Quantum Computation with Bose-Einstein Condensation and Capable of Solving NP-Complete and \#P Problems

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Abstract

It is proposed that quantum computation can be implemented on the basis of macroscopic quantum coherence of a many-body system, especially the Bose-Einstein condensation. Since a Bose-Einstein condensate is described by a non-linear Schrödinger equation, and the non-linearity is tunable, in principle one may build a quantum computer composed of both linear and non-linear gates. Consequently NP-complete and \#P problems can be solved. This idea is illustrated by representing the qubit as the atomic Bose-Einstein condensate trapped in a double-well potential.

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Because of “quantum parallelism” of many branches superposing a quantum state, a quantum computer is much more efficient than a classical computer in solving certain problems [1,2]. Experimental implementations are pursued or proposed in various physical systems [3–8]. Nevertheless, whether quantum computer, based on linear quantum mechanics, can solve all problems in class NP was shown to be unlikely to be resolved without a major breakthrough in complexity theory [9]. It was found that supplemented by nonlinear quantum gates, quantum computer can solve in polynomial time NP-complete and \#P problems [10]. However, up to now it largely remained as an academic interest since elementary quantum mechanics is accurately confirmed to be linear. Here we point out that this problem can be turned to be a practicality by exploiting macroscopic quantum coherence. Because of the spontaneous gauge symmetry breaking, a Bose-Einstein condensate is described by a Schrödinger equation which may be non-linear. This non-linearity is due to interaction between the bosons composing the condensate. As an emergent entity, a Bose-Einstein condensate can represent a qubit as far as a two-state system is constructed for it. Hence one may build a quantum computer consisting of both linear and non-linear gates. We illustrate this idea by using atomic Bose-Einstein condensate trapped in a double-well potential.

In most of the experimental implementations of quantum computation up to now, a qubit is represented by a single or a few particles. But representing qubits in terms of macroscopic quantum coherence of a many-body system, such as superconducting state or Bose-Einstein condensation, may have various advantages, including the simplification of the operations, easier manipulation, and being robust against some microscopic details and thus reduces errors. While quantum computation based on Josephson-junction [7] involves superconducting states, atomic Bose-Einstein condensates are more controllable at present day.

The class of NP-complete problems is a foundation of the computational complexity theory. This unfortunately includes thousands of practically interesting problems, such as travelling salesman, satisfiability, etc. NP stands for ‘non-deterministic polynomial time’. NP-complete problems are those for which a potential solution can be verified in polynomial
time, yet finding a solution appears to require exponential time in the worst case. The completeness means that if an efficient, i.e. polynomial-time, algorithm could be found for solving one of these problems, one would immediately have an efficient algorithm for all NP-complete problems. A fundamental conjecture in classical computation is that no such efficient algorithm exists. Abrams and Lloyd found that together with both linear and nonlinear gates, a quantum computer can solve NP-complete problem by efficiently determining if there exists an \( x \) for which \( f(x) = 1 \), and can solve \#P problems (including oracle problems) by efficiently determining the number of solutions [10]. However, it is an experimental fact that elementary quantum mechanics is linear to the available accuracy [11], while nonlinear elementary quantum theory [12] usually violates the second law of thermodynamics [13] and the theory of relativity [14]. On the other hand, their algorithm requires both linear and nonlinear operations, hence this requirement would not be satisfied as well by a nonlinear elementary quantum theory, which would exclude linear operation. We will show that a quantum computer composed of both linear and non-linear gates can be constructed based on macroscopic wavefunction, or spontaneous gauge symmetry breaking [15], because non-linearity can be tuned in this case.

For a Bose gas consisting of many bosons, one can define a boson field operator \( \hat{\psi}(\mathbf{r}) = \sum_\mathbf{k} \hat{a}_{\mathbf{k}} u_\mathbf{k} \), where \( \mathbf{k} \) is the momentum, \( \hat{a}_{\mathbf{k}} \) is the annihilation operator, \( u_\mathbf{k} \) is the single particle wavefunction. Because of Bose-Einstein statistics, at a low temperature, there may be a finite density of bosons in the zero-momentum \( (\mathbf{k} = 0) \) state. It is known that a general criterion for Bose-Einstein condensation is [16]

\[
\langle \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y}) \rangle \rightarrow \langle \psi(\mathbf{x}) \rangle^* \langle \psi(\mathbf{y}) \rangle, \text{ for } |\mathbf{x} - \mathbf{y}| \rightarrow \infty,
\]

where \( \langle \psi(\mathbf{x}) \rangle = \Phi(\mathbf{x}) \neq 0 \) is the macroscopic wavefunction of the condensate, \( \langle \cdots \rangle \) is thermal average. Broken gauge symmetry refers to the fact that when the condensate wavefunction is nonzero, the ground state depends on its phase, although the original many-particle Hamiltonian is invariant under a global (constant) phase change of \( \hat{\psi}(\mathbf{r}) \) [15,16]. The many-body Hamiltonian is
\[ \mathcal{H} = \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) \right] \psi(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') U(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}), \quad (2) \]

where \( m \) is the mass of the boson, \( V(\mathbf{r}) \) is the external potential, i.e. the trapping potential in case of the trapped atoms. The macroscopic wavefunction of the condensate \( \Phi(\mathbf{r}) \) is governed by a nonlinear Schrödinger equation, known as Gross-Pitaevskii equation:

\[ i\hbar \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) + g|\Phi(\mathbf{r}, t)|^2 \right) \Phi(\mathbf{r}, t), \quad (3) \]

where \( g = \int d\mathbf{r} U(\mathbf{r}) = 4\pi\hbar^2 a/m, \) \( a \) is the s-wave scattering length of a binary collision. The origin of the nonlinearity is the mean-field interaction energy between individual bosons. Therefore although the underlying fundamental quantum mechanics is linear, the macroscopic wavefunction of the Bose-Einstein condensate, as an emergent entity, is governed by the Gross-Pitaevskii equation. The main purpose of this letter is to point out the idea that this can be exploited to do quantum computation. The details below only serve as examples; there may be alternative schemes.

The algorithm given by Abram and Lloyd is based on the usual linear gates together with a one-bit nonlinear gate, whose repeated application drives nearby states apart exponentially rapidly; or two one-bit nonlinear gates, one of which maps both \(|0\rangle\) and \(|1\rangle\) to \(|0\rangle\), while the other maps \(x|0\rangle + y|1\rangle\) (for given \(x\) and \(y\)) to \(|0\rangle\).

By using many different Bose-Einstein condensates, each of which represents a qubit, we may build a quantum computer. Recently it was demonstrated experimentally by Ketterle’s group at Massachusetts Institute of Technology that the nonlinearity in the Gross-Pitaevskii equation, i.e. the inter-atom interactions, can be tuned [17]. Therefore we can build both linear and nonlinear gates in terms of Bose-Einstein condensates.

To represent the qubit, we need to construct a two-state system for each individual condensate. One way is to use a Mexican-hat-like symmetric double-well trapping potential \( V(\mathbf{r}) \) (FIG. 1), which was formed in the experiments [18]. The tunneling effect of the condensate has been investigated theoretically by numerous authors [19], and the work of Milburn et al. [20] is directly relevant to our situation. We may represent \(|0\rangle\) as the localised
state at one of the two wells, $|1\rangle$ as that at the other. Thus $|0\rangle = \phi_0(r) = v(r - r_0)$, $|1\rangle = \phi_1(r) = v(r - r_1)$, where $v(r - r_i)$ $(i = 0, 1)$ is the ground state for the local potential at the vicinity of the minimum of the well $i$, which may be a parabolic one $\tilde{V}^{(2)}(r - r_i)$. Note that because of the finiteness of the barrier in the double-well, $|0\rangle$ and $|1\rangle$ are not strictly orthogonal, but can be very nearly so, i.e. $|\langle 0|1 \rangle|^2 = \epsilon \ll 1$.

A general qubit $|n(t)\rangle$ is the superposition of $|0\rangle$ and $|1\rangle$, i.e. $|n(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$, and can be represented as a matrix

$$\begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix}.$$ 

In this case, the Gross-Pitaevskii equation can be transformed to a two-component equation:

$$i\hbar \frac{\partial}{\partial t} |n(t)\rangle = [E \hat{I} + \Omega \sigma_x + \kappa \hat{I} \otimes \begin{pmatrix} |C_0|^2 \\ |C_1|^2 \end{pmatrix}] |n(t)\rangle,$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix, $\Omega = \int dr \phi_0^*(r)[V(r) - \tilde{V}(r - r_0)\phi_1(r)]$ represents the tunneling effect, $\kappa = g \int dr |\phi_0(r)|^4$, $E$ is the ground state energy of each local potential near the vicinity of the minimum of the well, corresponding to $v(r - r_i)$.

When the inter-atom interaction turned off, $\kappa = 0$, Eq. (4) leads to an arbitrary one-bit unitary transformation, depending on the time span $\tau$: $|n(\tau)\rangle \rightarrow \exp[-i(E\hat{I} - \Omega \sigma_x)\tau/\hbar] |n(0)\rangle$. Therefore we can build 1-bit linear gates.

When the inter-atom interaction turned on, we have $\kappa \neq 0$ in Eq. (4), which represents a twisting rotation in the state space spanned by $|0\rangle$ and $|1\rangle$. By choosing appropriate time span, this can be exploited to build the nonlinear 1-bit gates which are needed.

To have a 2-bit linear gate, we need to introduce a coupling between two condensates, both without inter-atom interactions. This may be done by using dipole-dipole interaction between the two condensates. We may put together two double wells, each of which confines a condensate. We let wells close to each other in a face-to-face way, i.e., $|0\rangle_1$ is close to $|0\rangle_2$, and $|1\rangle_1$ is close to $|1\rangle_2$. If the condensates are trapped in optical traps, they have parallel electric dipole moments. If they are trapped in magnetic traps, they have parallel magnetic moments. In either case, because of the repulsion interaction between the electric
dipole moments or magnetic moments, the two condensates tend to move apart, i.e. the states $|0\rangle_1|1\rangle_2$ and $|1\rangle_1|0\rangle_2$ are more favorable than $|0\rangle_1|0\rangle_2$ and $|1\rangle_1|1\rangle_2$. Therefore the total Hamiltonian of the two-bit system is

$$\hat{H} = E^1 + \Omega^1 \sigma^1_x + E^2 + \Omega^2 \sigma^2_x + J \sigma^1_z \sigma^2_z,$$

where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For generality, we have distinguished $E$ and $\Omega$ between the two double-wells, of course they may be respectively equal. Depending on the time span, and combined with the 1-bit linear operation, one can generate a universal 2-bit linear gate, which can further compose any required linear gate. Alternatively the one-bit and two-bit operations controlled by Eq. (4), with $\kappa = 0$, and Eq. (5) can provide any required linear gate.

So far we have outlined the basic scheme of building a quantum computer capable of solving NP-complete and #P problem, by using Bose-Einstein condensates in double-wells. The two-state system of a condensate may also be provided by the internal hyperfine states, represented as $|\text{\bar{0}}\rangle$ and $|\text{\bar{1}}\rangle$. A general qubit is then $|m(t)\rangle = b_0|\text{\bar{0}}\rangle + b_1|\text{\bar{1}}\rangle \equiv \begin{pmatrix} b_0(t) \\ b_1(t) \end{pmatrix}$. Without coupling between the internal and spatial degrees of freedom, coupling the internal states with an electromagnetic field leads to [21]

$$i\hbar \frac{\partial}{\partial t} |m(t)\rangle = \left( \frac{\hbar}{2} \omega \sigma_x + \frac{\hbar}{2} \delta \sigma_z \right) |m(t)\rangle,$$

where $\omega$ is the Rabi frequency, $\delta$ is the detuning. The 2-bit gates may be built by using a method similar to that for the trapped ions [3]. A satisfactory way of introducing the nonlinearity is not clear, maybe an effective nonlinearity for the evolution of internal state could be constructed by using the coupling with the spatial degree of freedom. But further investigation is needed to see whether this coupling is an advantage or a disadvantage.

To summarize, I suggest using macroscopic quantum coherence, especially the Bose-Einstein condensation, to do quantum computation. The condensate, as an emergent entity, is described by a non-linear Schrödinger equation, and the non-linearity is tunable. Therefore
in principle one can build a quantum computer composed of both linear and non-linear gates, consequently the NP-complete and \#P problems can be solved. The qubit, i.e. the two-state system of a condensate, may be constructed based on the condensate trapped in a double-well potential. One may also exploit the internal hyperfine states of a condensate to construct a qubit, but further investigation is needed to see whether this is feasible. While detailed implementation remains to be explored, the general idea of using macroscopic quantum coherence to do quantum computation, including non-linear operation, may well be turned to a reality with the rapid advance of technique of manipulating the atomic Bose-Einstein condensation.

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FIGURES

FIG. 1. A qubit: a Bose-Einstein condensate trapped in a double-well potential. $|0\rangle$ is represented by the localized state at one well, while $|1\rangle$ is that at the other well.
REFERENCES

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