Unbroken supersymmetry in the Aharonov-Casher effect

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(April, 1999)

We consider the problem of the bound states of a spin 1/2 chargeless particle in a given Aharonov-Casher configuration. To this end we recast the description of the system in a supersymmetric form. Then the basic physical requirements for unbroken supersymmetry are established. We comment on the possibility of neutron confinement in this system.

PACS number(s): 03.65Ge, 03.65.Bz, 12.60.J, 11.30.P.

Aharonov and Casher [1] introduced a ‘dual’ to the well established Aharonov-Bohm effect [2]. The essence of the Aharonov-Bohm effect is the presence of the vector potential in the Lagrangian formulation used in quantum mechanics. A charged particle moving through a region close to a magnetic field experiences no Lorentz force but is modified by a non-zero vector potential in the equation of motion [2–5]. Based on the ‘symmetry’ of Maxwell’s equations Aharonov and Casher considered the interaction between a particle’s magnetic dipole moment and an electric field. A fully relativistic theory of the Aharonov-Casher effect has been given by Hagen [4] and He and McKellar [5] for spin 1/2 particles.

The AC phase shift was observed using neutron interferometry [6–8]. In the same year that the Aharonov-Casher effect was announced, M. V. Berry introduced the concept of the geometric or topological phase in quantum mechanics [9]. In cases where the adiabatic theorem can be invoked, a non-integrable (i.e. non-dynamic) phase is accumulated in the cyclic evolution of a Hamiltonian which is not simply connected. An important example of this geometric phase was the Aharonov-Bohm effect. Although classical examples have been found the topological nature of the AB and AC phase is an important argument for their

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quantum nature.

A significant investigation of the “duality” of AB and AC effects has been carried out to show the equivalence of these effects by transformation of one equation of motion into the other [4,10,11]. Other investigations of the Aharonov-Casher effect have included extensions into massive photon electrodynamics [12], non-locality [13], abstract geometry [14], gravity [15], and AB/AC interference [16,17].

Here we are concerned with another application of the AC effect. It deals with the conditions for finding the bound states of a system with unbroken supersymmetry. To this end we have to assume connectness in the configuration space in order to be able to define a normalizable ground state. The problem turns out to have exact supersymmetry only under the fulfillment of a condition for the magnitude of the charge distribution which generates the electric field. We also discuss the possibility of breaking supersymmetry by examining the requirements for the existence of lower energy bound states.

To start with, let us consider an infinite cylinder with uniform charge per unit volume $\rho$ centered along the $z$ axis, so that there exists an electric field

$$ E_<(r) = \rho r/2, \quad 0 \leq r \leq r_0; \quad E_>(r) = \rho r_0^2 r/2r^2, \quad r_0 \leq r < \infty, \quad (1) $$

where $r_0$ is the radius of the cylinder and for simplicity we have chosen $\hat{r} \cdot \hat{z} = 0$. Here $\hat{r}$ and $\hat{z}$ are unit vectors in the $r$ and $z$ directions respectively. The uncharged particles (v.gr. neutrons) are completely polarized along the positive $z$ direction. They move on a plane in the external field $E$. In this circumstance there is apparently no force on the neutrons but there exists a kind of Aharonov-Bohm effect [1,4,6]. Nevertheless, if the singularity in the $z$ axis is removed, as is implied in (1), the neutrons are allowed to penetrate the charged line. Therefore a new question is to be considered: It regards the problem of the possible bound states of the neutron in this new AC configuration.

To be specific, let us consider a spin 1/2 chargless particle with an anomalous magnetic moment $\kappa_n$. The Dirac equation can be written [1,4] in a covariant form ($\hbar = c = 1$) as

$$ \left( \gamma_\mu D^\mu - \frac{e\kappa_n}{2M_n} F^{\mu\nu} \sigma_{\mu\nu} - M_n \right) \Psi(r,t) = 0, \quad (2) $$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field tensor.
The Aharonov-Casher effective wave equation is obtained by making $A_0 \neq 0$, $B = 0$, with $\nabla \cdot \mathbf{E} = \rho$. For stationary states of energy $E$ we write
\[
\Psi_E(r,t) = \left( \phi(r) \right) e^{-iEt}.
\] (3)

Thus from (2) and (3) we get
\[
\frac{1}{2M_n} \sigma \cdot (\mathbf{p} + i\eta \mathbf{E}(r)) \sigma \cdot (\mathbf{p} - i\eta \mathbf{E}(r)) \phi(r) = \frac{\varepsilon}{2M_n} \phi(r),
\] (4)
\[
\frac{1}{2M_n} \sigma \cdot (\mathbf{p} - i\eta \mathbf{E}(r)) \sigma \cdot (\mathbf{p} + i\eta \mathbf{E}(r)) \chi(r) = \frac{\varepsilon}{2M_n} \chi(r),
\]
where $\sigma = (\sigma_1, \sigma_2)$, $\eta = e\kappa_n / 2M_n$ and $\varepsilon \equiv E^2 - M_n^2$. This set of uncoupled differential equations can be rewritten in the supersymmetric form
\[
H_{SS} = \{Q, Q^\dagger\}, \quad [H_{SS}, Q] = [H_{SS}, Q^\dagger] = 0,
\] (5)

with
\[
H_{SS} \Psi_E(r,t) = \frac{\varepsilon}{2M_n} \Psi_E(r,t).
\] (6)

Here
\[
Q \equiv \frac{1}{\sqrt{2M_n}} \tau^- \otimes \sigma \cdot (\mathbf{p} - i\eta \mathbf{E}(r))
\] (7)
is the supersymmetric charge and $\tau^- = (1/2) (\tau_1 - i\tau_2)$, where the $\tau_1$, $\tau_2$ are Pauli matrices.

Thus $H_{SS}$ is invariant under $N = 1$ supersymmetry. From (4) we find [18] that
\[
\{ \mathbf{p}^2 + \eta \tau_3 \otimes (\nabla \cdot \mathbf{E}(r)) + 2\sigma_3 (\mathbf{E}(r) \times \mathbf{p})_3 + \eta^2 \mathbf{E}^2(r) \} \Psi(r) = \varepsilon \Psi(r).
\] (8)

It is not surprising to find here a supersymmetric system since the Hamiltonian in (5) describes the interaction between a spin 1/2 particle with an electromagnetic field (spin 1). Note that the AC effect has also been discussed in the framework of $N = 2$ nonrelativistic supersymmetry [19].

Supersymmetry is unbroken if
\[
Q \phi^{(0)}(r) = 0, \quad Q^\dagger \phi^{(0)}(r) = 0,
\] (9)
where $\phi^{(0)}$ is the ground state of the system. In other words, the generators of $N = 1$ supersymmetry annihilate the vacuum state in order to have an exact symmetry. Furthermore, in a system with axial symmetry we have also the constraint
(\mathbf{E}(\mathbf{r}) \times \mathbf{p})_3 \phi^{(0)}(\mathbf{r}) = \frac{|\mathbf{E}(\mathbf{r})|}{r} L_3 \phi^{(0)}(\mathbf{r}) = 0 \quad (a \ s\text{-state}), \quad (10)

with \( L_3 = (\mathbf{r} \times \mathbf{p})_3 \) the \( z \) component of the orbital angular momentum operator. Here then we are concerned with states for which \( E^2 = M^2_n \), i.e., \( \varepsilon = 0 \).

The second equation (9) is satisfied identically since in the nonrelativistic limit the lower components \( \Psi_{E=M_n} \) vanish. The first one together with (10) yield

\[ \sigma \cdot (\mathbf{p} - i\eta \mathbf{E}(\mathbf{r})) \phi^{(0)}(r) = 0. \]  

Without lack of generality we can set

\[ \phi^{(0)}(r) \equiv \begin{pmatrix} \phi(r) \\ 0 \end{pmatrix}, \quad \chi^{(0)}(r) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]  

Then from (11) we find the first order differential equations

\[ \left( \frac{d}{dr} - \beta r \right) \phi_<(r) = 0, \quad 0 \leq r \leq r_0; \quad \left( \frac{d}{dr} - \frac{\beta r_0^2}{r} \right) \phi_>(r) = 0, \quad r_0 \leq r < \infty, \]  

where \( \beta \equiv \rho \eta/2 \). Thus

\[ \phi_<(r) = Ae^{-\beta r^2/2}, \quad 0 \leq r \leq r_0; \quad \phi_>(r) = Br^{-\beta r_0^2}, \quad r_0 \leq r < \infty, \]  

with \( A, B \) complex constants.

Next we demand continuity of the wavefunction and its derivative at \( r = r_0 \). Both conditions give the same information:

\[ Ae^{-\beta r_0^2/2} = Br_0^{-\beta r_0^2}. \]  

Furthermore, if \( \Psi_{E=M_n} \) belongs to the Hilbert space, \( \phi \) must be normalizable on the plane \([0, 2\pi] \times [0, \infty]:\)

\[ 2\pi \int_0^\infty |\phi(r)|^2 r dr = 2\pi \left\{ |A|^2 \int_0^{r_0} dr r e^{-\beta r^2} + |B|^2 \int_{r_0}^{\infty} dr r e^{-2\beta r_0^2} \right\} = 1. \]  

By using (15) we get

\[ |A|^2 = \frac{\beta (\beta r_0^2 - 1) e^{\frac{1}{2} \beta r_0^2}}{2\pi \left[ (\beta r_0^2 - 1) \sinh \left( \frac{1}{2} \beta r_0^2 \right) + \frac{1}{2} \beta r_0^2 e^{-\frac{1}{2} \beta r_0^2} \right]} . \]  

Notice that in (16) we must require that
This inequality constitutes a constraint on the possible values of \( \rho \) and \( r_0 \) (or equivalently for \( \lambda \equiv \rho \pi r_0^2 \)) if we want to preserve unbroken supersymmetry. Inserting \( c^2 \) in (18), we can estimate the minimum value of \( \lambda \) to be able to obtain a normalizable ground state:

\[
|\lambda|_{\text{min}} \simeq 4\pi M_n c^2 / |e\kappa_n| \simeq 20.62 \times 10^{-3} \quad [\text{C/cm}].
\]

As \( \lambda \) depends linearly on \( r_0^2 \), we can in principle set up a configuration with the required \( \lambda \).

Figure 1 shows the neutron density of probability \( |\phi|^2 \) as a function of the dimensionless parameter \( r/r_0 \) for different values of \( \beta > 1 \), in natural units. Notice that when \( \beta \) approaches 1, \( |\phi|^2 \) becomes flatter, i.e., there exists a larger probability that the neutron be outside the charged distribution than within it. This is also easily represented by the ratio of probabilities

\[
R_\beta \equiv \frac{W[r_0, \infty]}{W[0, r_0]} = \frac{\beta r_0^2}{2 (\beta r_0^2 - 1) \sinh \left( \frac{1}{2} \beta r_0^2 \right) e^{\frac{1}{2} \beta r_0^2}},
\]

where

\[
W[r_1, r_2] = 2\pi \int_{r_1}^{r_2} |\phi(r)|^2 r dr,
\]

for values of \( \beta > 1/r_0^2 \) (fig. 2).

The general eigenvalue problem for (8) reduces to the system of differential equations

\[
\left( -\nabla^2 - \eta (\nabla \cdot \mathbf{E}(r) \pm 2 (\mathbf{E}(r) \times \mathbf{p})_3) + \eta^2 \mathbf{E}^2(r) \right) \phi^{(1)}(r) = \varepsilon \phi^{(1)}(r),
\]

\[
\left( -\nabla^2 + \eta (\nabla \cdot \mathbf{E}(r) \pm 2 (\mathbf{E}(r) \times \mathbf{p})_3) + \eta^2 \mathbf{E}^2(r) \right) \chi^{(1)}(r) = \varepsilon \chi^{(1)}(r).
\]

We solve (21) by separation of variables:

\[
\phi(r) = \phi(r) \exp(im\varphi), \quad \chi(r) = \chi(r) \exp(im\varphi).
\]

Thus from (21) and (22) we get

\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} - 2\beta (1 \pm m) + \beta^2 r^2 \right) \phi^{(1)}(r) = \varepsilon \phi^{(1)}(r),
\]

\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} + 2\beta (1 \pm m) + \beta^2 r^2 \right) \chi^{(1)}(r) = \varepsilon \chi^{(1)}(r),
\]
for \( r \leq r_0 \), and
\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m (m \mp 2\beta r_0^2)}{r^2} + \left( \frac{\beta r_0^2}{r} \right)^2 \right) \phi^{(1)}_>(r) = \varepsilon \phi^{(1)}_>(r),
\]
\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m (m \pm 2\beta r_0^2)}{r^2} + \left( \frac{\beta r_0^2}{r} \right)^2 \right) \chi^{(1)}_>(r) = \varepsilon \chi^{(1)}_>(r),
\]
for \( r \geq r_0 \).

The radial solutions must be normalizable in the range \( 0 \leq r < \infty \), and we also demand continuity at \( r_0 \) on the corresponding solutions. For the upper component \( \phi^{(1)}_>(r) \) we get
\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{l^2}{r^2} - \frac{\epsilon_m^2}{r^2} + \frac{\beta^2}{2} + \frac{\epsilon_m}{2} \right) \phi^{(1)}_<(r) = 0,
\]
(23)

where \( \epsilon_m \equiv \epsilon + 2\beta (1 + m) \). Thus
\[
\phi^{(1)}_>(r) = C_1 F_1 \left( \frac{m + 1 - \epsilon_m/2\beta}{2}; m + 1; 2\beta r^2 \right) r^m e^{-\beta r^2/2},
\]
(24)

where \( C \) is a complex constant and \( F_1 \) is the confluent hypergeometric function.

For \( r \geq r_0 \) we have
\[
\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{l^2}{r^2} + \frac{\epsilon}{2} \right) \phi^{(1)}_>(r) = 0
\]
(25)

where \( l \equiv m - \beta r_0^2 \). Eq.(25) has two kinds of solutions: (a) non-normalizable scattering-like states for \( \epsilon > 0 \) (\( E^2 > M_n^2 \) :
\[
\phi^{(1)}_>(r) = C_1 J_l \left( \sqrt{-\epsilon} r \right) + C_2 Y_l \left( \sqrt{-\epsilon} r \right), \quad |m| > 1,
\]
(26)

where \( J_l(z) \) and \( Y_l(z) \) are the Bessel functions of first and second kind correspondingly; and (b) bound states for \( \epsilon < 0 \) (\( E^2 < M_n^2 \) :
\[
\phi^{(1)}_>(r) = D K_l \left( \sqrt{-\epsilon} r \right), \quad |m| \geq 0,
\]
(27)

where \( K_l(z) \) is the (normalizable) modified Bessel function of second kind. By matching (24) and (27) at \( r = r_0 \), we find the quantization condition for the remaining bound state energy levels. This problem is now under research. We note that the supersymmetric state (14) is obtained as a limit case (\( \epsilon \rightarrow 0 \) for \( m = 0 \)) of bound state solutions of the form (27). In fact, in this limit \( \phi^{(1)}_>(r) \propto e^{-\beta r^2/2}. \) The existence of further bound states would break exact supersymmetry (stated by (5) and (9)), since \( \varepsilon_{\text{min}} < 0. \)
From the above we can draw at least two main conclusions. First, the magnitude of the electric charge distribution has to be sufficiently large ($\lambda \gtrsim 4\pi M_n c^2 / |\epsilon \kappa_n|$) in order to generate a bound (ground) state. Second, we are not asserting that the neutron directly “feels” a force due to the electric field generated by the charge density. Rather, from the second term of the left hand side of (8), we state that the neutron tends to move toward regions where the gradient of the electric field increases. The third term in the same equation corresponds to the appearance of an induced electric dipole moment on the particle [1].

It is interesting to note that the fulfillment of condition (18) would allow neutron trapping by an electrostatic field as a result of a purely quantum mechanical effect.

Acknowledgments

This work was supported by Dirección de Investigación, Universidad de Concepción, through grants P.I. 96.11.19-1.0 and Fondecyt #1970995.

One of us (SB) is grateful to the School of Physics, University of Melbourne, Australia, for its warm hospitality. We are very thankful to Professors A. G. Klein and G. I. Opat for their valuable criticisms and helpful suggestions on the experimental and theoretical aspects of this paper.
Figure Captions

FIG. 1. The neutron ground state probability density $|\phi(r)|^2$ as a function of the dimensionless parameter $r/r_0$ for different values of $\beta > 1$. The units used are $\hbar = c = 1$.

FIG. 2. The ratio of probabilities $R_\beta \equiv W[r_0, \infty]/W[0, r_0]$ for values of the parameter $\beta > 1$ with $r_0 = 1$. Notice that the integration area under each of the three curves is different. However, $W[0, \infty] = 1$ for each one of them, since the integration is performed on the plane $[0, 2\pi] \times [0, \infty]$ where the measure is $2\pi r dr$. 


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