Non-conformal examples of AdS/CFT

Steven S. Gubser

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract

Asymptotically anti-de Sitter spacetimes with Poincaré invariance along the boundary can describe, via the AdS/CFT correspondence, either relevant deformations of a conformal field theory or non-conformal vacuum states. I consider examples of both types constructed in the framework of five-dimensional gauged supergravity. I explain the proof and motivation of a gravitational “c-theorem” which is independent of dimension. I show how one class of examples can be elevated to ten-dimensional geometries involving distributions of parallel D3-branes. For these cases some peculiar properties of two-point functions emerge, and I close with speculations on their physical origin.

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1 Introduction

The AdS/CFT correspondence [?, ?, ?] (for a review see [?]) offers us the opportunity to study not only exactly conformal field theories and their anti-de Sitter duals, but also deformations of conformal field theories corresponding to distortions of the anti-de Sitter geometry. A subtlety that was realized early on is that these distortions can represent either changes in the hamiltonian of the dual field theory or changes in the physical state. Which obtains in any particular case is determined by the asymptotics of the various fields involved near the boundary of anti-de Sitter space. Resolving this ambiguity has been one of the main obstacles to formulating the renormalization group (RG) equations clearly in an AdS/CFT setting. Very roughly, supergravity equations of motion are second order, admitting two solutions—one solution being a deformation of the hamiltonian and the other a change in the physical state—but RG equations are first order, and any translation of RG into AdS/CFT language must somehow specify which supergravity solution the RG picks out. Conventionally we think of RG as acting on the hamiltonian rather than the state, so we should be looking for a way to track deformations of the hamiltonian with the state held fixed.

An improved understanding of RG is one motivation of the work I will describe. The result which touches most directly on RG is a gravitational “c-theorem,” which I will exhibit in section 2 after some preliminaries to set conventions and make slightly more precise the distinction between states and deformations. I will then move on in section 3 to an interesting class of examples which represent states in the Coulomb branch of $\mathcal{N} = 4$ super-Yang-Mills theory. The supergravity geometries can be described wholly in terms of five-dimensional $\mathcal{N} = 8$ gauged supergravity, but the physics is clearest from the distribution of D3-branes in ten dimensions from which these geometries descend. They provide a simple example of consistent truncation. They also exhibit two surprising features: 1) the distribution of D3-branes is not always positive definite, and 2) the spectrum of supergravity excitations is often gapped or even discrete. The spectrum can be obtained through an analysis of two-point functions, and this will be the topic of section 4. This talk is mostly based on collaborative work with D. Freedman, K. Pilch, and N. Warner [?, ?].

2 A c-theorem from gravity

In order to have a clear methodology for computing physical quantities such as correlators, I will restrict attention to spaces which are asymptotically anti-de Sitter. Furthermore, all my examples will have 3+1-dimensional Poincaré invariance, so that
the five-dimensional geometry can be written in the form
\[ ds^2 = e^{2A(r)}(dt^2 - dx_1^2 - dx_2^2 - dx_3^2) - dr^2. \]  
(1)

With this choice of coordinates, \( A = r/L \) corresponds to pure \( AdS_5 \) (or more precisely a Poincaré patch of \( AdS_5 \)), with \( R_{\mu\nu} = \frac{4}{L^2}g_{\mu\nu} \). If \( A \) is not linear, then there must be some matter fields providing an \( r \)-dependent stress-energy tensor which supports the geometry. In order to have Poincaré invariance, essentially the only possibility is that the matter fields should be scalars. For the space to be asymptotically \( AdS_5 \), we need the scalars to approach constant values near the boundary. Those constant values must represent an extremum of the scalar potential, and by convention we will say the scalars vanish there. Near the boundary one can use a linearized approximation to the scalar equations of motion: for one scalar, \((\Box + m^2)\phi = 0\). The two independent solutions map to deformations and vacuum states in the following way:
\[
\begin{align*}
\phi &\sim e^{-\Delta_- r/L} &\iff \mathcal{L} \rightarrow \mathcal{L} + e^{\Delta_- r/L}\phi\mathcal{O} \\
\phi &\sim e^{-\Delta_+ r/L} &\iff \langle\mathcal{O}\rangle = e^{\Delta_+ r/L}\phi
\end{align*}
\]  
(2)

where \( \Delta_{\pm} = 2 \pm \sqrt{4 + (mL)^2} \) and \( \mathcal{O} \) is the color singlet operator (of dimension \( \Delta_+ \)) in the conformal field theory which is dual to \( \phi \). In words, the more singular asymptotics of \( \phi \) corresponds to adding a source for \( \mathcal{O} \) to the lagrangian, while the less singular solution corresponds to giving \( \mathcal{O} \) a VEV. These are dual concepts in the sense of Legendre transforms. It is possible in certain circumstances for the roles of \( \Delta_+ \) and \( \Delta_- \) to be interchanged \([?]\). This talk will focus on deformations and states of \( \mathcal{N} = 4 \) super-Yang-Mills, where the conventional identification of the more singular solution as a deformation and the less singular one as a VEV is correct for all the fields.

There is a general intuition in quantum field theory that there is a thinning out of degrees of freedom as one passes from the ultraviolet to the infrared. This holds equally whether the energy scales of the theory arise from the physical state (as in spontaneous symmetry breaking) or from the dynamics of the lagrangian (as in confinement). A first step, then, in finding the meaning of the renormalization group in the language of AdS/CFT is to see how and whether one can quantify the thinning out process.

In two dimensions there is a celebrated theorem of Zamolodchikov \([?]\) to the effect that renormalization group flows follow steepest descent trajectories of a so-called “c-function” of the couplings in the hamiltonian. At fixed points of the renormalization group the c-function coincides with the central charge in the Virasoro algebra, or equivalently with the coefficient \( c \) in the anomalous one-point function
\[
\langle T_\mu^\nu \rangle g_{\mu\nu} = -\frac{c}{24\pi}R
\]  
(3)
where $g_{\mu\nu}$ is an external metric and $R$ is its scalar curvature. The analogous expression in four dimensions, in the notation of [?], is

$$
\langle T^\mu_{\mu}\rangle_{g_{\mu\nu}} = \frac{c}{16\pi^2} W^2_{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} \tilde{R}_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}.
$$

The first term is the square of the Weyl tensor and the second is the topological Euler density. It has been conjectured by Cardy [?] that any renormalization group flow which begins and ends at a conformal fixed point must have a lower value of $a$ in the infrared than the ultraviolet.

On the other hand, it has been shown [?] using the prescription of [?, ?] that a pure AdS$_5$ geometry (that is, (1) with $A(r) = r/L$) leads to a one-point function of the form (4) with

$$a = c = \frac{\pi^2}{\kappa^2} A^3(r).$$

In arbitrary even boundary dimension $d$ a similar expression emerges from the analysis [?] with all anomaly coefficients in a fixed ratio and proportional to $1/A'(r)^{d-1}$, which again is constant since $A(r)$ is linear.

The remarks of the last two paragraphs suggest a natural “c-theorem” in AdS/CFT: can it be shown that in a geometry where $A(r)$ approaches linear behavior both as $r \to +\infty$ (near the boundary of AdS$_{d+1}$) and as $r \to -\infty$ (in the deep interior), the limiting value of $1/A'(r)^{d-1}$ is smaller in the deep interior than near the boundary? The answer is yes for a very straightforward reason: inspecting the curvature tensors for the metric (1), one finds that

$$-(d-1)A'' = R^t_t - R^r_r = G^t_t - G^r_r = \kappa^2_d (T^t_t - T^r_r) \geq 0
$$

where in the second to last step we have used Einstein’s equations, and in the last step we have assumed the null energy condition: $T_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0$ for any null vector $\xi^\alpha$. This energy condition is obeyed for all the matter fields in gauged supergravity, and to my knowledge also in string theory (I exclude from consideration orientifold planes, which are non-dynamical in the sense that their location is fixed). It should be noted that all curvature tensors as well as the stress tensors in (6) are $d+1$-dimensional quantities relating to the supergravity dual picture, whereas in (4) we were discussing $d$-dimensional quantities relating to the boundary quantum field theory. An equivalent form of (5) was considered independently in [?], and its monotonicity was checked for the non-supersymmetric flows studied there.

A compelling point of evidence in favor of the definition (5) is that it leads to the correct central charges for supersymmetric RG flows between conformal fixed points. The field theory techniques for analyzing these flows were developed in [?]. AdS/CFT examples have been studied in [?, ?, ?, ?, ?]. I will focus on the case worked out in
complete detail in [?], namely $\mathcal{N} = 4$ super-Yang-Mills theory deformed to an $\mathcal{N} = 1$ theory by adding a mass for one of the adjoint chiral superfields. This has already been discussed in the talk by N. Warner, so I will be brief.

The superpotential for the field theory in question is

$$ W = \text{tr} \phi_3 [\phi_1, \phi_2] + \frac{m}{2} \text{tr} \phi_3^2. \quad (7) $$

Using the techniques of [?], the infrared limit can be shown to be a strongly interacting fixed point with a quartic superpotential proportional to $\text{tr} [\phi_1, \phi_2]^2$. The anomaly coefficients $c$ and $a$ can be computed from anomalies in the divergence of the $R$-current: this is on account of the fact that in $\mathcal{N} = 1$ supersymmetry, $\partial_\mu R^\mu$ and $T_\mu^\mu$ are in the same multiplet. The result is that $c = a$ both in the infrared and the ultraviolet, and $c_{UV} = N^2/4$ while $c_{IR} = \frac{27}{32} c_{UV}$. 't Hooft anomaly matching is required to obtain the latter result. These predictions are precisely matched by (5) applied to the dual supergravity geometry [?], which was described in detail by N. Warner earlier in this conference. The first check that AdS/CFT predicted the correct value for $c_{IR}/c_{UV}$ was carried out in [?]. Subsequent work on the subject includes [?]. In the case of supersymmetric flows, it was possible to express the $c$-function as an explicit function of the supergravity scalars [?]: all $r$-dependence in $C(r)$ arose strictly through the scalar profiles. The scalars’ evolution followed the gradient flow of an appropriate power of the $c$-function. This has very much the flavor of Zamolodchikov’s $c$-theorem, but it seemed to rely on supersymmetry. However it was recently shown [?] that any solution to the equations of motion of

$$ S = \int d^4x dr \sqrt{|\text{det} g_{\mu\nu}|} \left[ -\frac{1}{4} R + \frac{1}{2} G_{I J} \partial^\mu \phi^I \partial_\mu \phi^J - V(\phi) \right], \quad (8) $$

of the form (1) with scalar profiles depending only on $r$ could be obtained as solutions to the first order equations

$$ \frac{d\phi^I}{dr} = \frac{1}{2} G^{IJ} \frac{\partial W(\phi)}{\partial \phi^J}, \quad \frac{dA}{dr} = -\frac{1}{3} W(\phi), \quad (9) $$

where $W(\phi)$ is a “superpotential” (not to be confused with the boundary field theory superpotential (7)) satisfying

$$ V(\phi) = \frac{1}{8} G^{IJ} \frac{\partial W(\phi)}{\partial \phi^I} \frac{\partial W(\phi)}{\partial \phi^J} - \frac{1}{3} W(\phi)^2. \quad (10) $$

The point is that, given $V(\phi)$ (10) can in principle be solved for $W(\phi)$. Thus the relation $C \propto 1/W(\phi)^3$, first derived in [?] for supersymmetric flows, is seen to extend to the

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*R. Myers has also noted that this solution generating technique [?] can be applied even in the absence of supersymmetry.
non-supersymmetric context [?]—only now $W(\phi)$ is not unique but rather depends on as many constants of integration as there are scalars. This gives me some new hope that the RG equations might yet be related explicitly to supergravity equations, with the aforementioned constants of integration specifying what physical quantities are held fixed in the renormalization. However I should note the drawback that $W(\phi)$ is not a single-valued function of the $\phi^I$. The graph of $W$ versus $\phi^I$ is a piecewise smooth, codimension one hypersurface in the region of $(W,\phi^I)$ space where $V(\phi) + \frac{1}{3}W^2 > 0$. It has discontinuous slope at the boundary of this region: it “reflects” off the boundary to form multiple sheets for the function $W(\phi)$.

3 States on the Coulomb branch of $\mathcal{N} = 4$ super-Yang-Mills

The main geometrical fact we have learned from the previous section is that the function $A(r)$ appearing in the metric (1) is concave down as a function of $r$: that is, $A''(r) \leq 0$. The inequality is saturated precisely for (locally) anti-de Sitter space. There are then three different possibilities for the behavior of $A(r)$ in geometries dual to field theories with conformal invariance broken either by VEV’s or relevant deformations. They are sketched in figure 1.

All three possibilities are realized in various contexts in the literature. Case (1) occurs for the renormalization group flow discussed in the previous section. Case (3) is perhaps the most generic, and occurs in various attempts to find a geometry dual to
confining gauge theories \cite{1,2,3}. It has the pathology of a naked timelike singularity at the radius where \( A(r) \to -\infty \). The borderline case (2) has unbounded curvatures as \( r \to -\infty \), but there are no genuine curvature singularities inside the Poincaré patch.\footnote{It is a largely open question what is the maximal analytic extension of the bulk geometries discussed in this lecture.} An example of (2) was obtained \cite{4} as a state on the Coulomb branch of \( \mathcal{N} = 4 \) super-Yang-Mills, and we will discuss it below. Earlier work on the subject includes \cite{5,6}, and the independent work of \cite{7} has some overlap with the results of \cite{4}.

The Coulomb branch of \( SU(N) \mathcal{N} = 4 \) super-Yang-Mills theory is parametrized by VEV's of the scalar fields \( X_I \), or equivalently the relative positions of the \( N \) D3-branes in the transverse six dimensions. We should be able to describe these states in terms of an asymptotically \( AdS_5 \) geometry with some scalar profiles specified near the boundary. Instead of specifying directly the eigenvalues of the \( X_I \), this approach amounts to specifying the quantities \( \text{tr} X_{I_1}X_{I_2}...X_{I_\ell} \). Actually these gauge-invariant traces do not wholly fix the eigenvalues of the \( X_I \), as we will see in examples.

The simplest case from the point of view of five-dimensional supergravity is to give VEV's only to those scalars which are in the same \( d = 5 \), \( \mathcal{N} = 8 \) supersymmetry multiplet as the graviton. There are 42 such scalars, of which 20 with \( (mL)^2 = -4 \) are dual to the operators \( \text{tr} X_{(i}X_{j)} \). They parametrize the coset \( SL(6, \mathbb{R})/SO(6) \). A explicit discussion of the five-dimensional geometries in question was given in the talk by N. Warner: they all preserve 16 supersymmetries and a \( SO(n) \times SO(6 - n) \) subgroup of \( SO(6) \) (by convention \( SO(1) \) is the trivial group). Here I will focus on the ten-dimensional origin of these geometries and on the spectrum of supergravity excitations which they give rise to.

One obvious point is that the ten-dimensional geometries must be near-horizon limits of multi-center D3-brane solutions: that is,

\[
d s^2_{10} = \frac{1}{\sqrt{H}} (d t^2 - d x_1^2 - d x_2^2 - d x_3^2) - \sqrt{H} (d y_1^2 + d y_2^2 + \ldots + d y_6^2)
\]

\[
H = \frac{L^4}{N} \sum_{i=1}^{N} \frac{1}{|\mathbf{y} - \mathbf{y}_i|^4} \approx \frac{L^4}{N} \int d^6 y' \frac{\sigma(\mathbf{y}')}{|\mathbf{y} - \mathbf{y}'|^4} \quad .
\]

Here \( L^4 = \frac{\kappa_{10} N}{2\pi^{5/2}} \) and \( \sigma(\mathbf{y}) \) is a continuous distribution normalized so that its integral is one. (In the supergravity approximation, discreteness of D3-branes is lost). The \( SO(n) \times SO(6 - n) \) symmetry of the five-dimensional supergravity solution must translate into a symmetry of the distribution \( \sigma(\mathbf{y}) \). It emerged from the analysis of \cite{4} that the distributions in question were

\[
n = 0 : \quad \sigma = \delta^6(\mathbf{y}) ,
\]

\[
n = 1 : \quad \sigma = \frac{2}{\pi^2} (1 - \frac{y_1^2}{\ell^2})^{1/2} \Theta(1 - \frac{y_1^2}{\ell^2}) \delta^5(\mathbf{y}_{2,3,4,5,6}) ,
\]
\[ n = 2 : \quad \sigma = \frac{1}{\pi \ell^2} \Theta(1 - \frac{y_1^2}{\ell^2} - \frac{y_2^2}{\ell^2}) \delta^{(4)}(\vec{y}_{3,4,5,6}), \]

\[ n = 3 : \quad \sigma = \frac{1}{\pi \ell^2} \left(1 - \sum_{i=1}^{3} \frac{y_i^2}{\ell^2}\right)^{-1/2} \Theta \left(1 - \sum_{i=1}^{3} \frac{y_i^2}{\ell^2}\right) \delta^{(3)}(\vec{y}_{4,5,6}), \quad (12) \]

\[ n = 4 : \quad \sigma = \frac{1}{\pi \ell^3} \delta \left(1 - \sum_{i=1}^{4} \frac{y_i^2}{\ell^2}\right) \delta^{(2)}(\vec{y}_{5,6}), \]

\[ n = 5 : \quad \sigma = \frac{1}{2\pi^3 \ell^5} \left[ - \left(1 - \sum_{i=1}^{5} \frac{y_i^2}{\ell^2}\right)^{-3/2} \Theta \left(1 - \sum_{i=1}^{5} \frac{y_i^2}{\ell^2}\right) \right. \]
\[ \left. + 2 \left(1 - \sum_{i=1}^{5} \frac{y_i^2}{\ell^2}\right)^{-1/2} \delta \left(1 - \sum_{i=1}^{5} \frac{y_i^2}{\ell^2}\right) \right] \delta(y_6), \]

where \( \Theta(x) \) is 1 for \( x > 0 \) and 0 for \( x < 0 \). It was observed in [?] that all of these distributions are limits (in the weak sense of being convolved against any smooth function) of

\[ n = 6 : \quad \sigma = \frac{1}{\pi \ell_1 \cdots \ell_6} \delta' \left(1 - \sum_{i=1}^{6} \frac{y_i^2}{\ell_i^2}\right), \quad (13) \]

where \( \delta'(x) = \frac{d}{dx} \delta(x) \). An obvious peculiarity of the \( n = 5 \) and \( n = 6 \) distributions is that \( \sigma(\vec{y}) \) is not a positive definite distribution. This means that there are “ghost” D3-branes with the opposite charge and tension of a normal D3-brane. Formally the force between ghost D3-branes and normal D3-branes vanishes, but the kinetic terms for fluctuations of a ghost D3-brane are negative. This is apparently invisible in five-dimensional supergravity; nevertheless I regard non-positive \( \sigma \) as unphysical. A second point to observe is that the geometry away from the \( n = 6 \) brane distribution, (13), is invariant if we change all the \( \ell_i^2 \) by the same additive constant. In particular, when all the \( \ell_i \)'s are equal, the geometry outside the \( \delta' \)-function shell in the \( n = 6 \) distribution is perfect \( \text{AdS}_5 \times S^5 \). This continues to hold when we replace the \( \delta' \)-function with an ordinary \( \delta \)-function (thus curing the pathology of ghost D3-branes). This geometry was considered in [?, ?]. The only gauge singlets that have VEV’s are also \( SO(6) \) singlets, the simplest being \( \text{tr} \sum_I X_I^2 \). These operators are not in short multiplets, and are expected to acquire large anomalous dimensions in the strong coupling limit. How this squares with the exactness of the classical moduli space of \( \mathcal{N} = 4 \) super-Yang-Mills is a source of some puzzlement for me.

Extracting the distributions (12) from the explicit five-dimensional geometries that N. Warner discussed is a slightly involved calculation. I will summarize only the most interesting part: the consistent truncation ansatz.

Consistent truncation is often mistaken for the imprecise statement that all but finitely many Kaluza-Klein modes can be consistently set to zero in reduction of supergravity theories on compact manifolds. In the case of \( S^5 \) it is difficult to see how this can be so, since both the modes that are kept and the modes that are cast out include
SO(6) singlets and non-singlets. A more straightforward way to think about consistent truncation is as a technique for generating exact solutions to a higher dimensional supergravity from the data of a lower-dimensional one.

Unlike for eleven dimensional supergravity on $S^7$ [?], or $S^4$ [?], the full solution-generating ansatz for ten-dimensional type IIB supergravity on $S^5$ is not known even in implicit form. We do however have the full ansatz when only scalars are excited; essentially it was stated in [?]. Briefly, the 42 scalars in the $\mathcal{N}=8 d=5$ supergraviton multiplet can be parametrized by a 27-bein $V_{AB}^{ab}$ for the coset $E_6(6)/USp(8)$. The indices $a$ and $b$ are in the fundamental 8 of $USp(8)$. In $V_{AB}^{ab}$ they are antisymmetrized and the symplectic trace is removed: this results in the 27-dimensional representation of $USp(8)$, which is also the fundamental representation of $E_6(6)$. A common convention is to use $z^{AB}$ to denote a vector in the 27 of $E_6(6)$ and $z_{AB}$ for the 27.

A crucial decomposition in the construction of $d=5 \mathcal{N}=8$ gauged supergravity [?] is

$$
E_6(6) \supset SL(6, \mathbf{R}) \times SL(2, \mathbf{R})
$$

$$27 = (6, 2) + (15, 1) \quad (14)
$$

where we have indicated in the second and third lines the way the 27 of $E_6(6)$ splits under this decomposition. Lowered indices $I, J$ refer to the 6 of $SL(6, \mathbf{R})$. They key fact in (14) which makes it possible to gauge the obvious $SO(6)$ subgroup of $SL(6, \mathbf{R})$ is that the 27 of $E_6(6)$ contains the adjoint of $SO(6)$ (it is the 15). That is important because the vector fields of ungauged $d=5 \mathcal{N}=8$ supergravity transform in the 27. These vectors correspond (roughly) to components of the metric with one leg in the non-compact dimensions and one along a Killing vector, $K^{mIJ}$, of $S^5$. (Here I use $m$ to denote a tangent space index to $S^5$; $IJ$ label the relevant element of the 15.) Up to a normalization, the $K^{mIJ}$ can be taken to be the projections onto $S^5$ of the differential operators $x^I \partial_J - x^J \partial_I$.

Now suppose we have some arbitrary solution to $d=5 \mathcal{N}=8$ gauged supergravity involving only the metric, $ds_5^2$, and the scalars, $V_{AB}^{ab}$. To construct the ten-dimensional Einstein metric, we proceed as follows. Define $K^{mab} = K^{mIJ}(\mathcal{V}^{-1})_{IJab}$. Solve the simultaneous equations

$$
\Delta^{-2/3} \tilde{g}^{mn} = K_{ab} K^{a}_{cd} \Omega^c \Omega^d
$$

$$
\Delta^2 = \det(\tilde{g}_{mn} g^{np}(0)) \quad (15)
$$

for $\Delta$ and the metric $\tilde{g}_{mn}$ (whose inverse is $\tilde{g}^{mn}$). The metric $g_{(0)mn}$ is the usual round metric on $S^5$. Then the ten-dimensional metric is

$$
ds_{10}^2 = \Delta^{-2/3} ds_5^2 + \tilde{g}_{mn} d\psi^m d\psi^n \quad (16)
$$

where $\psi^m$ are coordinates on $S^5$. General elements of $E_6(6)/USp(8)$ can lead to metrics
\( \tilde{g}_{mn} \) which are difficult to describe concisely. However for the subspace \( SL(6, \mathbb{R})/SO(6) \) parametrized by the 20 scalars dual to \( tr X_I X_J \), \( \Delta^{-2/3} \tilde{g}_{mn} \) is the metric of an ellipsoid.

Once the metric (16) has been obtained, it is relatively straightforward to find appropriate coordinates \( y_I \) in which the metric takes the form (11).

### 4 Two-point functions and the spectrum of supergravity excitations

Given a two-point function \( \langle O(x)O(0) \rangle \), there is a straightforward way of extracting the spectrum of excitations in the field theory that can be excited using the operator \( O \). Let us define

\[
\Pi(s) = \int d^4x e^{ip \cdot x} \langle O(x)O(0) \rangle ,
\]

where \( s = p^2 \). The function \( \Pi(s) \) will be analytic in the complex \( s \)-plane except possibly for branch cuts and/or poles on the real axis. The discontinuity across the real axis, \( \text{Disc} \Pi(s) = \Pi(s + i\epsilon) - \Pi(s - i\epsilon) \), is the spectral measure of the invariant masses of the states which \( O(0) \) can create from the vacuum. If we use the AdS/CFT prescription for computing Green’s functions \([?, ?]\), which reads, schematically,

\[
W_{\text{CFT}}[\phi_0] = \log \langle \exp \int d^4x \phi_0 O \rangle_{\text{CFT}} = \text{extremum} S_{\text{SUGRA}}[\phi] ,
\]

what one finds is that this spectral measure is precisely the spectrum of the linearized equation for excitations of \( \phi \) in the bulk spacetime.

To be more explicit, consider the example of the five-dimensional dilaton, which is dual to the dimension four operator \( O_4 = tr(F^2 + \bar{\psi} \psi - X \Box X + \ldots) \). (I have not been fastidious about numerical factors here). The five-dimensional wave equation is \( \Box \phi = 0. \)\(^1\) This can be solved using separation of variables: defining a radial variable \( z \) such that the five-dimensional metric takes the conformally flat form

\[
ds^2 = e^{2A(z)}(dt^2 - dx_1^2 - dx_2^2 - dx_3^2 - dz^2) ,
\]

we find

\[
\phi = e^{-ip \cdot x} e^{-3A(z)/2} R(z)
\]

\[
[-\partial_z^2 + V(z)] R(z) = p^2 R(z)
\]

\[
V(z) = \frac{3}{2} A''(z) + \frac{9}{4} A'(z)^2 .
\]

\(^1\)Actually all the supergravity calculations can be done equally efficiently starting from the ten-dimensional backgrounds. The five-dimensional dilaton is the s-wave of the ten-dimensional dilaton.
Figure 2: The various behaviors for $V(z)$ far from the boundary of $AdS_5$: a) Vanishes; b) Asymptotes to a finite value; c) Increases without bound; d) Decreases without bound.

The spectrum the Schrödinger operator $[-\partial_z^2 + V(z)]$ is precisely the spectrum of excitations created by $O_4$. The expression for $V$ in the last line of (20) has the form of supersymmetric quantum mechanics, so we know that the spectrum is bounded below, $p^2 \geq 0$. The qualitative behavior of the spectrum can be reasoned out directly from the shape of $V(z)$. The possibilities are shown in figure 2. The first is encountered for the $n = 1$ flow, and the spectrum for $p^2$ is $(0, \infty)$. The second is encountered for $n = 2$, and the spectrum is $(\ell^2/L^4, \infty)$. The third is encountered for $n = 3$, and the fourth for $n = 4, 5$; in these cases the spectrum is discrete. It is in fact calculable for $n = 4$, with the result $p^2 = 4\ell^2/L^4 j(j + 1)$ for $j = 1, 2, 3, \ldots$.

A more precise and exhaustive description of the spectrum and the explicitly calculable two-point functions can be found in [?, ?, ?]. The five- and ten-dimensional geometries were also studied in [?], and the linearized fluctuation equations were related to integrable models of the Calogero-Moser type.

The outstanding puzzle that emerges from the analysis of the spectrum is, how is Coulomb branch physics giving rise to mass gaps and discrete spectra? After all, the Coulomb branch is distinguished in that there are long range forces, which suggests massless excitations. I do not have a good answer, but I will make some remarks which tend to make the obvious answers less plausible and conclude with a (perhaps

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§ My lecture at Strings ’99 and the initial versions of [?] were slightly mistaken on this point. The error was caught in [?].
wild-eyed) suggestion of what might be going on.

The first obvious point is that there are only $O(N)$ truly massless modes, corresponding to the un-Higgsed $U(1)^{N-1}$ gauge group. We do not expect classical supergravity to see subleading effects in $N$. Thus perhaps the massive excitations we are seeing are color singlet combinations of the $O(N^2)$ massive modes, corresponding to strings stretched between different D3-branes in the distribution. In some sense I think this idea has to be right. But there are some difficulties left. First, the typical mass of a stretched string is $\ell/\alpha'$, where $\ell$ is the diameter of the brane distribution. But in supergravity the scale of the mass gap (or the discrete spectra) is instead $\ell/L^2 = \ell/(\alpha'\sqrt{g_{YM}^2N})$. The stretched strings are BPS states, so their masses are protected; but color-singlet bound states of them need not be BPS, and their masses could be smaller than the sum of the masses of the constituents because of binding energy. In fact, maybe the binding energy lowers the mass by the requisite factor $\sqrt{g_{YM}^2N}$.\footnote{To my knowledge this was first proposed by J. Polchinski.} In a lot of ways this sounds like the most plausible way out of our difficulties. However I am not wholly satisfied with it for the following reasons. First, the BPS mass spectrum of stretched strings extends continuously to 0 in the large $N$ approximation. A fraction on the order of $(g_{YM}^2N)^{-(1+\delta)/2}$ of the stretched strings have mass less than the supergravity mass gap, $\ell/L^2$. For all the positive definite mass distributions ($n \leq 4$ in (12)) we have $\delta \leq 2$, so this is still a large number of states. Binding energy by definition cannot be positive, so the existence of these states seems to contradict the existence of a mass gap. (One way out of this is to say that because their number is suppressed by a power of the 't Hooft coupling, supergravity simply misses them. It would be interesting to estimate more carefully the $\alpha'$ corrections in the regions of strong curvature and test whether this escape is plausible.) Second, the $n = 1$ distribution exhibited no mass gap in supergravity. What aspect of the field theory binding mechanism could be responsible for producing a gap only when $n > 1$?

A possible resolution to the second question is that what supergravity measures is not simply the correlator $\langle \mathcal{O} \mathcal{O} \rangle$ in a particular state on the Coulomb branch, but rather the averaged quantity $\langle \mathcal{O} \mathcal{O} \rangle$, where the averaging is over all discrete distributions of $N$ D3-branes which are consistent with the continuous distribution $\sigma$. Say $\sigma$ is a $p$-dimensional distribution in the transverse 6 dimensions: $p = 1$ for the case $n = 1$, where the D3-branes are distributed along a line segment; $p = 2$ for $n = 2$, where the D3-branes are distributed over a disk; and $p = 3$ for $n = 3$ and $n = 4$, where the branes are distributed in a 3-dimensional ball or across the surface of an $S^3$. Intuitively speaking, the branes are allowed to move a distance comparable to the nearest-neighbor distance, $\ell/N^{1/p}$. The suggestion advanced in [?] is that averaging over the ensemble of brane distributions that we get by allowing individual D3-branes to “wiggle” a distance $\ell/N^{1/p}$ leads to an altered lagrangian which includes trace-squared terms. The guess is
that these terms are sufficiently dominant for $n > 2$ to give rise to a discrete spectrum, but are insignificant for $n < 2$. A naive estimate is that their coefficient would go as $N^{1-2/p}$, as compared to 1 for the usual terms in the $\mathcal{N} = 4$ lagrangian. I would emphasize however the speculative nature of this idea. No successful calculation has been carried out to verify that ensemble averaging could change the behavior of the Green’s functions so radically.

5 Conclusions

The one universal result in non-conformal examples of the AdS/CFT correspondence is the c-theorem. In supergravity it says that the function $C(r) = \frac{1}{A'(r)^d}$ is monotonic in $r$ for the metric (1) in $d+1$ non-compact bulk dimensions. This is simply a consequence of the null energy condition, which is satisfied for all types of matter that arise in string theory (with the possible exception of orientifold planes, where there is a formally negative tension). In field theory, $C(r)$ serves as a useful approximate guide to the number of degrees of freedom. It is approximate partly because supergravity only knows about the leading order in $N$, but mostly because the correspondence between radius and energy scale is not known precisely except in the case of pure $AdS_5$. When the geometry approaches $AdS_5$, $C(r)$ is a constant proportional to the trace anomaly coefficients in the conformal field theory. This and its monotonicity make it the best candidate we have for a c-function in the AdS/CFT context.

There is a five-parameter family of states on the Coulomb branch of $\mathcal{N} = 4$ super-Yang-Mills theory which admits a dual description wholly in terms of $d = 5 \mathcal{N} = 8$ gauged supergravity. These are states where the VEV’s of the operators $\text{tr} X_{(i} X_{j)}$ are adjusted arbitrarily. The VEV’s of all the other symmetric traceless products can best be calculated by raising the five-dimensional solution up to ten dimensions via the consistent truncation ansatz. The five-dimensional geometries provide examples of “flows” where $C(r)$ goes to zero, either smoothly as $r \to -\infty$ or sharply at a finite value of $r$. We understand this qualitatively as reflecting the fact that at lower energies, fewer of the massive stretched string excitations are available. However the characteristic energy scale of correlators calculated in the supergravity geometry is $\ell/L^2$: smaller by a factor of $\sqrt{g_{YM}^2 N}$ than the characteristic energy scale of the Higgs VEV’s which define the Coulomb state. Strong gauge interactions could be responsible for color singlet combinations of BPS stretched string states giving up most of their mass to binding energy. I have suggested that averaging over ensembles of brane distributions may also play a role in determining the qualitative features of the spectra and their remarkable dependence on the dimension of the distribution.
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