Entangled Coherent State Qubits in an Ion Trap

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(October 15, 1999)

We show how entangled qubits can be encoded as entangled coherent states of two-dimensional centre-of-mass vibrational motion for two ions in an ion trap. The entangled qubit state is equivalent to the canonical Bell state, and we introduce a proposal for entanglement transfer from the two vibrational modes to the electronic states of the two ions in order for the Bell state to be detected by resonance fluorescence shelving methods.

The qubit, or quantum bit, is the fundamental component of information in quantum computation. In the case of a spin-1/2 system, for some axis of orientation, one can identify the up state with an on state, generally written as $|1\rangle_L \equiv |1\rangle$, for 1 indicating on and the $L$ subscript indicating that this is a logical state. The $|0\rangle_L \equiv |0\rangle$, or off state, then corresponds to down for this orientation. Qubits can thus be realised, in principle, in any spin-1/2 system, such as the electronic state of a two-level atom, the polarisation of a single photon, or the vibrational state of an ion which is restricted to either zero- or one-phonon excitations. The concept of a qubit is useful for quantum information considerations, but the qubit is also a useful construct for Bell inequality tests [1,2] and for considering the maximally entangled canonical Bell states.

It is not necessary to restrict a qubit encoding to systems with a two dimensional Hilbert space. For example a more exotic form of qubit can be constructed from superpositions of coherent states [3] and, as we show here, by employing entangled coherent states [4]. Despite both the nonorthogonality of coherent states and the unbounded Hilbert space, Bell inequality violations are possible in both limits $\alpha \rightarrow 0$ [4] and $\alpha \rightarrow \infty$ [5], for $\alpha$ the dimensionless amplitude of the coherent state. The $\alpha \rightarrow \infty$ limit is achieved by representing the entangled coherent states in a sub-space corresponding to two coupled spin-1/2 systems, and ideal canonical Bell states are realised in the $\alpha \rightarrow \infty$ limit. Entangled coherent states can include the entanglement of even and odd coherent states [6]), which can also be treated as coupled spin-1/2 systems. The advantage of entangled even and odd coherent states, as we show, is that the states are distinguishable by parity, so that heating which changes the vibrational quanta correspond to bit flip errors, which can be detected and corrected via the appropriate circuit [7].

Here we show how the desired entangled coherent states can be created for the two-dimensional centre-of-mass vibrational mode state of two trapped ions. This proposal involves the generalisation of experimental techniques for generating even coherent states for the motional state of one ion in one dimension [8]. The advantage of distinguishing the logical states by phonon number parity has been shown for the case of one-dimensional motion [9,3]. We demonstrate that these entangled coherent states can be represented as entangled qubit states, and, moreover, such a state is equivalent to a canonical Bell state up to unitary transformation with respect to one of the two vibrational modes, that is up to a local unitary transformation. In order to make measurements on the entangled coherent states we give a procedure for swapping entanglement from the vibrational to the internal electronic states of the ions which can then be read by resonance shelving methods.

The two-mode coherent state

$$|\alpha, \beta\rangle \equiv |\alpha\rangle_a \otimes |\beta\rangle_b$$

(1)

can be prepared in an entangled coherent state via the mutual phase-shift interaction $H_I = \hbar \chi a^\dagger ab^\dagger b$; this interaction has been studied in detail in the context of quantum nondemolition measurements [10] and for implementing phase gates for photon qubits [11]. In the ion trap, the two-mode coherent state corresponds to a two-dimensional Gaussian wavepacket for the centre-of-mass motion of the two trapped ions. The mutual phase-shift interaction between these two vibrational modes of freedom for the ion can be achieved by an appropriate Raman laser excitation [12].

After an interaction time $t = \pi/\chi$, the output state is [13]

$$|\psi\rangle = \frac{1}{2} (|\alpha\rangle_a \otimes |+\rangle_b + |-\alpha\rangle_a \otimes |-\rangle_b)$$
\[ \frac{1}{2} (|+\rangle_a \otimes |β\rangle_b + |−\rangle_a \otimes |−β\rangle_b), \tag{2} \]

denote the even and odd coherent states are defined by
\[ |±\rangle_a \equiv N_{±}(α)(|α\rangle_a ± |−α\rangle_a), \]
\[ |±\rangle_b \equiv N_{±}(β)(|β\rangle_b ± |−β\rangle_b), \tag{3} \]

we can henceforth ignore.

The state in eq (2) is equivalent, up to a local (single-oscillator) unitary transformation, to a Bell state for a Bell state \( |ψ\rangle \equiv |ψ_1\rangle \otimes |ψ_2\rangle \), where we assume that \( D(β) = e^{iαβ} |α + β\rangle \) are real amplitudes, and we assume that \( α = −iβ \) is real to obtain
\[ D(iε)|α⟩ \cong e^{iαε} |α + iε⟩. \tag{11} \]

If we let \( θ = αε \) be fixed, with \( ε \rightarrow 0 \) and \( α \rightarrow ∞ \), then we obtain the rotation (9) for
\[ |ψ_0(θ)⟩ \sim D(iε)|0⟩ , |ψ_1(θ)⟩ \sim D(iε)|1⟩. \tag{12} \]

The fidelity approaches unity exponentially with respect to \( ε^2 \).

The Bell state can thus be created for the state of the two-dimensional vibrational mode. However, direct detection of the Bell state is not possible with current technology. An entanglement transfer from the vibrational mode to the internal electronic states of the ions would allow detection of the entanglement due to the existence of the Bell state. The electronic state of an ion can be rotated and read with current technology.

In order to transfer entanglement from vibrational to electronic degrees of freedom, we need to be able to effect the transfer
\[ (c_0|0⟩ + c_1|1⟩)|0⟩_e \rightarrow |0⟩_e (c_0|0⟩_e + c_1|1⟩_e), \tag{14} \]

for \( \{ |0⟩_e, |1⟩_e \} \) the two electronic states of the ion. The transfer (14) is achieved via the swap operation
\[ |0⟩_e \otimes |0⟩_e \rightarrow |0⟩_e \otimes |0⟩_e, \tag{15} \]
\[ |0⟩_e \otimes |1⟩_e \rightarrow |1⟩_e \otimes |0⟩_e, \tag{16} \]
\[ |1⟩_e \otimes |0⟩_e \rightarrow |0⟩_e \otimes |1⟩_e, \tag{17} \]
\[ |1⟩_e \otimes |1⟩_e \rightarrow |1⟩_e \otimes |1⟩_e. \tag{18} \]

The swap operation can be realised via a sequence of three controlled CNot gates. The vibrational qubit is the control and the electronic qubit is the target for the first and third gates, and the reverse holds for the second qubit. We now discuss how to realise these two types of CNot gates.

In the first case, where the vibrational qubit is the control, it is necessary for the electronic qubit to be prepared in the ground state and to become excited if and only if the vibrational qubit contains an odd number of phonons. This transformation is achieved via the unitary transformation \[ U_{ve} = e^{−iπa^\dagger aσ_z}, \tag{19} \]

where the even and odd coherent states are defined by
\[ |±\rangle_a \equiv N_{±}(α)(|α⟩_a ± |−α⟩_a), \]
\[ |±\rangle_b \equiv N_{±}(β)(|β⟩_b ± |−β⟩_b), \tag{3} \]
which can be achieved by employing Raman pulses at the carrier frequency.

The second CNot gate reverses the roles of the vibrational and electronic qubits. Therefore, phonon number at rate processes; one corresponds to an upward transition in the constant rate the same, at least initially. The mean value of the ampli
cussed in this paper these two rates are approximately aple model of heating for a vibrational mode with anni
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therefore produces the desired entanglement swap.

In current ion trap experiments heating of the vibra
tional mode, though small, cannot be neglected. A sim
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the state. In other words, heating leads to bit-flip errors.

Up to a fidelity of $\exp(-\epsilon^2)$, the pure Bell state $\rho = \frac{1}{2}|\phi^+\rangle \langle \phi^+| + \frac{1}{2}|\psi^+\rangle \langle \psi^+|$, (25) with $|\psi^+\rangle = |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle$ (26) another of the four maximally entangled Bell states, which is orthogonal to the desired state.

The state given by (25) must violate the spin Bell inequ
ity $[1,2]$ $B = \left| E(\theta_1, \theta_2) + E(\theta_1, \theta'_2) + E(\theta'_1, \theta_2) - E(\theta'_1, \theta'_2) \right| \leq 2$ (27)
for $\delta$ sufficiently small. The spin correlation function $E(\theta_1, \theta_2)$ is given by

$$E(\theta_1, \theta_2) = \text{Tr} \left[ \rho \prod_{i=1}^2 \hat{V}_{i}^{1,\frac{\pi}{2}}(\theta_i) \sigma_z \hat{V}_{i}^{-1,\frac{\pi}{2}}(\theta_i) \right]$$ (28)

where $\hat{V}_{i}^{1,\frac{\pi}{2}}(\theta_i)$ is the Cirac and Zoller single–qubit rotation operator [17] on the $i$th ion obtained by applying a carrier pulse with phase $\theta_i$ for a time $t = \pi/2\Omega$ (with $\Omega$ is the Rabi frequency). In practice such a measurement is achieved by applying a single qubit rotation $\hat{V}_{i}^{1,\frac{\pi}{2}}(\theta_1)$ to the first ion and a rotation $\hat{V}_{2}^{1,\frac{\pi}{2}}(\theta_2)$ to the second ion. A simultaneous measurement of $\sigma_z$ on both ions via the shelving fluorescence technique [18] then determines the correlation function over many runs.

For the density matrix given by (25), it is easily shown that

$$E(\theta_1, \theta_2) = (1 - \delta) \cos \theta_1 \theta_2 + \delta \cos \theta_1 \theta_2$$; (29)
hence, the spin Bell inequality, for the choice of optimal angles with respect to violating the inequality, reduces to

$$B = 2\sqrt{2}(1 - \delta).$$ (30)
A violation of this inequality is possible for $B > 2$, that is $\delta < 1 - \frac{1}{\sqrt{2}}$. Whereas we may be able to violate techni
cally the Bell inequality, it is not a loophole–free test due to the limited temporal separation of the ions. It does how
ever completely close the detection loophole.

To summarise, we have described how entangled qubits can be encoded as entangled coherent states of two–di
mensional centre-of-mass vibrational motion for two ions in an ion trap. The entangled qubit state is equiva
lent to the canonical Bell state, and by transferring the entanglement from the two vibrational modes to the ele
tronic states of the two ions, the Bell state can be detected by resonance fluorescence shelving methods.

\[ \frac{1}{2} \left( 1 - \delta \right) |\phi^+\rangle \langle \phi^+| + \frac{1}{2} |\psi^+\rangle \langle \psi^+|, \] (25)