} N_{\bar{n}_{1}} (1 - N_{\bar{n}_{1}'}) \delta_{\bar{n}_{2}, \bar{n}_{2}'} \]  

(29)

and

\[ \langle g_{\bar{n}_{1}'}^{\dagger} \bar{e}_{L} \cdot \bar{f}_{\bar{n}_{1}}^{\dagger} \bar{e}_{\mu} \cdot \bar{f}_{\bar{n}_{2}}^{\dagger} g_{\bar{n}_{2}'} \rangle = \delta_{\bar{n}_{1}, \bar{n}_{2}} \delta_{\bar{n}_{1}', \bar{n}_{2}'} N_{\bar{n}_{1}'}^2. \]  

(30)

The difference between bosons and fermions appears in the last term of Eq. (29); \(1 - N\) for fermions and \(1 + N\) for bosons \[8\].

Inserting the above expressions in Eq. (28), and performing tedious, but elementary calculations we finally obtain analytic expressions for the spectra. In particular, the coherent part is

\[ C_{\text{coh}}(\bar{k}, \mu) = S_{\text{coh}}(\bar{\omega}) \left| \sum_{\bar{n}} N_{\bar{n}} \eta_{\bar{n}}(\bar{k} - \bar{k}) \right|^2, \]  

(31)

identical in form as the bosonic system, but now with \(N_{\bar{n}}\) representing the mean occupation number for a fermionic system.

The incoherent part of the spectrum is

\[ C_{\text{in}}(\bar{k}, \mu) = S_{\text{coh}}(\bar{\omega}) \sum_{\bar{n}} \delta N_{\bar{n}}^2 |\eta_{\bar{n}}(\bar{k} - \bar{k})|^2 \]

\[ + S_{\text{coh}}(\bar{\omega}) \sum_{\bar{n}} \sum_{\bar{n}'} N_{\bar{n}} (1 - N_{\bar{n}'}) |\eta_{\bar{n}'}(\bar{k} - \bar{k})|^2 \]

\[ + N S_{\text{in}}(\bar{\omega}). \]  

(32)

Again the difference appears in the \(1 - N\) term on the second line of Eq. (32) compared with \(1 + N\) in the case of bosons \[8\]. We note that the incoherent spectrum (32) consists of three parts coming from: i) quantum dispersion of the occupation numbers \(\delta N_{\bar{n}}^2 = \langle (\bar{g}_{\bar{n}}^2 g_{\bar{n}})^2 \rangle - \langle \bar{g}_{\bar{n}}^2 g_{\bar{n}} \rangle^2\), ii) processes of creation of the \(n\)-th wave packet accompanied by annihilation of the \(\bar{n}^2\)-th one for \(\bar{n} \neq \bar{n}'\), and iii) the single atom incoherent spectrum. Obviously, both coherent and incoherent spectra reflect quantum statistical properties of atoms since they depend on \(N_{\bar{n}}\)’s, which are described by the Fermi-Dirac distribution for our system of fermions. The first two terms in Eq. (32) for the incoherent spectrum depend explicitly on the statistical properties of atoms through \(\delta N_{\bar{n}}^2\) and the factor \((1 - N_{\bar{n}'})\).

The total number of emitted photons can be obtained by integrating the spectrum,

\[ N_{\text{tot}} = \sum_{\mu} \int d\bar{k} C(\bar{k}, \mu), \]  

(33)

which could also be divided into coherent and incoherent parts. By fixing the direction of \(\bar{k}\) and integrating over the azimuthal angle \(\varphi\) one can also define an angular distribution of photons \(dN_{\text{tot}}(\theta)\) [and, correspondingly \(dN_{\text{coh}}(\theta)\) and \(dN_{\text{in}}(\theta)\)]

\[ dN_{\text{tot}}(\theta) = dN_{\text{coh}}(\theta) + dN_{\text{in}}(\theta) \]

\[ = \sin(\theta) d\theta \sum_{\mu} \int_{0}^{2\pi} d\varphi \int k^2 dk C(\bar{k}, \mu), \]  

(34)

where \(\theta\) is the angle between \(\bar{k}\) and \(\bar{k}_L\). We can also choose to define an integrated spectrum by fixing \(|\bar{k}|\), and integrating the spectrum over the full solid angle.

IV. NUMERICAL CALCULATION OF THE SPECTRUM

The analytical expressions for both the coherent and incoherent spectra as obtained in Eqs. (31) and (32) are not directly applicable to straight numerical summations as they involve triple and six-fold sums respectively. In the previously work for trapped bosons, the analogous expressions were in terms of a power series of the fugacity \(z\). A subsequent resummation of an auxiliary series provided a fast convergent numerical approach. However, this same technique is only applicable when \(z < 1\).
corresponding to a high temperature limit for the Fermi gas. We therefore needed a new method to calculate the spectrum at low temperatures when \( z \gg 1 \), which is exactly the region where the interesting effects of quantum statistics becomes important.

The coherent spectrum as expressed earlier in Eq. (31) consists of a triple sum. Similar to the case for bosons [8], when \( z < 1 \), we can write the spectrum as a power expansion of \( z \)

\[
C_{\text{coh}}(\vec{k}, \mu) = S_{\text{coh}}(\varpi) \left[ \sum_{i=1}^{\infty} (-1)^{i-1} \frac{z^i}{(1-e^{-i\beta\omega_n})^3} \right] \times \exp \left[ \frac{-1}{2} (\vec{k} - \vec{k}_L)^2 a^2 \coth(\beta\omega_n/2) \right] .
\]

(35)

Thus we have transformed a triple sum into a single sum. Also for very high temperatures when \( z \ll 1 \), the above single sum converges quickly. On the other hand, the quantum degenerate low temperature limit for fermions corresponds to \( z \to \infty \) when zero temperature is approached. We thus use an alternative method where the original expression for the spectrum, Eq. (31), is written in terms of the Laguerre and the generalized Laguerre polynomials \( L_n(\cdot) \) and \( L_n^m(\cdot) \). The properties of these polynomials are then used to reduce the triple sum into a single sum over the generalized Laguerre polynomials. The new form is then (see Appendix A for details),

\[
C_{\text{coh}}(\vec{k}, \mu) = S_{\text{coh}}(\varpi) e^{-(\vec{k} - \vec{k}_L)^2 a^2} \times \left[ \sum_{n=0}^{\infty} P(n) L_n^m[(\vec{k} - \vec{k}_L)^2 a^2] \right]^{2} ,
\]

(36)

where

\[
P(n) = \frac{ze^{-\beta\omega_n n}}{1 + ze^{-\beta\omega_n n}}
\]

(37)

is the mean occupation number of fermions in any of the degenerate energy states with principle quantum number \( n \).

By using this new expression we achieve enormous savings in computational time over the initial triple sum. However the added complexity of generating the generalized Laguerre polynomials implies that this method is not as fast as the power expansion method which was only valid for \( z < 1 \) [8].

The evaluation of the incoherent spectrum is a more difficult numerical problem as we see that the second term of Eq. (32) involves a six-sum. This is due to the fact that incoherent photon emissions corresponds to different final and initial motional energy levels (unlike the coherent case where the two are the same). For the case of \( z < 1 \) we can again use the power expansion approach to write the incoherent spectrum as

\[
C_{\text{inc}}(\vec{k}, \mu) = N S_{\text{coh}}(\varpi) + N S_{\text{in}}(\varpi)
\]

with

\[
f(\beta, l_1, l_2) = \frac{(1 - e^{-l_1 \beta \omega_n})(1 - e^{-l_2 \beta \omega_n})}{1 - e^{-(l_1 + l_2) \beta \omega_n}} .
\]

(39)

We note that the minus sign \( '-' \) in front of the third term in Eq. (38) is due to quantum statistics. In the bosonic case as expressed earlier [8] a plus sign was obtained. This approach helped to reduce the six-sum into a double sum. Again this limit is only applicable for high temperatures (when average energy per atom is much greater than the Fermi temperature). We were unable to find a suitable transformation to reduce the number of sums in the general case when \( z \geq 1 \). However, some alternative simplifications are possible. We note that the first and second terms of Eq. (32) can be combined into a single term.

\[
C_{\text{inc}}(\vec{k}, \mu) = S_{\text{coh}}(\varpi) \sum_{n_x, n_y, n_z, m_x, m_y, m_z} \sum_{l\beta \omega_n} P_{\text{inc}}(n_x + n_y + n_z, m_x + m_y + m_z)
\]

\[
\times \left[ I_{n_x, m_x} I_{n_y, m_y} I_{n_z, m_z} \right]^{2} + N S_{\text{in}}(\varpi) ,
\]

(40)

with

\[
P_{\text{inc}}(n, m) = \frac{ze^{-\beta\omega_n n}}{(1 + ze^{-\beta\omega_n n})},
\]

(41)

and

\[
I_{n, m} = \langle n | e^{-i\delta k_j R_j} | m \rangle .
\]

(42)

Where we denote the \( j \)-th component of \( \delta \vec{k} = \vec{k} - \vec{k}_L \) by \( \delta k_j \). The number of sums can be reduced by exploiting the symmetry of the scattering geometry. Because the scattering is symmetric about the axis along the incoming laser direction \( \vec{k}_L \). We can set one of the \( \delta \vec{k} \) components in the plane perpendicular to the laser axis to be zero, i.e. choosing the laser to be aligned with the \( z \)-axis we can set \( k_y = 0 \). The incoherent spectrum with the aid of this symmetry and the identity Eq. (A3) is then

\[
C_{\text{inc}}(\vec{k}, \mu) = S_{\text{coh}}(\varpi) e^{-|\delta \vec{k}|^2 a^2} \times \sum_{n_x, n_y, n_z, m_x, m_y, m_z} \sum_{l\beta \omega_n} P(n_x + n_y + n_z, m_x + m_y + m_z)
\]

\[
\times \left[ (m_x!m_z! / n_x!n_z!)(\delta k_x^2 a^2)^{n_x-m_x}(\delta k_z^2 a^2)^{n_z-m_z}
\right]
\]

\[
\times \left[ I_{n_x-m_x}^{n_x-m_x} (\delta k^2 a^2) L_{n_x-m_x}^{n_x-m_x} (\delta k^2 a^2)^2 \right]^{2} + N S_{\text{in}}(\varpi) .
\]

(43)
with

\[ P(n, m) = \sum_{y} P_{\text{inc}}(n + y, m + y). \] (44)

This finally reduces the six-sum to a four-sum. Based on our model and numerical investigations, the scattering should mainly be arranged for parameter regimes such that the most dramatic features of the spectrum are observed from the coherent spectrum. This four-sum inefficiency of calculating the incoherent spectrum is therefore not a major problem, as only selected points are calculated as a check to the validity of our approximation.

V. DISCUSSION OF THE SPECTRUM

In this section we discuss three methods for analyzing the spectrum. 1) we look at the part of the spectrum that reveals quantum statistics. This represents all the collective effects on the spectrum. 2) we integrate over either the frequency (angle) to obtain an angular (frequency) spectrum respectively. 3) the total number of scattered energy levels around the Fermi sea.

A Form functions

A useful part of the spectrum is the component that describes the quantum statistics as opposed to the single atom component of the spectrum. We call this component the “form function” as it is indeed related to the Fourier transform of the average density profile of the degenerate gas. Temperature dependent behavior of the scattering spectrum respectively. The ground state trap size is \( kT/E \)

Figures 1-3 represent the coherent form functions for several different temperatures of \( kT/E_F = 1.36, 0.60, \) and 0.0016 with a system consisting of one million atoms. In all numerical simulations we have set the dimensionless parameter \( k_{L, a} = 12.5 \). For a resonant transition wavelength of \( \sim 800 \) nm, our choice corresponds to magnetic trapping frequencies of about \( 2\pi \times 300 \) Hz, \( \pi \times 30 \) Hz, and \( 2\pi \times 23 \) Hz for \(^6\)Li, \(^{40}\)K, and \(^{86}\)Rb respectively. The ground state trap size is \( \sim 1.6 \) (\mu m). We see that the form function becomes broader as the temperature drops. At high temperatures phase matching effects become dominant where destructive interference attenuates the scattering except for a narrow cone in the forward direction. As \( kT/d \) drops below the Fermi energy \( E_F \), phase matching becomes less important and the role of the quantum statistics of the gas becomes increasingly significant. As a rule of thumb, the effects of quantum statistics come into play when \( kT/d \) is less than one-half of \( E_F \), consistent with the effects seen in evaporative cooling [19] and recent experimental studies [6]. We note that the temperature in Fig. 3 is over one hundred times cooler than in Fig. 2, but the width of the form function only changes by a factor less than two. Once the quantum statistics effects become significant, lowering the temperature does little to the shape of the form function as the lowest energy levels become close to being “stacked”, and start to block further filling due to the Pauli exclusion principle. Any additional cooling will therefore only result in filling of the abundant higher energy levels around the Fermi sea.

The form function for the incoherent spectrum when \( z < 1 \) is retrieved from Eq. (38)

\[ \mathcal{F}_{\text{inc}}^2(\omega - \omega_L, \theta) = \sum_{l_1, l_2 = 1}^{\infty} \frac{(-z)^{(l_1 + l_2)}}{[1 - e^{-(l_1 + l_2)}\beta a^2]^2} \times \exp \left[ -\frac{(\delta k^2 a^2)}{2} (\delta k^2 a^2) \right]. \] (47)

However, a more natural definition of the corresponding form function for the general case in Eq. (43) would be

\[ \mathcal{F}_{\text{inc}}^2(\omega - \omega_L, \theta) = e^{-|\delta k^2 a^2|} \times \sum_{n_x, n_z, m_x, m_z} P(n_x + n_z, m_x + m_z) \times \left[ \frac{a!}{n_x! n_z!} \right] (\delta k^2 a^2)(\delta k^2 a^2) \times \left[ \frac{m_x! m_z!}{n_x! n_z!} \right] (\delta k^2 a^2)(\delta k^2 a^2), \] (48)

which has also absorbed the \( N S_{\text{coh}}(\pi) \) term. We can denote this latter function with a prime, and it is related to \( \mathcal{F}_{\text{inc}}^2 \) through

\[ \mathcal{F}_{\text{inc}}^2 = N - \mathcal{F}_{\text{inc}}^2. \] (49)

Figures 4-5 display the incoherent form functions for two different temperatures \( kT/E_F = 1.36 \) and 0.0016 again for one million atoms. Note that it would have taken an impractical amount of computing time to generate the form function for \( kT/E_F = 0.60 \). By far the most striking feature of these figures is the appearance of a crater in
Fig. 5 at very cold temperatures resulting in the volcano-like structure. This feature is not prominently present in the high temperature situation of Fig. 4, but persist at all lower temperature limit calculations. Also we see that the form functions are asymmetric along the frequency domain but remains symmetric in angular one. This angular symmetry is basically due to our previous assumption on the cylindrical symmetry of the scattering. But the asymmetry along the frequency direction is essentially due to the Fermi inhibition for scattering into an already occupied atomic state. This simplified view is consistent with our observation that scattered photons is biased towards the lower frequencies below resonance, which corresponds to filling higher motional energy state in the ground state manifold. We do not have a more detailed physical explanation for this asymmetry at this time.

To further examine the crater feature at low temperatures in greater detail we next consider a smaller system of one thousand atoms. Figures 6-9 show the gradual change in the crater depth as we go from low to high temperatures. Figures 6 and 7 are very similar in appearance even though their temperatures at \( kT/E_F = 0.0014 \) and 0.0099 differ by about seven times. It seems that the volcano shaped structure of the form function is relatively insensitive to changes in temperature far below \( E_F \). However, significant changes occur at \( kT/E_F = 0.11 \), which is clearly displayed in Fig. 8, at a temperature only about 10 percent hotter than in Fig. 7, the volcano’s overall width has broadened somewhat and its height has increased by several times. Note the change in scale of the vertical axis. The overall scattering has also increased significantly. We cannot tell from these two figures whether the crater depth has changed but this is clearly displayed in Fig. 9. At a relatively high temperature of \( kT/E_F = 0.28 \), the crater has become shallower and the volcano girth has widen drastically. Asymmetry can also be seen in these figures with the same bias as the one million atom situation of Figs. 4 and 5. By inspecting the relationship between \( F_{\text{inc}}^2 \) and \( F_{\text{inc}}^2 \) in the regime of \( z < 1 \) described by Eq. (49), we see that the crater is due to the minus sign in front of \( F_{\text{inc}}^2 \) since we know that the analogous expression for a bosonic system is \( N + F_{\text{inc}}^2 \). The minus sign in the fermionic case originates from the quantum statistics. Consistent with our naïve explanation given earlier. Thus we further confirm that the presence of the crater at low temperatures is a manifestation of the fermionic nature of the system of atoms. This crater becomes shallower as the temperature is increased when the quantum statistical properties of the system become insignificant. This however is far from a full explanation as this change of sign appears in the limited regime of \( z < 1 \). For the general situation there is no clear clues.

Since we can easily calculate the height of the form functions at the center \((0,0)\) we have also studied the crater depth as a function of temperature. The height at the center for one thousand atoms versus the fugacity \( z \) is plotted in Figures 10 and 11. We find it to be more convenient to plot against the fugacity instead of the temperature. Their is a one-to-one relationship between the fugacity and temperature with larger fugacities corresponding to lower temperatures. Figure 10 displays results for \( z > 1 \). As high fugacities (cold atoms), we see that as the fugacity increases the height at the center drops smoothly. This suggests that the craters become deeper as one cools the atoms. Figure 11 concentrates on the hotter regime within \( z < 1 \). It displays the same characteristics as the first plot in Fig. 11, i.e. \( F_{\text{inc}}^2(0,0) \) decreases as \( z \) grows. Within the regime of \( z < 1 \), we have used two methods to calculate the form function. The power expansion method is denoted by the crosses whereas the circles represent results calculated by the general method. As one can see they agree very well. This last observations are also consistent with our understanding outlined above. This interesting effect is definite worthy of further investigations as it may lead to potential diagnostic tools for Fermi degeneracy.

### B. Angular and frequency spectrum

The angular and frequency spectra are of interest as they can be easily observed in an experiment. We have already defined an angular spectrum of photons \( dN_{\text{tot}}(\theta) \) previously by Eq. (34). Note that the domains of the variables; the azimuthal angle \( \varphi \), polar angle \( \theta \) and \( k \) the magnitude of \( \vec{k} \) are: \( \varphi \in [0, 2\pi] \), \( \theta \in [0, \pi] \) and \( k \in [-\infty, +\infty] \). It is desirable to look at the spectrum of the coherent and incoherent components of the spectrum separately as their behavior are significantly different. We do not explicitly write down the expressions for the coherent situation as it is trivial; it involves simply an integration and sum over the polarizations of the product of the single atom contribution \( S_{\text{coh}} \) and the coherent form function \( F_{\text{coh}}^2 \). The incoherent situation is, on the other hand, a little more complicated. Using the expression for \( C_{\text{in}} \) from Eq. (38) we have the following expression for the \( z < 1 \) regime,

\[
dN_{\text{in}}(\theta) = 2\pi \alpha \sin \theta d\theta \\
\times N \left\{ \int_{-\infty}^{+\infty} dk W_{\text{coh}}(\omega) k^2 [N - F_{\text{inc}}^2(\theta, k)] \right. \\
+ \left. \int_{-\infty}^{+\infty} dk k^2 W_{\text{inc}}(\omega) \right\} 
\]

where \( \alpha \) is defined by

\[
\alpha = \frac{3 \cdot \gamma \cdot e}{8\pi^2 \gamma L \cdot \omega k^2}.
\]

For \( z > 1 \) we simply replace the \( N - F_{\text{inc}}^2 \) by \( F_{\text{coh}}^2 \). To obtain the analogous expression for the angle integrated frequency spectrum one simply integrate over \( \theta \) instead of \( k \) in Eq. (50).
Figures 12 and 13 show the coherent angular and frequency spectrum for ten thousand atoms with a laser pulse width of $\tau_L = 10$ ps. As shown in Fig. 12, the coherent angular spectrum is narrow with a range from about $-2$ to 2 degrees. The solid, dashed, and dash-dotted curves correspond to temperatures of $kT/E_F = 0.0015$, 0.041, and 0.96 respectively. For the two cooler temperatures, the solid and dashed curves almost coincide. They are broader and greater in magnitude than the hotter dash-dotted curve. Figure 13 shows the frequency spectrum for the same three temperatures using the same curve formats. Here the two cooler curves again coincide while the hotter one diminishes in magnitude. The overall features for the coherent spectra are the increase in magnitude and broadening of the scattering as the temperature drops. As explained previously in the form function section, the effects of phase matching becomes diminished and quantum statistical effects becomes more dominant as the temperature cools leading to an observable broadening and increased scattering. The corresponding figures for one million atoms are not shown as they are simply scaled versions of Figs. 12 and 13 with the vertical scales being roughly a hundred times greater than the ten thousand atom case.

Finally Figures 14 and 15 show the incoherent angular and frequency spectrum again for ten thousand atoms with a laser pulse width of $\tau_L = 10$ ps. The incoherent angular spectrum is far broader than the coherent one with a range from about $-180$ to 180 degrees, i.e. the complete polar range. We used the same curve formats as before for the three temperatures. We see that the hottest (dashed-dotted) curve has the greatest magnitude whereas the cooler two have diminished magnitudes. All three curves have the same angular range. In Fig. 15 we see that the two cooler curves (solid and dashed) follow one another closely. The dashed-dotted curve, on the other hand, possesses a dip at resonance frequency. The main feature of the incoherent spectra is the decrease in magnitude of the scattering as the temperature drops, opposite to that of the coherent case. The dip at zero detuning shows up at high temperatures. Because these spectrum consists of incoherent processes, unlike the case of coherent scattering at high temperatures, there is no phase matching effects. The overall scattering rate diminishes as the trapped gas sample gets colder. The dip in the frequency spectrum is mathematically due to the requirement that all the three temperature curves meet at $\omega - \omega_0$ equal zero. For this particular point the form functions and hence the spectra does not depend on the temperature, reminiscent of some kind of optical theorem [8].

### C. Number of scattered photons

The total number of scattered photons is also directly observable in an experiment. By calculating the total number of scattered photons from a simple pulse excitation we can study its dependence on the temperature. We again separate these into coherent and incoherent components. As the number of scattered photons scales as $N^2$ for the coherent case and as $N$ for the incoherent one. For larger numbers of atoms one expects that the scattering will be dominated by the coherent scattering in the short pulse limit. Similarly, for low atom numbers, the incoherent scattering will become dominate and thus be observable when only few atoms are scattered.

Due to the stringent computation requirement, we were not able to go to very large numbers where the coherent scattering dominates. We show instead only results for ten thousand and one million atoms. In the smaller number case of 10,000 atoms, the incoherent scattering dominates whereas for the higher number one (1,000,000 atoms) it is comparable with the coherent scattering. One can of course choose shorter pulses to also monitor their dependence on $\tau_L$ but we set the pulse duration to 10 ps. The coherent scattering becomes more dominant at lower temperatures for the system of one million atoms. Our preference for a dominate coherent scattering is due to the more sensitive nature of the coherent scattering to changes in temperature. This is shown clearly in the following figures. Figures 16 and 17 show coherent and incoherent scattering as a function of the temperature for ten thousand and one million atoms respectively. The coherent results are plotted as circles while squares denote the incoherent ones. For comparison the triangle plots are calculated using the Maxwell-Boltzmann distribution (MBD) for a classical gas. The coherent curve displays temperature-dependent sensitivity across the full range of temperatures both above and below $E_F$. The incoherent curve, on the other hand, is basically flat across most temperatures decreasing slightly below $E_F$. As is expected, the MBD and FDD coherent curves are the same at high temperatures since quantum statistical nature of the atom becomes less important as $kT/E_F$ is greater than one. They start to deviate from each other between $kT/E_F$ equal to one-half and one. This becomes very pronounced at very low temperatures when $kT/E_F$ is less than one-tenth. The FDD curve (circles) displays a flattening off at very low temperatures. This flattening off can be more clearly seen in the insert of Fig. 17. The MBD curve, on the other hand, does not share this feature, but it displays a rapid increase in scattering at these low temperatures. Thus the flattening off is caused by the fermionic nature of the atoms. Previous work on a bosonic system displayed a dramatic increase scattering at these low temperatures. This flattening originates from the inhibition of spontaneous emission far below the Fermi temperatures where the lower levels are “stacked” so that spontaneous emission into these levels are forbidden by the Pauli’s exclusion principle [15].
VI. CONCLUSIONS

We have studied theoretically the scattering of intense short laser pulses off a system of cold atoms. We have presented a detailed theory of such processes. We have demonstrated that by scattering pulses of area $2\pi K$ one may observe signatures of fermionic degeneracy of a system of trapped atoms. In the regime of validity of our theory, $2\pi K$ pulses leave the system of trapped atoms practically unperturbed. At high temperatures when the average energy per atom is many times the Fermi temperature $T_F$ the coherent scattering is very weak and occurs in a very narrow cone in the forward direction due to phase matching effects, which is the same effect as for the case of bosons [8]. As the temperature $kT$ becomes of order less than one half of $E_F$, angular distributions of scattered coherent photons broadens as the influence of the phase matching effects are reduced. Incoherent scattering is reduced at still cooler temperatures and the angular range of the scattering doesn’t significantly change with temperature.

The number of scattered photons increases as the system of atoms are cooled but the rate of this increase tails off as the fermionic nature sets in to suppress photon scattering events leading to atoms into already occupied motional states. This temperature dependent property is compared with results calculated for atoms obeying the Maxwell-Boltzmann statistics. In this case of distinguishable classical atoms, the number of scattered photons follow the fermionic system at high temperatures, as it should, but starts to deviate at low temperatures.

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APPENDIX A: COHERENT SPECTRUM

Starting from Eq. (31) we explicitly write down the triple sum expression for the coherent spectrum as

$$C_{\text{coh}}(\vec{k}, \mu) = S_{\text{coh}}(\vec{\omega}) \left| \sum_{n_x, n_y, n_z} N_{n_x, n_y, n_z} I_{n_x} I_{n_y} I_{n_z} \right|^2,$$  \hspace{1cm} (A1)

where we have defined

$$I_{n_j} = \langle n_j | e^{-i\delta k_j R_j} | n_j \rangle. \hspace{1cm} (A2)$$

Since $R_j$ is the position operator in the $j$-component ($j = x, y, z$), the above matrix element is simply the diagonal elements of the displacement operator, $\langle n_j | D(-i\delta k_j a) | n_j \rangle$ (with $a$ the annihilation operator for the position operator $R_j$). We can then write this matrix element in terms of the Laguerre polynomials using the following identity [20],

$$\langle n_j | D(\xi) | m \rangle = \frac{m!}{m!} e^{-|\xi|^2/2} \xi^{n-m} L_{n-m}^{|m|}(\xi^2). \hspace{1cm} (A3)$$

to obtain

$$\langle n_j | D(\xi_j) | n_j \rangle = e^{-|\xi_j|^2/2} L_{n_j}(\xi_j^2), \hspace{1cm} (A4)$$

with $\xi_j = -i\delta k_j a$. We can now re-write Eq. (A1) as

$$C_{\text{coh}}(\vec{k}, \mu) = S_{\text{coh}}(\vec{\omega}) e^{-|\xi|^2} \left| \sum_{n_x, n_y, n_z} \sum_{n_x + n_y + n_z = n} N_{n_x, n_y, n_z} \times L_{n_x}(\xi_x^2) L_{n_y}(\xi_y^2) L_{n_z}(\xi_z^2) \right|^2. \hspace{1cm} (A5)$$

where $\xi = (\xi_x, \xi_y, \xi_z)$. We now rearrange the order of summation to obtain

$$C_{\text{coh}}(\vec{k}, \mu) = S_{\text{coh}}(\vec{\omega}) e^{-|\xi|^2} \left| \sum_{n} \sum_{n_x + n_y + n_z = n} \sum_{n_x, n_y, n_z} N_{n_x, n_y, n_z} \times L_{n_x}(\xi_x^2) L_{n_y}(\xi_y^2) L_{n_z}(\xi_z^2) \right|^2. \hspace{1cm} (A6)$$

Since for the spherical symmetric trap under consideration, the mean occupation number $N_{n_x, n_y, n_z}$ only depends on the sum of $n_x$, $n_y$, and $n_z$ ($n = n_x + n_y + n_z$), we can take it out of the inner sum and express it as

$$P(n) = \frac{ze^{-\beta \omega}}{1 + ze^{-\beta \omega}}, \hspace{1cm} (A7)$$

We then have

$$C_{\text{coh}}(\vec{k}, \mu) = S_{\text{coh}}(\vec{\omega}) e^{-|\xi|^2} \left| \sum_{n} P(n) \sum_{n_x + n_y + n_z = n} \sum_{n_x, n_y, n_z} N_{n_x, n_y, n_z} \times L_{n_x}(\xi_x^2) L_{n_y}(\xi_y^2) L_{n_z}(\xi_z^2) \right|^2. \hspace{1cm} (A8)$$

Using the summation theorem of Laguerre polynomials [21] to replace the inner sum with a single generalized Laguerre polynomial we finally obtain for the coherent spectrum,

$$C_{\text{coh}}(\vec{k}, \mu) = S_{\text{coh}}(\vec{\omega}) \left| \sum_{n} \sum_{n_x + n_y + n_z = n} P(n) L_{n}^2(|\xi|^2) \right|^2 e^{-|\xi|^2}. \hspace{1cm} (A9)$$
FIG. 1. The coherent form function for one million trapped fermionic atoms at a temperature of $kT/E_F = 1.36$.

FIG. 2. The same as Fig. 1 but at a temperature of $kT/E_F = 0.60$.

FIG. 3. The same as Fig. 1 but at a temperature of $kT/E_F = 0.0016$. 
FIG. 4. The incoherent form function for one million trapped fermionic atoms at a temperature of $kT/E_F = 1.36$.

FIG. 5. The same as Fig. 4 but at a temperature of $kT/E_F = 0.0016$.

FIG. 6. The incoherent form function for one thousand trapped fermionic atoms at a temperature of $kT/E_F = 0.0014$.

FIG. 7. The same as Fig. 6 but at a temperature of $kT/E_F = 0.0099$.  


FIG. 8. The same as in Fig. 6 but at a temperature of $kT/E_F = 0.11$.

FIG. 9. The same as in Fig. 6 but at a temperature of $kT/E_F = 0.28$.

FIG. 10. The incoherent form function at $(0,0)$ as a function of the fugacity $z$ for $z > 1$ with one thousand trapped fermionic atoms.

FIG. 11. The same as Fig 10 but for $z$ less than one. The cross markers represent calculations performed using the power series expansion method.
FIG. 12. The differential coherent scattering versus angle for ten thousand trapped fermionic atoms. The solid, dashed, and dash-dotted curves represent temperatures of $kT/E_F = 0.0015$, 0.041 and 0.96 respectively.

FIG. 13. The differential coherent scattering as a function of frequency for ten thousand trapped fermionic atoms. Follows the same curve format and temperatures as Fig. 12.

FIG. 14. The same as Fig. 12 but for differential incoherent scattering.

FIG. 15. The same as Fig. 13 but for differential incoherent scattering.
FIG. 16. The total number of coherent (circles) and incoherent (squares) scattered photons as a function of temperature for ten thousand trapped fermionic atoms. Coherent scattering calculated for a Maxwell-Boltzmann system is plotted as triangles. A 10 ps laser pulse width was used in the calculations.

FIG. 17. The same calculations as Fig. 16 but with one million trapped fermionic atoms. The insert shows the very cold temperature regime.
\[ F_{coh}^2 / N^2 \]

\[ \omega - \omega_0 \text{ [units of } \gamma \text{]} \]

\[ \theta \text{ [degrees]} \]
The graph represents the function $F_{inc}^{2}/N^2$ as a function of $\omega - \omega_0$ [units of $\gamma$] and $\theta$ [degrees]. The values range from $10^{-7}$ to $10^8$. The graph shows a surface with peaks and troughs, indicating the variation of the function across the given ranges.
$\frac{F_{\text{inc}}^2}{N^2}$

$\omega - \omega_0$ [units of $\gamma$]

$\theta$ [degrees]
\[ \frac{F_{inc}^2}{N^2} \]

\[ \omega - \omega_0 \text{ [units of } \gamma] \]

\[ \theta \text{ [degrees]} \]
$F_{inc}^2/N^2$
$dN_{\text{coh}}(\theta)$
\[ dN_{\text{coh}}(\omega - \omega_0) \]
\( dN_{\text{inc}}(\theta) \) vs. \( \theta [\text{degree}] \)
\[ dN_{\text{inc}}(\omega-\omega_0) \]

\[ \omega-\omega_0 \text{ [units of } \gamma \text{]} \]

\[ x \times 10^6 \]
Number of scattered photons vs. $kT/E_F$