The branch process of Cosmic strings

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In the light of \( \phi \)-mapping method and the topological tensor current theory, the topological structure and the topological quantization of topological defects are obtained under the condition that Jacobian \( J(\phi/v) \neq 0 \). When \( J(\phi/v) = 0 \), it is shown that there exists the crucial case of branch process. Based on the implicit function theorem and the Taylor expansion, the generation, annihilation and bifurcation of the linear defects are detailed in the neighborhoods of the limit points and bifurcation points of \( \phi \)-mapping, respectively.

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I. INTRODUCTION

There have been rapid and exciting developments over the last decades on the interface between particle physics and cosmology\(^1\). Particle physicists pursuing the goal of unification would like to test their theories at energy scales far beyond those available now or in the future in terrestrial accelerators. An obvious place to look is to the very early Universe, where conditions of extreme temperature and density obtained. Meanwhile, cosmologists have sought to understand feature of the Universe currently observed by tracing their history back to that very early period. An exciting outcome of the interplay between particle physics and cosmology is the cosmic string theory\(^2\)–\(^4\). It is strongly believed to solve the short-distance problems of quantum gravity at the Plank scale by providing a fundamental length \( l_{str} = \sqrt{\hbar c/T} \), where \( T \) is the string tension, and provides a bridge between the physics of the very small and the very large. The research in the topic of cosmic strings can help to explain some of the largest–scale structure seen in the Unverse today.

Past researches have mostly focused on the dynamical properties of the cosmic strings\(^5\)–\(^7\), and all of them are based on some particular models. In our previous work\(^8\), by making use of the \( \phi \)-mapping topological current theory\(^9\), which play an important role in discussing the topological invariant and structure of the physical system\(^9\)–\(^13\), we have studied the topological structure and topological quantization of cosmic strings without any concrete model. In this paper, based on our previous work, we will study the generating, annihilating, colliding, splitting and merging of cosmic strings in topology viewpoint, and give the branch conditions of cosmic strings without any concrete model.

This paper is organized as follows: In section 2, a brief review of the topological structure and the topological quantization of cosmic strings is given. In section 3, by virtue of the implicit function theorem, the creation and annihilation of cosmic strings at the limit points is discussed. The bifurcation behaviour of strings is detailed in the neighborhood of bifurcation point in section 4.

II. TOPOLOGICAL STRUCTURE AND TOPOLOGICAL QUANTIZATION OF COSMIC STRINGS

Cosmic strings are linear defects\(^14\) in four dimensional space-time \( X \), analogous to those topological defects found in some condensed matter systems such as vortex lines in liquid helium, flux tubes in type-II superconductors or disclination lines in liquid crystal, and they are closely related to the torsion tensor of the Riemann–Cartan manifold\(^10\)\(^,\)\(^15\)–\(^17\).

In vierbein theory, the torsion tensor is expressed by

\[
T^A_{\mu \nu} = D_\mu e^A_\nu - D_\nu e^A_\mu, \quad \mu, \nu, A = 1, 2, 3, 4
\]  

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where \( \epsilon_{\mu}^A \) is the vierbein field. As in our previous work\(^8\), by analogy with the 't Hooft's viewpoint\(^1\), to establish a physical observable theory of space–time defect, we must first define a gauge invariant antisymmetrical 2–order tensor from torsion tensor with respect of a unit vector field \( N^A(x) \) as follows

\[
T_{\mu\nu} = T_{\mu\nu}^A N^A + \epsilon_{\mu}^A D_\nu N^A - \epsilon_{\mu}^A D_\nu N^A = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

(2)

where \( A_\mu = \epsilon_{\mu}^A N^A \) is a kind of \( U(1) \) gauge potential. This shows that the antisymmetrical tensor \( T_{\mu\nu} \) expressed in terms of \( A_\mu \) is just the \( U(1) \) like gauge field strength (i.e. the curvature on \( U(1) \) principle bundle with base manifold \( X \) ), which is invariant for the \( U(1) \)–like gauge transformation

\[
A_\mu'(x) = A_\mu(x) + \partial_\mu \Lambda(x)
\]

(3)

where \( \Lambda(x) \) is an arbitrary function.

In order to study the string theory, we should extend the traditional concept of topological currents\(^9\) which have been used to study the topological properties of point like defects\(^13\).\(^19\), and introduce a topological tensor current of second order from torsion. From the above discussions, we can define a topological tensor current \( j^{\mu\nu} \) as the dual tensor of \( T_{\lambda\rho} \) as follow

\[
j^{\mu\nu} = \frac{1}{2 \sqrt{|g|}} \epsilon^{\mu\nu\lambda\rho} T_{\lambda\rho} = \frac{1}{2 \sqrt{|g|}} \epsilon^{\mu\nu\lambda\rho} (\partial_\lambda A_\rho - \partial_\rho A_\lambda).
\]

(4)

Very commonly, topological property of a physical system is much more important and worth investigating medicul-ly. It is our conviction that, in order to get a topological result, one should input the topological information from the beginning. Two useful tools—\( \phi \)-mapping method and composed gauge potential theory\(^9\).\(^11\) just do the work. As mentioned in our previous works\(^8\).\(^10\), the decomposition of \( A_\mu(x) \) can be expressed by

\[
A_\mu(x) = \frac{L_p}{2\pi} \epsilon_{ab} n^a(x) \partial_\mu n^b(x); \quad n^a(x) = \frac{\phi^a(x)}{||\phi(x)||}
\]

(5)

where \( \phi^a(x) \) is the order parameter field of cosmic strings, and \( L_p = \sqrt{\hbar G/c^4} \) is the Planck length introduced to make the both sides of the formula with the same dimension\(^10\). With the decomposition of \( A_\mu \) in (5), \( j^{\mu\nu} \) can be expressed in terms of \( n^a \) by

\[
j^{\mu\nu} = \frac{L_p}{2\pi} \frac{1}{\sqrt{|g|}} \epsilon^{\mu\nu\lambda\rho} \epsilon_{ab} \partial_\lambda n^a \partial_\rho n^b,
\]

(6)

which shows that \( j^{\mu\nu} \) is just an antisymmetric and identically conserved 2-order topological tensor current. Because of the topological property of \( n^a \), we input the topological information successfully. Obviously, \( n^a(x) n^a(x) = 1 \), and \( n^a(x) \) is a section of the sphere bundle \( S(X) \). The zero points of \( \phi^a(x) \) are just the singular points of \( n^a(x) \).

By making use of the expression of \( n^a \) in (5) and the Laplacian relation in \( \phi \)-space

\[
\partial_\mu \partial_\mu \ln ||\phi|| = 2\pi \delta(\phi), \quad \partial_\mu = \frac{\partial}{\partial \phi^a},
\]

the topological tensor current \( j^{\mu\nu} \) can be rewritten in a compact form

\[
j^{\mu\nu} = \frac{1}{\sqrt{|g|}} L_p \delta(\phi) J^{\mu\nu}(\phi^a(x)),
\]

(7)

where \( J^{\mu\nu}(\phi) \) is the general Jacobian determinants

\[
\epsilon^{ab} J^{\mu\nu}(\phi^a(x)) = \epsilon^{\mu\nu\lambda\rho} \partial_\lambda \phi^a \partial_\rho \phi^b.
\]

(8)

It is obvious that \( j^{\mu\nu} \) is non-zero only when \( \phi = 0 \).

Suppose that for the system of equations

\[
\phi^1(x) = 0, \quad \phi^2(x) = 0,
\]

there are \( l \) different solutions, when the solutions are regular solutions of \( \phi \) at which the rank of the Jacobian matrix \( [\partial_\mu \phi^a] \) is 2, the solutions of \( \phi(x) = 0 \) can be expressed parameterizedly by

2
\[ x^\mu = z^\mu(u^1, u^2), \quad i = 1, \ldots, l, \]

where the subscript \( i \) represents the \( i \)-th solution and the parameters \( u^I (I = 1, 2) \) span a 2-dimensional submanifold with the metric tensor \( g_{IJ} = g^{\mu\nu} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\nu}{\partial u^J} \), which is called the \( i \)-th singular submanifold \( N_i \) in \( X \). For each \( N_i \), we can define a normal submanifold \( M_i \) in \( X \) which is spanned by the parameters \( v^A (A = 1, 2) \) with the metric tensor \( g_{AB} = g^{\mu\nu} \frac{\partial x^\mu}{\partial v^A} \frac{\partial x^\nu}{\partial v^B} \), and the intersection point of \( M_i \) and \( N_i \) is denoted by \( p_i \). By virtue of the implicit function theorem, at the regular point \( p_i \), it should be hold true that the Jacobian matrix \( J(\bar{\phi}^I) \) satisfies

\[ J(\bar{\phi}^I) = \frac{D(\phi^1, \phi^2)}{D(v^1, v^2)} \neq 0. \] (10)

More deeper calculation can lead to the total expansion of the string current

\[ j^{\mu\nu} = \frac{L_p}{\sqrt{g_x}} \sum_{i=1}^{l} \beta_i \eta_i (v^1, v^2) \delta(N_i) J^{\mu\nu}(\bar{\phi}^I), \] (11)

or in terms of parameters \( y^A = (v^1, v^2, u^I, u^J) \)

\[ j^{AB} = \frac{L_p}{\sqrt{g_y}} \sum_{i=1}^{l} \beta_i \eta_i (v^1, v^2) \delta(N_i) J^{AB}(\bar{\phi}^I). \] (12)

where \( \beta_i \) is a positive integer called the Hopf index\(^{20} \) of \( \phi \)-mapping on \( M_i \) and \( \eta_i = \text{sign} J(\bar{\phi}^I)_{p_i} = \pm 1 \) is the Brouwer degree\(^{20} \) of \( \phi \)-mapping. \( \delta(N_i) \) is the \( \delta \)-function on singular submanifold \( N_i \).\(^{8,21} \) with the expression

\[ \delta(N_i) = \int_{N_i} \frac{1}{\sqrt{g_x}} \delta^2(\vec{x} - z(u^1, u^2)) \sqrt{g_y} d^2u. \]

From the above equation, we conclude that the inner structure of \( j^{\mu\nu} \) or \( j^{AB} \) is labelled by the total expansion of \( \delta(\bar{\phi}^I) \), which includes the topological information \( \beta_i \) and \( \eta_i \). It is obvious that, in (9), when \( u^1 \) and \( u^2 \) are taken to be time-like evolution parameter and space-like string parameter, respectively, the inner structure of \( j^{\mu\nu} \) or \( j^{AB} \) just represents \( l \) strings moving in the 4-dimensional Riemann-Cartan manifold \( X \). The 2-dimensional singular submanifolds \( N_i (i = 1, \ldots, l) \) are their world sheets. The Hopf indices \( \beta_i \) and Brouwer degree \( \eta_i \) classify these strings. In detail, the Hopf indices \( \beta_i \) characterize the absolute values of the topological quantization and the Brouwer degrees \( \eta_i = +1 \) correspond to strings while \( \eta_i = -1 \) to antistrings.

### III. THE BRANCH PROCESS OF STRINGS AT LIMIT POINTS

But from the above discussion, we know that the results mentioned above are obtained under the condition \( J(\phi/v)_{p_i} \neq 0 \). When this condition fails, i.e. the Brouwer degrees \( \eta_i \) are indefinite, what will happen? In what follows, we will study the case when \( J(\phi/v)_{p_i} = 0 \). It often happens when the zero of \( \phi \) includes some branch points, which lead to the bifurcation of the topological current.

In order to discuss the evolution of these strings and to simplify our study, we select the parameter \( u^1 \) as the evolution parameter \( t \), and let the string parameter \( u^2 = \sigma \) be fixed. In this case, the Jacobian matrices are reduced to

\[ J^{A4} = J^A, \quad J^{AB} = 0, \quad J^3 = J^{34} = J(\phi/v), \quad A, B = 1, 2, 3, \]

for \( y^4 = u^2 = \sigma \). The branch points are determined by

\[
\begin{align*}
\phi^1(u^1, v^2, t, \sigma) &= 0 \\
\phi^2(u^1, v^2, t, \sigma) &= 0 \\
\phi^3(u^1, v^2, t, \sigma) &\equiv J(\bar{\phi}^I) = 0
\end{align*}
\] (13)

for the fixed \( \sigma \), and they are denoted as \( (t^*, p_i) \). In \( \phi \)-mapping theory usually there are two kinds of branch points, namely the limit points and the bifurcation points\(^{22} \), each kind of them corresponds to different cases of branch process.
First, in this section, we study the case that the zeros of the order parameter field $\vec{\phi}$ includes some limit points which satisfying

$$J^A(\vec{\phi}_y)|_{(t^*, p_i)} \neq 0, \quad A = 1 \text{ or } 2$$  \hspace{1cm} (14)

For simplicity, we consider $A = 1$ only.

For the purpose of using the implicit function theorem to study the branch properties of strings at the limit points, we use the Jacobian $J^1(\vec{\phi}_y)$ instead of $J(\vec{\phi}_y)$ to search for the solutions of $\vec{\phi} = 0$. This means we have replaced $v^1$ by $t$. For clarity we rewrite the first two equations of (13) as

$$\phi^a(t, v^2, v^1, \sigma) = 0, \quad a = 1, 2. \hspace{1cm} (15)$$

Taking account of (14) and using the implicit function theorem, we have a unique solution of the equations (15) in the neighborhood of the limit point $(t^*, p_i)$

$$t = t(v^1, \sigma), \quad v^2 = v^2(v^1, \sigma)$$

with $t^* = t(p^1_i, \sigma)$. In order to show the behavior of the strings at the limit points, we will investigate the Taylor expansion of (16) in the neighborhood of $(t^*, p_i)$.

Taking account of (14) and using the implicit function theorem, we have a unique solution of the equations (15) in the neighborhood of the limit point $(t^*, p_i)$

$$\frac{dv^1}{dt} = J^1(\vec{\phi}_y)|_{(t^*, p_i)} = \infty$$

i.e.

$$\frac{dt}{dv^1}|_{(t^*, p_i)} = 0.$$  

Then, the Taylor expansion of $t = t(v^1, \sigma)$ at the limit point $(t^*, p_i)$ is

$$t = t(p^1_i, \sigma) + \frac{dt}{dv^1}|_{(t^*, p_i)}(v^1 - p^1_i) + \frac{1}{2} \left( \frac{dt}{dv^1} \right)^2|_{(t^*, p_i)}(v^1 - p^1_i)^2$$

$$= t^* + \frac{1}{2} \left( \frac{dt}{dv^1} \right)^2|_{(t^*, p_i)}(v^1 - p^1_i)^2.$$  

Therefore

$$t - t^* = \frac{1}{2} \left( \frac{dt}{dv^1} \right)^2|_{(t^*, p_i)}(v^1 - p^1_i)^2 \hspace{1cm} (17)$$

which is a parabola in $v^1 - t$ plane. From (17) we can obtain two solutions $v^1_{(1)}(t, \sigma)$ and $v^1_{(2)}(t, \sigma)$, which give the branch solutions of strings at the limit points. If $\frac{dt}{dv^1}|_{(t^*, z_i)} > 0$, we have the branch solutions for $t > t^*$ (Fig 1(a)), otherwise, we have the branch solutions for $t < t^*$ (Fig 1(b)). Since the topological current of strings is identically conserved, the topological quantum numbers of these two generated strings must be opposite at the limit point, i.e. $\beta_1 \eta_1 + \beta_2 \eta_2 = 0$, the former is related to the creation of cosmic strings and antistrings in pair at the limit points, and the latter to the annihilation of the cosmic strings.

**IV. THE BRANCH PROCESS OF STRINGS AT BIFURCATION POINTS**

In the following, let us turn to consider the case of bifurcation point in which the additional restrictions are

$$J^1(\vec{\phi}_y)|_{(t^*, p_i)} = 0, \quad J^2(\vec{\phi}_y)|_{(t^*, p_i)} = 0. \hspace{1cm} (18)$$

These two restrictive conditions will lead to an important fact that the function relationship between $t$ and $v^1$ or $v^2$ is not unique in the neighborhood of bifurcation point $(t^*, p_i)$. The equation
\[
\frac{dv}{dt} = J^1(\frac{\phi}{v}) \left|_{(t^*, p_1)} \right.
\]

which under restraint of (18) directly shows that the direction of the integral curve of (19) is indefinite at the point \((t^*, p_1)\). This is why the very point \((t^*, p_1)\) is called a bifurcation point of the multistring current. With the aim of finding the different directions of all branch curves at the bifurcation point, we suppose that
\[
\frac{\partial \phi^1}{\partial v^2} \left|_{(t^*, p_1)} \right. \neq 0.
\]

From \(\phi^1(v^1, v^2, t, \sigma) = 0\), the implicit function theorem says that there exists one and only one function relationship
\[
v^2 = v^2(v^1, t, \sigma)
\]
with the partial derivatives \(f_1^2 = \partial v^2 / \partial v^1\), \(f_2^2 = \partial v^2 / \partial t\). Substituting (21) into \(\phi^1\), we have
\[
\phi^1(v^1, u^2(v^1, t, \sigma), t, \sigma) \equiv 0
\]
which gives
\[
\frac{\partial \phi^1}{\partial v^2} f_1^2 = - \frac{\partial \phi^1}{\partial v^1}, \quad \frac{\partial \phi^1}{\partial v^2} f_2^2 = - \frac{\partial \phi^1}{\partial t},
\]
\[
\frac{\partial \phi^1}{\partial v^2} f_1^2 = -2 \frac{\partial^2 \phi^1}{\partial v^2 \partial v^1} f_1^2 - \frac{\partial^2 \phi^1}{\partial (v^2)^2} (f_1^2)^2 - \frac{\partial^2 \phi^1}{\partial v^1 \partial t},
\]
\[
\frac{\partial \phi^1}{\partial v^2} f_1^2 = -2 \frac{\partial^2 \phi^1}{\partial v^2 \partial v^1} f_1^2 - \frac{\partial^2 \phi^1}{\partial (v^2)^2} (f_1^2)^2 - \frac{\partial^2 \phi^1}{\partial v^1 \partial t},
\]
where
\[
f_1^2 = \frac{\partial v^2}{(\partial v^2)^2}, \quad f_2^2 = \frac{\partial v^2}{\partial v^1},
\]
\[
f_1^2 = \frac{\partial v^2}{(\partial v^2)^2}, \quad f_2^2 = \frac{\partial v^2}{\partial v^1}.
\]

From these expressions we can calculate the values of \(f_1^2, f_2^2, f_1^2, f_2^2\) and \(f_2^2\) at \((t^*, p_1)\).

In order to explore the behavior of the string at the bifurcation points, let us investigate the Taylor expansion of
\[
F(v^1, t, \sigma) = \phi^2(v^1, v^2(v^1, t, \sigma), t, \sigma)
\]
in the neighborhood of \((t^*, p_1)\), which according to the Eqs.(13) must vanish at the bifurcation point, i.e.
\[
F(t^*, p_1) = 0.
\]

From (23), the first order partial derivatives of \(F(v^1, t, \sigma)\) with respect to \(v^1\) and \(t\) can be expressed by
\[
\frac{\partial F}{\partial v^1} = \frac{\partial \phi^2}{\partial v^1} + \frac{\partial \phi^2}{\partial v^2} f_1^2, \quad \frac{\partial F}{\partial t} = \frac{\partial \phi^2}{\partial v^1} + \frac{\partial \phi^2}{\partial v^2} f_2^2.
\]

Making use of (22), (25) and Cramer’s rule, it is easy to prove that the two restrictive conditions (18) can be rewritten as
\[
J^1(\phi) \left|_{(t^*, p_1)} \right. = \left( \frac{\partial F}{\partial v^1} \frac{\partial \phi^1}{\partial v^2} \right) \left|_{(t^*, p_1)} \right. = 0,
\]
\[
J^1(\phi) \left|_{(t^*, p_1)} \right. = \left( \frac{\partial F}{\partial t} \frac{\partial \phi^1}{\partial v^2} \right) \left|_{(t^*, p_1)} \right. = 0.
\]
which give
\[ \frac{\partial F}{\partial v_1}(t^*, p_i) = 0, \quad \frac{\partial F}{\partial t}(t^*, p_i) = 0 \]  
(26)

by considering (20). The second order partial derivatives of the function $F$ are easily to find out to be

\[ \frac{\partial^2 F}{(\partial v_1)^2} = \frac{\partial^2 \phi^2}{(\partial v_1)^2} + 2 \frac{\partial^2 \phi^2}{\partial v_1 \partial v_1} f_1^2 + \frac{\partial^2 \phi^2}{\partial v_1^2} f_1^2 + \frac{\partial^2 \phi^2}{(\partial v_1)^2} f_1^2 \]

\[ \frac{\partial^2 F}{\partial v_1 \partial t} = \frac{\partial^2 \phi^2}{\partial v_1 \partial t} + \frac{\partial^2 \phi^2}{\partial v_2 \partial v_1} f_1^2 + \frac{\partial^2 \phi^2}{\partial v_1^2} f_1^2 + \frac{\partial^2 \phi^2}{(\partial v_1)^2} f_1^2 \]

\[ \frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 \phi^2}{\partial t^2} + 2 \frac{\partial^2 \phi^2}{\partial v_2 \partial t} f_1^2 + \frac{\partial^2 \phi^2}{\partial v_2^2} f_1^2 + \frac{\partial^2 \phi^2}{(\partial v_2)^2} f_1^2 \]

which at $(t^*, p_i)$ are denoted by

\[ A = \frac{\partial^2 F}{(\partial v_1)^2}|_{(t^*, p_i)}, \quad B = \frac{\partial^2 F}{\partial v_1 \partial t}|_{(t^*, p_i)}, \quad C = \frac{\partial^2 F}{\partial t^2}|_{(t^*, p_i)}. \]  
(27)

Then, taking notice of (24), (26) and (27), we can obtain the Taylor expansion of $F(v^1, t, \sigma)$ in the neighborhood of the bifurcation point $(t^*, p_i)$

\[ F(v^1, t, \sigma) = \frac{1}{2} A(v^1 - p_1^1)^2 + B(v^1 - p_1^1)(t - t^*) + \frac{1}{2} C(t - t^*)^2 \]

which by (23) is the behavior of $\phi^2$ in this region. Because of the second equation of (13), we get

\[ A(v^1 - p_1^1)^2 + 2B(v^1 - p_1^1)(t - t^*) + C(t - t^*)^2 = 0 \]

which leads to

\[ A \left( \frac{dv^1}{dt} \right)^2 + 2B \frac{dv^1}{dt} + C = 0 \]  
(28)

and

\[ C \left( \frac{dt}{dv^1} \right)^2 + 2B \frac{dt}{dv^1} + A = 0. \]  
(29)

The different directions of the branch curves at the bifurcation point are determined by (28) or (29). There are four possible cases:

Case1 ($A \neq 0$): For $\Delta = B^2 - AC > 0$, from (28) we get two different solutions

\[ \frac{dv^1}{dt} \bigg|_{1, 2} = -B \pm \sqrt{B^2 - AC} \]

\[ A, \]  
(30)

which is shown in Fig. 2, where two cosmic strings collide at the bifurcation point $(t^*, p_i)$. This shows that two cosmic strings meet and then depart at the bifurcation point.

Case2 ($A \neq 0$): For $\Delta = B^2 - AC = 0$, there is only one solution

\[ \frac{dv^1}{dt} = -B/A, \]  
(31)

which includes three important cases shown in Fig. 3. Firstly, two cosmic strings tangentially collide at the bifurcation point (Fig. 3(a)). Secondly, two cosmic strings merge into one cosmic string at the bifurcation point (Fig. 3(b)). Thirdly, one cosmic string splits into two cosmic strings at the bifurcation point (Fig. 3(c)).
Case3 \((A = 0, C \neq 0)\): For \(\Delta = B^2 - AC > 0\), from (29) we have
\[
\frac{dt}{dv^1}|_{1,2} = -\frac{B \pm \sqrt{B^2 - AC}}{C} = \begin{cases} 
0 \\
-\frac{2B}{C}
\end{cases}.
\] (32)

As shown in Fig. 4, there are two important cases: (a) One cosmic string splits into three cosmic strings at the bifurcation point (Fig. 4(a)). (b) Three cosmic strings merge into one at the bifurcation point (Fig. 4(b)).

Case4 \((A = C = 0)\): The equations (28) and (29) give respectively:
\[
\frac{dv^1}{dt} = 0, \quad \frac{dt}{dv^1} = 0.
\] (33)

This case is obvious as in Fig. 5, which is similar to the third situation.

The remainder component \(dv^2/dt\) can be given by
\[
\frac{dv^2}{dt} = f^2_1 \frac{dv^1}{dt} + f^2_t
\]
where partial derivative coefficients \(f^2_1\) and \(f^2_t\) have been calculated in (22).

At the end of this section, we conclude that in our string theory there exist the crucial case of branch process. This means that, when an original string moves through the bifurcation point in the early universe, it may split into two strings moving along different branch curves. Since the topological current of strings is identically conserved, the sum of the topological quantum numbers of these two splitted strings must be equal to that of the original string at the bifurcation point, i.e.
\[
\beta_i \eta_1 + \beta_i \eta_2 = \beta_i \eta
\]
for fixed \(i\). This can be looked upon as the topological reason of string splitting.

V. CONCLUSION

In this paper, with the gauge potential decomposition and the so called \(\phi\)-mapping method, we obtain the topological current to describe the strings in 4–dimensional Riemann–Cartan manifold. In the early universe, by discussing the properties of the zero points of the vector field \(\phi\) and the expansion of the delta function \(\delta(\vec{\phi})\), we get the topological quantization of the strings under the condition that the Jacobian \(J(\vec{\phi}) \neq 0\), and pointed out that the singular manifolds are just the evolution manifolds of these strings. When the Jacobian \(J(\vec{\phi}) = 0\), i.e. at the critical points of \(\phi\)-mapping, it is shown that there exist the crucial case of branch process. Based on the implicit function theorem and the Taylor expansion, the origin and bifurcation of the strings are detailed in the neighborhoods of the limit points and bifurcation points of \(\phi\)-mapping respectively, i.e. the branch solutions at the limit points and the different directions of all branch curves at the bifurcation points are calculated out. Because the topological current of these strings is identically conserved, the topological charges of these strings will keep to be constant during the branch processes, which means that the topological quantum numbers of the two generated string currents must be opposite at the limit point and, at the bifurcation point, the sum of the topological quantum numbers of the splitted strings must be equal to that of the original.

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**FIGURES’ CAPTIONS**

Fig. 1. (a) The creation of two cosmic strings. (b) Two cosmic strings annihilate in collision at the limit point.

Fig. 2. Two cosmic strings collide with different directions of motion at the bifurcation point.

Fig. 3. Cosmic strings have the same direction of motion. (a) Two cosmic strings tangentially collide at the bifurcation point. (b) Two cosmic strings merge into one cosmic string at the bifurcation point. (c) One cosmic string splits into two cosmic strings at the bifurcation point.

Fig. 4. (a) One cosmic string splits into three cosmic strings at the bifurcation point. (b) One cosmic string merges into one cosmic string at the bifurcation point.

Fig. 5. This case is similar to Fig. 5. (a) Three cosmic strings merge into one cosmic string at the bifurcation point. (b) One cosmic string splits into three cosmic strings at the bifurcation point.
Fig. 1(a)
Fig. 1(b)
Fig. 2
Fig. 3(a)
Fig. 3(b)

\[ t^* \]

\[ \text{Coordinate} \]

\[ v' \text{Coordinate} \]
$v'$ Coordinate

$t^*$ Coordinate

Fig. 3(c)
Fig. 5(a)
Fig. 5(b)