Abstract

Dynamical systems theory is especially well-suited for determining the possible asymptotic states (at both early and late times) of cosmological models, particularly when the governing equations are a finite system of autonomous ordinary differential equations. We begin with a brief review of dynamical systems theory. We then discuss cosmological models as dynamical systems and point out the important role of self-similar models. We review the asymptotic properties of spatially homogeneous perfect fluid models in general relativity. We then discuss some results concerning scalar field models with an exponential potential (both with and without barotropic matter). Finally, we discuss some isotropic cosmological models derived from the string effective action.
1 Introduction

The governing equations of the most commonly studied cosmological models are a system of autonomous ordinary differential equations (ODEs). Since our main goal is to give a qualitative description of these models, a dynamical systems approach is undertaken. Usually, a dimensionless (logarithmic) time variable, $\tau$, is introduced so that the models are valid for all times (i.e., $\tau$ assumes all real values). A normalised set of variables are then chosen for a number of reasons. First, this normally leads to a compact dynamical system. Second, these variables are well-behaved and often have a direct physical interpretation. Third, due to a symmetry in the equations, one of the equations decouples (in general relativity the expansion is used to normalize the variables in ever expanding models whence the Raychaudhuri equation decouples) and the resulting simplified reduced system is then studied. The singular points of the reduced system then correspond to dynamically evolving self-similar cosmological models. More precisely, using the dimensionless time variable and a normalised set of variables, the governing ODEs define a flow and the evolution of the cosmological models can then be analysed by studying the orbits of this flow in the physical state space, which is a subset of Euclidean space. When the state space is compact, each orbit will have a non-empty $\alpha$-limit set and $\omega$-limit set, and hence there will be a both a past attractor and a future attractor in the state space.

1.1 Self-similarity

Self-similar solutions of the Einstein field equations (EFE) play an important role in describing the asymptotic properties of more general models. The energy-momentum tensor of a perfect fluid given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab},$$

(1.1)

where $u^a$ is the normalized fluid 4-velocity, $\mu$ is the density and $p$ is the pressure satisfying a linear barotropic equation of state of the form

$$p = (\gamma - 1)\mu,$$

(1.2)

where $\gamma$ is a constant. The existence of a self-similarity of the first kind can be invariantly formulated in terms of the existence of a homothetic vector [1]. For a general spacetime a proper homothetic vector (HV) is a vector field $\xi$ which satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = 2g_{\mu\nu},$$

(1.3)

where $g_{\mu\nu}$ is the metric and $\mathcal{L}$ denotes Lie differentiation along $\xi$. An arbitrary constant on the right-hand-side of (1.3) has been rescaled to unity. If this constant is
zero, i.e., $\mathcal{L}_\xi g_{uv} = 0$, then $\xi$ is a Killing vector. A homothetic motion or homothety captures the geometric notion of “invariance under scale transformations”.

Self-similar models are often related to the asymptotic states of more general models [2]. In particular, self-similar models play an important role in the asymptotic properties of spatially homogeneous models, spherically symmetric models, $G_2$ models and silent universe models [3]. We will focus on spatially homogeneous models here [4, 5]. We note that the self-similar Bianchi models of relevance below are transitively self-similar (in the sense that the orbits of the $H_4$ are the whole spacetime). Self-similar spherically symmetric models have been studied by many authors and have been recently reviewed in Carr and Coley [6]. $G_2$ models, which contain two commuting spacelike KV acting orthogonally transitively, have been discussed in Wainwright and Ellis [7] (hereafter denoted WE). Other exact homothetic models, including for example, plane-symmetric models have been discussed in Kramer et al. [5] and Carr and Coley [6].

**Exact solutions**

Let us review some exact self-similar solutions that are of particular importance.

- **Minkowski space ($M$):**
  \[
  ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \\
  \xi = t\partial_t + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.
  \]

  In addition to flat Minkowski spacetime, all ($k = 0, \pm 1$) Friedmann–Robertson–Walker (FRW) models admit a timelike HV in the special case of stiff matter ($\gamma = 2$). Otherwise, only the $k = 0$ models admit a HV, and this occurs for all such models in which $p = (\gamma - 1)\mu$ and hence the scale function has power-law depend on time.

- **$k = 0$, FRW ($F_0$):**
  \[
  ds^2 = -dt^2 + t^{4/\gamma} (dx^2 + dy^2 + dz^2) \\
  \xi = t\partial_t + \left(1 - \frac{2}{3\gamma}\right) \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)
  \]
  where
  \[
  p = (\gamma - 1)\mu; \quad \mu = \frac{4}{3\gamma^2} t^{2/3} \left( u = \frac{\partial}{\partial t} \right).
  \]

  All transitively self-similar orthogonal spatially homogeneous perfect fluid solutions (with $\frac{2}{3} < \gamma \leq 2$) and spatially homogeneous vacuum solutions are summarized in Hsu and Wainwright [2]. In particular, the Kasner vacuum solution is self-similar.

- **Kasner (vacuum) ($K$):**
  \[
  ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2
  \]
with
\[ \sum p_i = \sum p_i^2 = 1 \]  
(1.10)
\[ \xi = t \partial_t + (1 - p_1)x \partial_x + (1 - p_2)y \partial_y + (1 - p_3)z \partial_z. \]  
(1.11)

1.2 Brief Survey of Techniques in Dynamical Systems

This section will briefly review some of the results of dynamical systems theory.

**Definition 1** A *singular point* of a system of autonomous ODEs
\[ \dot{x} = f(x) \]  
(1.12)
is a point \( \bar{x} \in \mathbb{R}^n \) such that \( f(\bar{x}) = 0 \).

**Definition 2** Let \( \bar{x} \) be a singular point of the DE (1.12). The point \( \bar{x} \) is called a *hyperbolic* singular point if \( \text{Re}(\lambda_i) \neq 0 \) for all eigenvalues, \( \lambda_i \), of the Jacobian of the vector field \( f(x) \) evaluated at \( \bar{x} \). Otherwise the point is called *non-hyperbolic*.

**Definition 3** Let \( x(t) = \phi_a(t) \) be a solution of the DE (1.12) with initial condition \( x(0) = a \). The *flow* \( \{g^t\} \) is defined in terms of the solution function \( \phi_a(t) \) of the DE by
\[ g^t a = \phi_a(t). \]

**Definition 4** The orbit through \( a \), denoted by \( \gamma(a) \) is defined by
\[ \gamma(a) = \{x \in \mathbb{R}^n | x = g^t a, \text{ for all } t \in \mathbb{R} \}. \]

**Definition 5** Given a DE (1.12) in \( \mathbb{R}^n \), a set \( S \subseteq \mathbb{R}^n \) is called an invariant set for the DE if for any point \( a \in S \) the orbit through \( a \) lies entirely in \( S \), that is \( \gamma(a) \subseteq S \).

**Definition 6** Given a DE (1.12) in \( \mathbb{R}^n \), with flow \( \{g^t\} \), a subset \( S \subseteq \mathbb{R}^n \) is said to be a trapping set of the DE if it satisfies: 1. \( S \) is a closed and bounded set, 2. \( a \in S \) implies that \( g^t a \in S \) for all \( t \geq 0 \).

Qualitative analysis of a system begins with the location of singular points. Once the singular points of a system of ODEs are obtained, it is of interest to consider the dynamics in a local neighbourhood of each of the points. Assuming that the vector field \( f(x) \) is of class \( C^1 \) the process of determining the local behaviour is based on the linear approximation of the vector field in the local neighbourhood of the singular point \( \bar{x} \). In this neighbourhood
\[ f(x) \approx Df(\bar{x})(x - \bar{x}) \]  
(1.13)
where $Df(\bar{x})$ is the Jacobian of the vector field at the singular point $\bar{x}$. The system (1.13) is referred to as the *linearization of the DE at the singular point*. Each of the singular points can then be classified according to the eigenvalues of the Jacobian of the linearized vector field at the point.

The classification then follows from the fact that if the singular point is hyperbolic in nature the flows of the non-linear system and it’s linear approximation are *topologically equivalent* in a neighbourhood of the singular point. This result is given in the form of the following theorem:

**Theorem 1: Hartman-Grobman Theorem** Consider a DE: $\dot{x} = f(x)$, where the vector field $f$ is of class $C^1$. If $\bar{x}$ is a hyperbolic singular point of the DE then there exists a neighbourhood of $\bar{x}$ on which the flow is topologically equivalent to the flow of the linearization of the DE at $\bar{x}$.

Given a linear system of ODEs:

$$\dot{x} = Ax,$$

(1.14)

where $A$ is a matrix with constant coefficients, it is a straightforward matter to show that if the eigenvalues of the matrix $A$ are all positive the solutions in the neighbourhood of $\bar{x} = 0$ all diverge from that point. This point is then referred to as a source. Similarly, if the eigenvalues all have negative real parts all solutions converge to the singular point $\bar{x} = 0$, and the point is referred to as a sink. Therefore, it follows from topological equivalence that if all eigenvalues of the Jacobian of the vector field for a non-linear system of ODEs have positive real parts the point is classified as a source (and all orbits diverge from the singular point), and if the eigenvalues all have negative real parts the point is classified as a sink.

In most cases the eigenvalues of the linearized system (1.13) will have eigenvalues with both positive, negative and/or zero real parts. In these cases it is important to identify which orbits are attracted to the singular point, and which are repelled away as the independent variable (usually $t$) tends to infinity.

For a linear system of ODEs, (1.14), the phase space $\mathbb{R}^n$ is spanned by the eigenvectors of $A$. These eigenvectors divide the phase space into three distinct subspaces; namely:

- The **stable subspace** $E^s = \text{span}(s_1, s_2, \ldots s_{ns})$
- The **unstable subspace** $E^u = \text{span}(u_1, u_2, \ldots u_{nu})$
- The **centre subspace** $E^c = \text{span}(c_1, c_2, \ldots c_{nc})$

where $s_i$ are the eigenvectors who’s associated eigenvalues have negative real part, $u_i$ those who’s eigenvalues have positive real part, and $c_i$ those who’s eigenvalues have zero eigenvalues. Flows (or orbits) in the stable subspace asymptote in the future to
the singular point, and those in the unstable subspace asymptote in the past to the singular point.

In the non-linear case, the topological equivalence of flows allows for a similar classification of the singular points. The equivalence only applies in directions where the eigenvalue has non-zero real parts. In these directions, since the flows are topologically equivalent, there is a flow tangent to the eigenvectors. The phase space is again divided into stable and unstable subspaces (as well as centre subspaces). The stable manifold $W^s$ of a singular point is a differential manifold which is tangent to the stable subspace of the linearized system ($E^s$). Similarly, the unstable manifold is a differential manifold which is tangent to the unstable subspace ($E^u$) at the singular point. The centre manifold, $W^c$, is a differential manifold which is tangent to the centre subspace $E^c$. It is important to note, however, that unlike the case of a linear system, this centre manifold will contain all those dynamics not classified by linearization (i.e., the non-hyperbolic directions). In particular, this manifold may contain regions which are stable, unstable or neutral. The classification of the dynamics in this manifold can only be determined by utilizing more sophisticated methods, such as centre manifold theorems or the theory of normal forms (see [8]).

Unlike a linear system of ODEs, a non-linear system allows for singular structures which are more complicated than that of the singular points, fixed lines or periodic orbits. These structures include, though are not limited to, such things as heteroclinic and/or homoclinic orbits and non-linear invariant sub-manifolds (for definitions see [8]). Sets of non-isolated singular points often occur in cosmology and therefore their stability will be examined more rigorously.

**Definition 7:** A set of non-isolated singular points is said to be normally hyperbolic if the only eigenvalues with zero real parts are those whose corresponding eigenvectors are tangent to the set.

Since by definition any point on a set of non-isolated singular points will have at least one eigenvalue which is zero, all points in the set are non-hyperbolic. A set which is normally hyperbolic can, however, be completely classified as per its stability by considering the signs of the eigenvalues in the remaining directions (i.e., for a curve, in the remaining $n - 1$ directions) [9].

The local dynamics of a singular point may depend on one or more arbitrary parameters. When small continuous changes in the parameter result in dramatic changes in the dynamics, the singular point is said to undergo a bifurcation. The values of the parameter(s) which result in a bifurcation at the singular point can often be located by examining the linearized system. Singular point bifurcations will only occur if one (or more) of the eigenvalues of the linearized systems are a function of the parameter. The bifurcations are located at the parameter values for which the real part of an eigenvalue is zero.
The future and past asymptotic states of a non-linear system may be represented by any singular or periodic structure. In the case of a plane system (i.e., in two-dimensional phase space), the possible asymptotic states can be given explicitly. This result is due to the limited degrees of freedom in the space, and the fact that the flows (or orbits) in any dimensional space cannot cross. The result is given in the form of the following theorem:

**Theorem 2: Poincare-Bendixon Theorem:** Consider the system of ODEs \( \dot{x} = f(x) \) on \( \mathbb{R}^2 \), with \( f \in C^2 \), and suppose that there are at most a finite number of singular points (i.e., no non-isolated singular points). Then any compact asymptotic set is one of the following: 1. a singular point, 2. a periodic orbit, 3. the union of singular points and heteroclinic or homoclinic orbits.

This theorem has a very important consequence in that if the existence of a closed (i.e., periodic, heteroclinic or homo-clinic) orbit can be ruled out it follows that all asymptotic behaviour is located at a singular point.

The existence of a closed orbit can be ruled out by many methods, the most common is to use a consequence of Green’s Theorem, as follows:

**Theorem 3: Dulac’s Criterion:** If \( D \subseteq \mathbb{R}^2 \) is a simply connected open set and \( \nabla(Bf) = \frac{\partial}{\partial x_1}(Bf_1) + \frac{\partial}{\partial x_2}(Bf_2) > 0 \), or \(< 0\) for all \( x \in D \) where \( B \) is a \( C^1 \) function, then the DE \( \dot{x} = f(x) \) where \( f \in C^1 \) has no periodic (or closed) orbit which is contained in \( D \).

A fundamental criteria of the Poincare-Bendixon theorem is that the phase space is two-dimensional. When the phase space is of a higher dimension the requirement that orbits cannot cross does not result in such a decisive conclusion. The behaviour in such higher-dimensional spaces is known to be highly complicated, with the possibility of including such phenomena as recurrence and strange attractors [10]. For that reason, the analysis of non-linear systems in spaces of three or more dimensions cannot in general progress much further than the local analysis of the singular points (or non-isolated singular sets). The one tool which does allow for some progress in the analysis of higher dimensional systems is the possible existence of monotonic functions.

**Theorem 4: LaSalle Invariance Principle:** Consider a DE \( \dot{x} = f(x) \) on \( \mathbb{R}^n \). Let \( S \) be a closed, bounded and positively invariant set of the flow, and let \( Z \) be a \( C^1 \) monotonic function. Then for all \( x_0 \in S \),

\[
W(x_0) \subset \{ x \in S | \dot{Z} = 0 \}
\]

where \( W(x_0) \) is the forward asymptotic states for the orbit with initial value \( x_0 \); i.e., a \( W \)-limit set [11].

This principle has been generalized to the following result:
Theorem 5: Monotonicity Principle (see [12]). Let \( \phi_t \) be a flow on \( \mathbb{R}^n \) with \( S \) an invariant set. Let \( Z : S \to \mathbb{R} \) be a \( C^1 \) function whose range is the interval \((a, b)\), where \( a \in \mathbb{R} \cup \{-\infty\}, \ b \in \mathbb{R} \cup \{\infty\} \) and \( a < b \). If \( Z \) is decreasing on orbits in \( S \), then for all \( X \in S \),

\[
\omega(x) \subseteq \{ s \in \bar{S} \setminus S | \lim_{y \to s} Z(y) \neq b \},
\]

\[
\alpha(x) \subseteq \{ s \in \bar{S} \setminus S | \lim_{y \to s} Z(y) \neq a \},
\]

where \( \omega(x) \) and \( \alpha(x) \) are the forward and backward limit set of \( x \), respectively (i.e., the \( w \) and \( \alpha \) limit sets.)

2 Spatially Homogeneous Perfect Fluid Models

Many people have studied self-similar spatially homogeneous models, both as exact solutions and in the context of qualitative analyses (see WE and Coley [13] and references therein). Exact spatially homogeneous solutions were first displayed in early papers [14]; however, it was not until after 1985 that many of them were recognized by Wainwright [15] and Rosquist and Jantzen [16, 17] as being self-similar. The complete set of self-similar orthogonal spatially homogeneous perfect fluid and vacuum solutions were given by Hsu and Wainwright [2] and they have also been reviewed in WE. Kantowski-Sachs models were studied by Collins [18].

Spatially homogeneous models have attracted considerable attention since the governing equations reduce to a relatively simple finite-dimensional dynamical system, thereby enabling the models to be studied by standard qualitative techniques. Planar systems were initially analyzed by Collins [19, 20] and a comprehensive study of general Bianchi models was made by Bogoyavlenski and Novikov [21] and Bogoyavlenski [22] and more recently (using automorphism variables and Hamiltonian techniques) by Jantzen and Rosquist [16, 17, 23, 24]. Perhaps the most illuminating approach has been that of Wainwright and collaborators [2, 25, 26], in which the more physically or geometrically natural expansion-normalized (dimensionless) configuration variables are used. In this case, the physically admissible states typically lie within a bounded region, the dynamical system remains analytic both in the physical region and its boundaries, and the asymptotic states typically lie on the boundary represented by exact physical solutions rather than having singular behaviour. We note that the physically admissible states do not lie in a bounded region for Bianchi models of types VII\(_0\), VIII and IX; see WE for details.

Wainwright utilizes the orthonormal frame method [4] and introduces expansion-normalized (commutation function) variables and a new “dimensionless” time variable to study spatially homogeneous perfect fluid models satisfying \( p = (\gamma - 1)\mu \).

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The equations governing the models form an $N$-dimensional system of coupled autonomous ODEs. When the ODEs are written in expansion-normalized variables, they admit a symmetry which allows the equation for the time evolution of the expansion $\theta$ (the Raychaudhuri equation) to decouple. The reduced $N - 1$-dimensional dynamical system is then studied. At all of the singular points of the reduced system, $\dot{\theta}$ is proportional to $\theta^2$ and hence all such points correspond to transitively self-similar cosmological models [2]. This is why the self-similar models play an important role in describing the asymptotic dynamics of the Bianchi models. For orthogonal Bianchi models of class A, the resulting reduced state space is five-dimensional [25]. Orthogonal Bianchi cosmologies of class B were studied by Hewitt and Wainwright [26] and are governed by a five-dimensional system of analytic ODEs with constraints.

Two perfect-fluid models were studied by Coley and Wainwright [27]. In further work, imperfect fluid Bianchi models were studied under the assumption that all physical quantities satisfy “dimensionless equations of state”, thereby ensuring that the singular points of the resulting reduced dynamical system are represented by exact self-similar solutions. Models satisfying the linear Eckart theory of irreversible thermodynamics were studied by Burd and Coley [28] and Coley and van den Hoogen [29], those satisfying the truncated causal theory of Israel-Stewart by Coley and van den Hoogen [30], and those satisfying the full (i.e., non-truncated) relativistic Israel-Stewart theory by Coley et al. [31]. Self-similar solutions also play an important role in describing the dynamical behaviour of cosmological models close to the Planck time in general relativity with scalar fields [33, 34], in scalar-tensor theories of gravity [34], and particularly in the low-energy limit in supergravity theories from string theory and other higher-dimensional gravity theories.

2.1 Some Simple Examples

The expansion scalar $\theta$, shear scalar $\sigma \ (\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab})$, and Ricci 3-curvature (orthogonal to $u^a$) $^3R$, are defined in Ellis [35].

Basic equations:

Raychaudhuri eqn. : $\dot{\theta} = -\frac{1}{3}\theta^2 - 2\sigma^2 - \frac{1}{3}(\mu + 3p)$. \hspace{1cm} (2.1)

Conservation eqn. : $\dot{\mu} = -(\mu + p)\theta$. \hspace{1cm} (2.2)

Generalized Friedmann eqn. : $\theta^2 = -\frac{1}{3}^3R + 3\sigma^2 + 3 \mu \theta$. \hspace{1cm} (2.3)

Ricci identity: for $\dot{\sigma}$ \hspace{1cm} [35]. \hspace{1cm} (2.4)

We define the expansion-normalized variables ($\theta > 0$):

$\Omega = \frac{3\mu}{\theta^2}$, \hspace{1cm} density parameter \hspace{1cm} (2.5)
\[ \beta = 2\sqrt{3\frac{\sigma}{\theta}}, \text{ normalized rate of shear} \quad (2.6) \]

\[ \frac{d\tau}{dt} = \frac{1}{3} \theta, \quad [\tau \to -\infty \text{ as } t \to 0^+] \quad (2.7) \]

A dash (dot) denotes differentiation with respect to \( \tau(t) \). The equation of state is given by Eq. (1.2), where \( 1 \leq \gamma \leq 2 \) for normal matter and models with \( 0 \leq \gamma < \frac{2}{3} \) are of interest in connection with inflationary models of the universe (see, for example, Wald [36]). The weak energy condition implies that \( \mu \geq 0 \).

**A. FRW**: The metric is given by

\[ ds^2 = -dt^2 + R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \quad (2.8) \]

where \( k \) is the curvature constant. Here \( \sigma = 0 \) (\( \theta = 3\dot{R}/R \)), and the equations reduce to the single ODE:

\[ \Omega' = (2 - \gamma)\Omega(1 - \Omega), \quad (2.9) \]

where \( \Omega \geq 0 \) and \( 1 - \Omega = \frac{3}{2} 3R\theta^{-2} \), so that \( \Omega < 1, = 1, > 0 \) according to whether \( k < 0, = 0, > 0 \) (models open, flat or closed), respectively.

- Singular points (\( \gamma \neq 2 \)):
  - \( \Omega = 1 \) (\( F_0 \)); past attractor (flat FRW model describes the dynamics near the big bang).
  - \( \Omega = 0 \) (\( M \)); future attractor (open models approach Milne form of Minkowski space at later times).

[For closed models \( \theta = 0 \) at the point of maximum expansion, so the models are only valid up to this point. The models recollapse and asymptotically approach the flat FRW model \( F_0 \) to the future].

**B. Bianchi V**: The metric is given by

\[ ds^2 = -dt^2 + A^2(t)dx^2 + e^{2x}(B^2(t)dy^2 + A^4(t)B^{-2}(t)dz^2), \quad (2.10) \]

where \( \theta = 3\dot{A}A \), \( \sigma^2 = \left( \dot{A} \right) - \left( \frac{\dot{B}}{B} \right) \), and \( ^3R = -\frac{6}{A^2} \). The Ricci identity leads to

\[ \dot{\sigma} = -\theta \sigma. \quad (2.11) \]

The EFE then reduce to the plane autonomous system:

\[ \Omega' = (2 - 3\gamma)\Omega(1 - \Omega) + \beta^2\Omega \]

\[ \beta' = \frac{\beta}{2}[(3\gamma - 2)\Omega + \beta^2 - 4], \quad (2.12) \]
where
\[ \beta^2 + 4\Omega - 4 = -\frac{1}{A^2\theta^2} \leq 0 \]
(from the generalized Friedmann eqn.) and
\[ \Omega \geq 0, \]
so that phase space is compact.

- invariant sets: \( \beta = 0 \) (FRW, dealt with above).
  \[ \beta^2 + 4\Omega - 4 = 0 \] (Bianchi I, boundary)

In the Bianchi I (boundary) case we have a single ODE and the models evolve from \( K \) to \( F_0 \). In addition, the equations above are symmetric about the \( \Omega \)-axis.

- singular points \((\Omega, \beta)\) \[ \gamma \neq 2 \]:
  - \((0, 0)\) (M): future attractor
  - \((1, 0)\) (\(F_0\)): saddle (past attractor for open FRW models, intermediate asymptote otherwise).
  - \((0, 2)\) (\(K_+\)): past attractor (initial Kasner cigar singularity).
  - \((0, -2)\) (\(K_-\)): past attractor (initial Kasner cigar singularity).

In the special case \( \gamma = 2 \), the boundary \( \beta^2 + 4\Omega - 4 = 0 \) becomes a line of repelling singular points. The corresponding Bianchi I stiff matter models evolve from the general self-similar (stiff matter) Bianchi I solution of Jacobs [38] to the Milne (flat) universe [19].

### 2.2 Asymptotic States of Bianchi Models

We now discuss the asymptotic states of Bianchi models, again assuming the linear equation of state (1.2). We will summarize the work of Wainwright and Hsu [25] and Hewitt and Wainwright [26], who studied the asymptotic states of orthogonal spatially homogeneous models in terms of attractors of the associated dynamical system for class A and class B models, respectively. Due to the existence of monotone functions, it is known that there are no periodic or recurrent orbits in class A models. Although “typical” results can be proved in a number of Bianchi type B cases, these are not “generic” due to the lack of knowledge of appropriate monotone functions. In particular, there are no sources or sinks in the Bianchi invariant sets \( B^\pm_\alpha \) (VIII) or \( B^\pm(IX) \).

The key results are as follows:
A large class of orthogonal spatially homogeneous models (including all class B models) are asymptotically self-similar at the initial singularity and are approximated by exact perfect fluid or vacuum self-similar power-law models. Examples include self-similar Kasner vacuum models or self-similar locally rotationally symmetric (class III) Bianchi type II perfect fluid models [19, 37, 39]. However, this behaviour is not generic; general orthogonal models of Bianchi types IX and VIII have an oscillatory behaviour with chaotic-like characteristics, with the matter density becoming dynamically negligible as one follows the evolution into the past towards the initial singularity. Ma and Wainwright [40] show that the orbits of the associated cosmological dynamical system are negatively asymptotic to a lower two-dimensional attractor. This is the union of three ellipsoids in $\mathbb{R}^5$ consisting of the Kasner ring joined by Taub separatrices; the orbits spend most of the time near the Kasner vacuum singular points. Clearly the self-similar Kasner models play a primary role in the asymptotic behaviour of these models.

Exact self-similar power-law models can also approximate general Bianchi models at intermediate stages of their evolution (e.g., radiation Bianchi VII$_h$ models [39]). Of special interest are those models which can be approximated by an isotropic solution at an intermediate stage of their evolution (e.g., those models whose orbits spend a period of time near to a flat Friedmann singular point). This last point is of particular importance in relating Bianchi models to the real Universe, and is discussed further in general terms in WE (see, especially, Chapter 15) and specifically for Bianchi VII$_h$ models in Wainwright et al. [41]. In particular, the flat Friedmann singular point is universal in that it is contained in the state space of each Bianchi type. Isotropic intermediate behaviour has also been found in tilted Bianchi V models [42], and it appears that many tilted models have isotropic intermediate behaviour (see WE).

Self-similar solutions can describe the behaviour of Bianchi models at late times (i.e., as $t \to \infty$). Examples include self-similar flat space and self-similar homogeneous vacuum plane waves [19, 15]. All models expand indefinitely except for the Bianchi type IX models. The question of which Bianchi models can isotropize was addressed in the famous paper by Collins and Hawking [43], in which it was shown that, for physically reasonable matter, the set of homogeneous initial data that give rise to models that isotropize asymptotically to the future is of zero measure in the space of all homogeneous initial data (see also WE).
All vacuum models of Bianchi (B) types IV, V, VI$_h$ and VII$_h$ are asymptotic to plane wave states to the future. Type V models tend to the Milne form of flat spacetime [26]. Typically, and perhaps generically [26], non-vacuum models are asymptotic in the future to either plane-wave vacuum solutions [39] or non-vacuum Collins type VI$_h$ solutions [19].

Bianchi (A) models of types VII$_o$ (non-vacuum) and VIII expand indefinitely but are found to have oscillatory (though non-chaotic) behaviour in the Weyl curvature (see, for example, [44]). Bianchi type IX models obey the “closed universe recollapse” conjecture [45]. All orbits in the Bianchi invariant sets $B^+_0(VII_0)$ ($\Omega > 0$), $B^+_0(VIII)$ and $B^\pm(IX)$ are positively departing; in order to analyse the future asymptotic states of such models it is necessary to compactify phase-space. The description of these models in terms of conventional expansion-normalized variables is only valid up to the point of maximum expansion (where $\theta = 0$).

3 Scalar Field Models

3.1 Background

A variety of theories of fundamental physics predict the existence of scalar fields [46, 47, 49], motivating the study of the dynamical properties of scalar fields in cosmology. Indeed, scalar field cosmological models are of great importance in the study of the early universe, particularly in the investigation of inflation [47, 50, 48]. Recently there has also been great interest in the late-time evolution of scalar field models. ‘Quintessential’ scalar field models (or slowly decaying cosmological constant models) [51, 52] give rise to a residual scalar field which contributes to the present energy-density of the universe that may alleviate the dark matter problem and can predict an effective cosmological constant which is consistent with observations of the present accelerated cosmic expansion [53, 54].

Models with a self-interaction potential with an exponential dependence on the scalar field, $\phi$, of the form

$$V = \Lambda e^{k\phi}, \quad (3.1)$$

where $\Lambda$ and $k$ are positive constants, have been the subject of much interest and arise naturally from theories of gravity such as scalar-tensor theories or string theories [49]. Recently, it has been argued that a scalar field with an exponential potential is a strong candidate for dark matter in spiral galaxies [55] and is consistent with observations of current accelerated expansion of the universe [56].

A number of authors have studied scalar field cosmological models with an exponential potential within general relativity. Homogeneous and isotropic FRW models were studied by Halliwell [57] using phase-plane methods. Homogeneous but
anisotropic models of Bianchi types I and III (and Kantowski-Sachs models) were studied by Burd and Barrow [58], Bianchi type I models were studied by Lidsey [59] and Aguirregabiria et al. [60], and Bianchi models of types III and VI were studied by Feinstein and Ibáñez [61]. A qualitative analysis of Bianchi models with $k^2 < 2$ (including standard matter satisfying standard energy conditions) was completed by Kitada and Maeda [62]. The governing differential equations in spatially homogeneous Bianchi cosmologies containing a scalar field with an exponential potential reduce to a dynamical system when appropriate expansion-normalized variables are defined. This dynamical system was studied in detail in [63] (where matter terms were not considered).

One particular solution that is of great interest is the flat, isotropic power-law inflationary solution which occurs for $k^2 < 2$. This power-law inflationary solution is known to be an attractor for all initially expanding Bianchi models (except a subclass of the Bianchi type IX models which will recollapse) [62, 63]. Therefore, all of these models inflate forever; there is no exit from inflation and no conditions for conventional reheating.

Recently cosmological models which contain both a scalar field with an exponential potential and a barotropic perfect fluid with a linear equation of state given by (1.2), where $\gamma$ is in the physically relevant range $2/3 < \gamma \leq 2$, have come under heavy analysis. One class of exact solutions found for these models has the property that the energy density due to the scalar field is proportional to the energy density of the perfect fluid, and hence these models have been labelled matter scaling cosmologies [64, 65, 66]. These matter scaling solutions are spatially flat isotropic models and are known to be late-time attractors (i.e., stable) in the subclass of flat isotropic models [65] and are clearly of physical interest. In addition to the matter scaling solutions, curvature scaling solutions [67] and anisotropic scaling solutions [68] are also possible. A comprehensive analysis of spatially homogeneous models with a perfect fluid and a scalar field with an exponential potential has recently been undertaken [32].

Although the exponential potential models are interesting models for a variety of reasons, they have some shortcomings as inflationary models [47, 48]. While Bianchi models generically asymptote towards the power-law inflationary model in which the matter terms are driven to zero for $k^2 < 2$, there is no graceful exit from this inflationary phase. Furthermore, the scalar field cannot oscillate and so reheating cannot occur by the conventional scenario. In recent work [69] interaction terms were included, through which the energy of the scalar field is transferred to the matter fields. These terms were found to affect the qualitative behaviour of these models and, in particular, lead to interesting inflationary behaviour.


3.2 Isotropisation

In the famous paper by Collins and Hawking [43] it was proven that within the set of spatially homogeneous cosmological models (which satisfy reasonable energy conditions) those which approach isotropy at infinite times is of measure zero; that is, in general anisotropic models do not isotropize as they evolve to the future. Since we presently observe the universe to be highly isotropic, we therefore need an explanation of why our universe has evolved the way it has. This problem, known as the isotropy problem, can be easily solved with an idea popularized by Guth [50]. If the early universe experiences a period of inflation, then all anisotropies are essentially pushed out of our present observable light-cone and are therefore not presently observed. The Cosmic No-Hair Conjecture asserts that under appropriate conditions, any universe model will undergo a period of inflation and will consequently isotropize.

A significant amount of work on the Cosmic No-hair Conjecture has already been done for spatially homogeneous (Bianchi) cosmologies [36, 70, 71, 72, 73]. For instance, Wald [36] has proven a version of the Cosmic No-Hair Conjecture for spatially homogeneous spacetimes with a positive cosmological constant; namely, he has shown that all initially expanding Bianchi models asymptotically approach a spatially homogeneous and isotropic model, except the subclass of Bianchi type IX models which recollapse.

Anisotropic models with scalar fields and with particular forms for the scalar field potential have also been investigated. Heusler [71] has analyzed the case in which the potential function passes through the origin and is concave up and, like Collins and Hawking [43] has found that the only models that can possibly isotropize to the future are those of Bianchi types I, V, VII and IX.

As noted above, Kitada and Maeda [72, 62] have proven that if \( k < \sqrt{2} \), then all initially expanding Bianchi models except possibly those of type IX must isotropize. Let us consider what happens in the case \( k > \sqrt{2} \). In Ibáñez et al. [73] it was proven, using results from Heusler’s paper [71], that the only models that can possibly isotropize when \( k > \sqrt{2} \) are those of Bianchi types I, V, VII, or IX. Since the Bianchi I, V and VII\(_h\) models are restricted classes of models, the only general spatially homogeneous models that can possibly isotropize are consequently of types VII\(_h\) or IX. Here we shall study the possible isotropization of the Bianchi type VII\(_h\) models when \( k > \sqrt{2} \).

3.2.1 The Bianchi VII\(_h\) Equations

The Bianchi type VII\(_h\) models belong to the Bianchi type B models as classified by Ellis and MacCallum [4]. Hewitt and Wainwright [26] have derived the equations describing the evolution of the general Bianchi type B models. We shall utilize these
equations, adjusted so that they describe a model with a minimally coupled scalar field $\phi$ with an exponential potential $V(\phi) = \Lambda e^{k\phi}$. The energy-momentum tensor describing a minimally coupled scalar field is given by

$$T_{ab} = \phi_a \phi_b - g_{ab} \left( \frac{1}{2} \phi_c \phi^c + V(\phi) \right),$$

where, for a homogeneous scalar field, $\phi = \phi(t)$. In this case we can formally treat the energy-momentum tensor as a perfect fluid with velocity vector $u^a = \phi^a / \sqrt{-\phi_c \phi^c}$, where the energy density, $\mu_\phi$, and the pressure, $p_\phi$, are given by

$$\mu_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (3.2)$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi). \quad (3.3)$$

Our variables are the same as those used by Hewitt and Wainwright [26], with the addition of

$$x \equiv \sqrt{\frac{3}{2}} \frac{\dot{\phi}}{\theta}, \quad y \equiv \frac{\sqrt{3}V}{\theta} \quad (3.4)$$

to describe the scalar field. We note that $\Omega_\phi \equiv 3\mu_\phi / \theta^2 = x^2 + y^2$.

The dimensionless evolution equations are then [26]:

$$\Sigma_+ = (q - 2) \Sigma_+ - 2 \tilde{N}, \quad (3.5)$$

$$\tilde{\Sigma} = 2(q - 2) \tilde{\Sigma} - 4 \Delta N_+ - 4 \Sigma_+ \tilde{A}, \quad (3.6)$$

$$\Delta = 2(q + \Sigma_+ - 1) \Delta + 2(\tilde{\Sigma} - \tilde{N}) N_+, \quad (3.7)$$

$$\tilde{A} = 2(q + 2 \Sigma_+) \tilde{A}, \quad (3.8)$$

$$N_+ = (q + 2 \Sigma_+) N_+ + 6 \Delta, \quad (3.9)$$

$$x' = (q - 2) x - \sqrt{\frac{3}{2}} ky^2, \quad (3.10)$$

where the prime denotes differentiation with respect to the new dimensionless time $\tau$, where $\frac{d}{d\tau} = \frac{2}{\theta}$, and

$$\tilde{N} = \frac{1}{3} (N_+^2 - \ell \tilde{A}), \quad (3.11)$$

$$q = 2 \Sigma_+^2 + 2 \tilde{\Sigma} + 2 x^2 - y^2. \quad (3.12)$$

There also exists the constraint,

$$\tilde{\Sigma} \tilde{N} - \Delta^2 - \tilde{A} \Sigma_+^2 = 0, \quad (3.13)$$

and the equations are subject to the conditions

$$\tilde{A} \geq 0, \quad \tilde{\Sigma} \geq 0, \quad \tilde{N} \geq 0. \quad (3.14)$$
The generalized Friedmann equation, written in dimensionless variables, becomes
\[ y^2 = 1 - \Sigma_+^2 - \tilde{\Sigma} - \tilde{A} - \tilde{N} - x^2, \]  
(3.15)
which serves to define \( y \), and the evolution of \( y \) is governed by
\[ y' = (q + 1 + \sqrt{\frac{3}{2}kx})y. \]  
(3.16)
Eqs. (3.10) and (3.16) are equivalent to the Klein-Gordon equation
\[ \ddot{\phi} + \theta \dot{\phi} + V'(\phi) = 0 \]  
written in dimensionless variables. The parameter \( \ell = \frac{1}{h} \) defines the group parameter \( h \) in the Bianchi VII\(_h\) models.

The variables \( \Sigma_+ \) and \( \tilde{\Sigma} \) describe the shear anisotropy. The variables \( \tilde{A}, N_+ \) and \( \tilde{N} \) describe the spatial curvature of the models. The variable \( \Delta \) describes the relative orientation of the shear and spatial curvature eigenframes.

We are not interested here in the complete qualitative behaviour of the cosmological models [74] but simply whether the Bianchi VII\(_h\) models isotropize to the future when \( k > \sqrt{2} \). This question can be easily answered by examining the stability of the isotropic singular points of the above six-dimensional dynamical system (3.5–3.10).

### 3.2.2 Stability Analysis

All of the isotropic singular points lie in the invariant set \( FRW \) defined by \( \{ \Sigma_+ = 0, \tilde{\Sigma} = 0, \tilde{N} = 0, \Delta = 0 \} \). Therefore, we shall find all of the isotropic singular points and determine whether any are stable attractors or sinks [75].

The singular point
\[ \Sigma_+ = 0, \tilde{\Sigma} = 0, \Delta = 0, A = 1, N_+ = \sqrt{\ell}, x = 0 \]  
(3.17)
implies \( y = 0 \) and represents the negatively curved Milne vacuum model. The linearization of the dynamical system in the neighborhood of this singular point has eigenvalues
\[ 0, 2, -2, -4, -2 + 4\sqrt{-\ell}, -2 + 4\sqrt{-\ell}. \]  
(3.18)
Therefore, this singular point is a saddle.

The singular point(s)
\[ \Sigma_+ = 0, \tilde{\Sigma} = 0, \Delta = 0, A = 0, N_+ = 0, x = 1 \text{ or } -1 \]  
(3.19)
imply that \( y = 0 \) and represent flat non-inflationary FRW model(s). The eigenvalues in both cases are
\[ 0, 0, 2, 2, 4, 6 + \sqrt{6k}. \]  
(3.20)
These singular points are unstable with an unstable manifold of at least dimension 4.

The singular point

\[ \Sigma_+ = 0, \tilde{\Sigma} = 0, \Delta = 0, A = 0, N_+ = 0, x = -\frac{\sqrt{6}}{6}k \]  

implies that \( y = \sqrt{1 - \frac{k^2}{6}} \). The eigenvalues are

\[ \frac{k^2 - 6}{2}, \frac{k^2 - 6}{2}, k^2 - 6, k^2 - 4, k^2 - 2, \frac{k^2 - 2}{2} \]  

For \( k < \sqrt{2} \), this singular point represents the usual power-law inflationary attractor. If \( \sqrt{2} < k < 2 \), then the singular point has an unstable manifold of dimension 4. If \( 2 < k < \sqrt{6} \), then the singular point has an unstable manifold of dimension 3. This singular point does not exist if \( k > \sqrt{6} \).

The singular point

\[ \Sigma_+ = 0, \tilde{\Sigma} = 0, \Delta = 0, A = 1 - \frac{2}{k^2}, N_+ = \sqrt{\ell \left(1 - \frac{2}{k^2}\right)}, x = -\frac{\sqrt{6}}{3k} \]  

denoted \( F \), implies that \( y = \frac{2\sqrt{3}}{3k} \) and represents a non-inflationary negatively curved FRW model. The eigenvalues are

\[ -1 + \sqrt{\frac{8}{k^2} - 3}, \quad -1 - \sqrt{\frac{8}{k^2} - 3}, \]
\[ -2 + \frac{\sqrt{2}}{k} \sqrt{(k^2 - 4\ell k^2 + 8\ell) + E}, \quad -2 + \frac{\sqrt{2}}{k} \sqrt{(k^2 - 4\ell k^2 + 8\ell) - E}, \]
\[ -2 - \frac{\sqrt{2}}{k} \sqrt{(k^2 - 4\ell k^2 + 8\ell) + E}, \quad -2 - \frac{\sqrt{2}}{k} \sqrt{(k^2 - 4\ell k^2 + 8\ell) - E}, \]

where

\[ E \equiv \sqrt{(k^2 + 4\ell k^2 - 8\ell)^2 + 32\ell(2 - k^2)}. \]

After some algebra it can be shown that if \( k > \sqrt{2} \) (note that \( \ell > 0 \) in the Bianchi VII\( _h \) models) then all of the eigenvalues have negative real parts. Therefore, if \( k > \sqrt{2} \), then this singular point is a stable attractor. (Note that this singular point does not exist if \( k < \sqrt{2} \).) In other words, there exists an open set of initial conditions in the set of anisotropic Bianchi VII\( _h \) (with a scalar field and exponential potential) initial data for which the corresponding cosmological models asymptotically approach an isotropic and negatively curved FRW model.
3.2.3 Discussion

We have shown that within the set of all spatially homogeneous initial data, there exists an open set of initial data describing the Bianchi type VII\textsubscript{h} models (having a scalar field with an exponential potential and $k > \sqrt{2}$) such that the models approach isotropy at infinite times. This compliments the results of Kitada and Maeda [72, 62], who showed that all ever-expanding spatially homogeneous models (including the Bianchi VII\textsubscript{h} models) with $k < \sqrt{2}$ approach isotropy to the future. In other words, there exists a set of spatially homogeneous initial data of non-zero measure for which models will isotropize to the future for all positive values of $k$. Of course, there also exists a set of spatially homogeneous initial data of non-zero measure for which models will not isotropize to the future when $k > \sqrt{2}$ (e.g., the Bianchi VIII models).

If $k < \sqrt{2}$, then all models will inflate as they approach the power-law inflationary attractor represented by Eq. (3.21). For $k > \sqrt{2}$, the stable singular point $F$, given by Eq. (3.23), which does not exist for $k < \sqrt{2}$, is isotropic and resides on the surface $q = 0$. This means that the corresponding exact solution is marginally non-inflationary. However, this does not mean that the corresponding cosmological models are not inflating as they asymptotically approach this singular state. As orbits approach $F$ they may have $q < 0$ or $q > 0$ (or even $q = 0$) and consequently the models may or may not be inflating. If they are inflating, then the rate of inflation is decreasing as $F$ is approached (i.e., $q \to 0$). When $\sqrt{2} < k < \sqrt{8}/3$, we find that $F$ is node-like, hence there is an open set of models that inflate as they approach $F$ and an open set which do not. When $k > \sqrt{8}/3$, $F$ is found to be spiral-like, and so it is expected that orbits experience regions of both $q < 0$ and $q > 0$ as they wind their way towards $F$. As in Kitada and Maeda [72, 62], the inclusion of matter in the form of a perfect fluid is not expected to change the results of the analysis provided the matter satisfies appropriate energy conditions.

3.3 Stability of Matter Scaling Solutions

Spatially homogeneous scalar field cosmological models with an exponential potential and with barotropic matter may also be important even if the exponential potential is too steep to drive inflation. For example, there exist ‘scaling solutions’ in which the scalar field energy density tracks that of the perfect fluid (so that at late times neither field is negligible) [64]. In particular, in [65] a phase-plane analysis of the spatially flat FRW models showed that these scaling solutions are the unique late-time attractors whenever they exist. The cosmological consequences of these scaling models have been further studied in [66]. For example, in such models a significant fraction of the current energy density of the Universe may be contained in the homogeneous scalar field whose dynamical effects mimic cold dark matter; the tightest constraint on these
cosmological models comes from primordial nucleosynthesis bounds on any such relic density [64, 65, 66].

Clearly these matter scaling models are of potential cosmological significance. It is consequently of prime importance to determine the genericity of such models by studying their stability in the context of more general spatially homogeneous models.

### 3.3.1 The Matter Scaling Solution

The governing equations for a scalar field with an exponential potential $V = V_0 e^{k\phi}$ evolving in a flat FRW model containing a separately conserved perfect fluid which satisfies the barotropic equation of state

$$p_\gamma = (\gamma - 1)\mu_\gamma,$$

where $0 < \gamma < 2$ here, are given by

$$\dot{\theta} = -\frac{3}{2}(\gamma \mu_\gamma + \dot{\phi}^2), \quad \mu_\gamma = -\gamma \theta \mu_\gamma, \quad \dot{\phi} = -\theta \dot{\phi} - kV,$$

subject to the Friedmann constraint

$$\theta^2 = 3(\mu_\gamma + \frac{1}{2} \dot{\phi}^2 + V),$$

where an overdot denotes ordinary differentiation with respect to time $t$, and units have been chosen so that $8\pi G = 1$. We note that the total energy density of the scalar field is given by Eq. (3.2).

Defining $x$ and $y$ by Eq. (3.4) and again using the logarithmic time variable, $\tau$, Eqs. (3.25) – (3.27) can be written as the plane-autonomous system [65]:

$$x' = -3x - \sqrt{\frac{3}{2}} k y^2 + \frac{3}{2} x [2x^2 + \gamma (1 - x^2 - y^2)], \quad (3.29)$$

$$y' = \frac{3}{2} y \left[ -\sqrt{\frac{2}{3}} - k x + 2x^2 + \gamma (1 - x^2 - y^2) \right], \quad (3.30)$$

where

$$\Omega \equiv \frac{\mu_\gamma}{\theta^2}, \quad \Omega_\phi \equiv \frac{\mu_\phi}{\theta^2} = x^2 + y^2; \quad \Omega + \Omega_\phi = 1,$$

which implies that $0 \leq x^2 + y^2 \leq 1$ for $\Omega \geq 0$, so that the phase-space is bounded.

A qualitative analysis of this plane-autonomous system was given in [65]. The well-known power-law inflationary solution for $k^2 < 2$ [47, 62] corresponds to the singular
point \( x = -k/\sqrt{6} \), \( y = (1 - k^2/6)^{1/2} \) \((\Omega_\phi = 1, \Omega = 0)\) of the system (3.29)/(3.30), which is shown to be stable (i.e., attracting) for \( k^2 < 3\gamma \) in the presence of a barotropic fluid. Previous analysis had shown that when \( k^2 < 2 \) this power-law inflationary solution is a global attractor in spatially homogeneous models in the absence of a perfect fluid (except for a subclass of Bianchi type IX models which recollapse).

In addition, for \( \gamma > 0 \) there exists a scaling solution corresponding to the singular point

\[
x = x_0 = -\sqrt{\frac{3\gamma}{2k}}, \quad y = y_0 = \left[3(2 - \gamma)\gamma/2k^2\right]^{1/2},
\]

(3.32)

whenever \( k^2 > 3\gamma \). The linearization of system (3.29)/(3.30) about the singular point (3.32) yields the two eigenvalues with negative real parts

\[
-\frac{3}{4}(2 - \gamma) \pm \frac{3}{4k}\sqrt{(2 - \gamma)[24\gamma^2 - k^2(9\gamma - 2)]}
\]

(3.33)

when \( \gamma < 2 \). The singular point is consequently stable (a spiral for \( k^2 > 24\gamma^2/(9\gamma - 2) \), else a node) so that the corresponding cosmological solution is a late-time attractor in the class of flat FRW models in which neither the scalar field nor the perfect fluid dominates the evolution. The effective equation of state for the scalar field is given by

\[
\gamma_\phi \equiv \frac{(\mu_\phi + p_\phi)}{\mu_\phi} = \frac{2x_0^2}{x_0^2 + y_0^2} = \gamma,
\]

which is the same as the equation of state parameter for the perfect fluid. The solution is referred to as a matter scaling solution since the energy density of the scalar field remains proportional to that of the barotropic perfect fluid according to \( \Omega/\Omega_\phi = k^2/3\gamma - 1 \) [64]. Since the scaling solution corresponds to a singular point of the system (3.29)/(3.30) we note that it is a self-similar cosmological model [6].

### 3.3.2 Stability of the Matter Scaling Solution

Let us study the stability of the matter scaling solution with respect to anisotropic and curvature perturbations within the class of spatially homogeneous models [76, 32].

**Bianchi I models**

In order to study the stability of the scaling solution with respect to shear perturbations we shall first investigate the class of anisotropic Bianchi I models, which are the simplest spatially homogeneous generalizations of the flat FRW models and have non-zero shear but zero three-curvature. The governing equations in the Bianchi I models are Eqs. (3.26) and (3.27), and Eq. (3.28) becomes

\[
\theta^2 = 3 \left( \mu_\gamma + \frac{1}{2} \dot{\phi}^2 + V \right) + \Sigma^2,
\]

(3.34)
where $\Sigma^2 \equiv 3\Sigma_0^2 R^{-6}$ is the contribution due to the shear, where $\Sigma_0$ is a constant and $R$ is the scale factor. Eq. (3.25) is replaced by the time derivative of Eq. (3.34).

Using the definitions (3.4) and (3.31) we can deduce the governing ODEs. Due to the $\Sigma^2$ term in (3.34) we can no longer use this equation to substitute for $\mu_{\gamma}$ in the remaining equations, and we consequently obtain the three-dimensional autonomous system:

$$x' = -3x - \sqrt{\frac{3}{2}ky^2} + \frac{3}{2}x[2 + (\gamma - 2)\Omega - 2y^2],$$  \hspace{1cm} (3.35) \\
y' = \frac{3}{2}y\left\{ \sqrt{\frac{2}{3}kx + 2 + (\gamma - 2)\Omega - 2y^2} \right\}, \hspace{1cm} (3.36) \\
\Omega' = 3\Omega\{ (\gamma - 2)(\Omega - 1) - 2y^2 \}, \hspace{1cm} (3.37)$$

where Eq. (3.34) yields

$$1 - \Omega - x^2 - y^2 = \Sigma^2 \theta^{-2} \geq 0,$$  \hspace{1cm} (3.38) 

so that we again have a bounded phase-space.

The matter scaling solution, corresponding to the flat FRW solution, is now represented by the singular point

$$x = x_0, \quad y = y_0, \quad \Omega = 1 - \frac{3\gamma}{k^2}. \hspace{1cm} (3.39)$$

The linearization of system (3.35) – (3.37) about the singular point (3.39) yields three eigenvalues, two of which are given by (3.33) and the third has the value $-3(2 - \gamma)$, all with negative real parts when $\gamma < 2$. Consequently the scaling solution is stable to Bianchi type I shear perturbations.

**Curved FRW models**

In order to study the stability of the scaling solution with respect to curvature perturbations we shall first consider the class of FRW models, which have curvature but no shear. Again Eqs. (3.26) and (3.27) are valid, but in this case Eq. (3.28) becomes

$$\theta^2 = 3(\mu_{\gamma} + \frac{1}{2}\phi^2 + V) + K,$$  \hspace{1cm} (3.40) 

where $K \equiv -9kR^{-2}$ and $k$ is a constant that can be scaled to $0, \pm 1$. Eq. (3.25) is again replaced by the time derivative of Eq. (3.40).

As in the previous case we cannot use Eq. (3.40) to replace $\mu_{\gamma}$, and using the definitions (3.4) and (3.31) we obtain the three-dimensional autonomous system:

$$x' = -3x - \sqrt{\frac{3}{2}ky^2} + \frac{3}{2}x \left[ \left( \gamma - \frac{2}{3} \right) \Omega + \frac{2}{3}(1 + 2x^2 - y^2) \right], \hspace{1cm} (3.41)$$

21
\[
y' = \frac{3}{2}y \left\{ \sqrt{\frac{2}{3}} k x + \left( \gamma - \frac{2}{3} \right) \Omega + \frac{2}{3} (1 + 2x^2 - y^2) \right\},
\]
\[
\Omega' = 3\Omega \left\{ \left( \gamma - \frac{2}{3} \right) (\Omega - 1) + \frac{2}{3} (2x^2 - y^2) \right\},
\]
where
\[
1 - \Omega - x^2 - y^2 = KH^{-2}.
\]
The phase-space is bounded for \( K = 0 \) or \( K < 0 \), but not for \( K > 0 \).

The matter scaling solution again corresponds to the singular point (3.39). The linearization of system (3.41) – (3.43) about this singular point yields the two eigenvalues with negative real parts given by (3.33) and the eigenvalue \( 3\gamma - 2 \). Hence the scaling solution is only stable for \( \gamma < \frac{2}{3} \). For \( \gamma > \frac{2}{3} \) the singular point (3.39) is a saddle with a two-dimensional stable manifold and a one-dimensional unstable manifold.

Consequently the scaling solution is unstable to curvature perturbations in the case of realistic matter \( (\gamma \geq 1) \); i.e., the scaling solution is no longer a late-time attractor in this case. However, the scaling solution does correspond to a singular point of the governing autonomous system of ODEs and hence there are cosmological models that can spend an arbitrarily long time ‘close’ to this solution. Moreover, since the curvature of the Universe is presently constrained to be small by cosmological observations, it is possible that the scaling solution could be important in the description of our actual Universe. That is, not enough time has yet elapsed for the curvature instability to have effected an appreciable deviation from the flat FRW model (as in the case of the standard perfect fluid FRW model). Hence the scaling solution may still be of physical interest.

**Bianchi VII\(h\) models**

To further study the significance of the scaling solution it is important to determine its stability within a general class of spatially homogeneous models such as the (general) class of Bianchi type VII\(h\) models (which are perhaps the most physically relevant models since they can be regarded as generalizations of the negative-curvature FRW models). The Bianchi VII\(h\) models are sufficiently complicated that a simple coordinate approach (similar to that given above) is not desirable. In subsection 3.2.1 the Bianchi VII\(h\) spatially homogeneous models with a minimally coupled scalar field with an exponential potential (but without a barotropic perfect fluid) were studied by employing a group-invariant orthonormal frame approach with expansion-normalized state variables governed by a set of dimensionless evolution equations (constituting a ‘reduced’ dynamical system) with respect to a dimensionless time subject to a non-linear constraint [26]. A barotropic perfect fluid can easily be included [32].

The reduced dynamical system is seven-dimensional (subject to a constraint). The scaling solution is again a singular point of this seven-dimensional system. This
singular point, which only exists for $k^2 > 3\gamma$, has two eigenvalues given by (3.33) which have negative real parts for $\gamma < 2$, two eigenvalues (corresponding to the shear modes) proportional to $(\gamma - 2)$ which are also negative for $\gamma < 2$, and two eigenvalues (essentially corresponding to curvature modes) proportional to $(3\gamma - 2)$ which are negative for $\gamma < \frac{2}{3}$ and positive for $\gamma > \frac{2}{3}$ [76]. The remaining eigenvalue (which also corresponds to a curvature mode) is equal to $3\gamma - 4$. Hence for $\gamma < \frac{2}{3}$ ($k^2 > 3\gamma$) the scaling solution is again stable. However, for realistic matter ($\gamma \geq 1$) the corresponding singular point is a saddle with a four- or five-dimensional stable manifold (depending upon whether $\gamma > 4/3$ or $\gamma < 4/3$, respectively).

4 String Models

There has been considerable interest recently in the cosmological implications of string theory. String theory introduces significant modifications to the standard, hot big bang model based on conventional Einstein gravity. Early-universe cosmology provides one of the few environments where the predictions of the theory can be quantitatively investigated. The evolution of the very early universe below the string scale is determined by ten–dimensional supergravity theories [46, 77]. All theories of this type contain a dilaton, a graviton and a two–form potential in the Neveu–Schwarz/Neveu–Schwarz (NS–NS) bosonic sector. If one considers a Kaluza–Klein compactification from ten dimensions onto an isotropic six–torus of radius $e^\beta$, the effective action is given by

$$S = \int d^4x \sqrt{-g} e^{-\Phi} \left[ R + (\nabla \Phi)^2 - 6 (\nabla \beta)^2 - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right], \quad (4.1)$$

where the moduli fields arising from the compactification of the form–fields on the internal dimensions and the graviphotons originating from the compactification of the metric have been neglected [78]. In Eq. (4.1), $R$ is the Ricci curvature of the spacetime with metric $g_{\mu\nu}$ and $g \equiv \det g_{\mu\nu}$, the dilaton field, $\Phi$, parametrizes the string coupling, $g^2_s \equiv e^\Phi$, and $H_{\mu\nu\lambda} \equiv \partial_{[\mu} B_{\nu\lambda]}$ is the field strength of the two–form potential $B_{\mu\nu}$. The volume of the internal dimensions is parametrized by the modulus field, $\beta$.

In four dimensions, the three–form field strength is dual to a one–form:

$$H^{\mu\nu\lambda} \equiv e^\Phi \epsilon^{\mu\nu\lambda\kappa} \nabla_\kappa \sigma, \quad (4.2)$$

where $\epsilon^{\mu\nu\lambda\kappa}$ is the covariantly constant four–form. In this dual formulation, the field equations (FE) can be derived from the action

$$S = \int d^4x \sqrt{-g} e^{-\Phi} \left[ R + (\nabla \Phi)^2 - 6 (\nabla \beta)^2 - \frac{1}{2} e^{2\Phi} (\nabla \sigma)^2 \right], \quad (4.3)$$

where $\sigma$ is interpreted as a pseudo–scalar ‘axion’ field [79].
It can be shown that the action (4.3) is invariant under a global $SL(2, \mathbb{R})$ transformation on the dilaton and axion fields [79]. The general FRW cosmologies derived from Eq. (4.3) have been found by employing this symmetry [80]. However, the symmetry is broken when a cosmological constant is present [81] and the general solution is not known in this case. The purpose here is to determine the general structure of the phase space of solutions for the wide class of string cosmologies that contain a cosmological constant in the effective action. This is particularly relevant in light of recent high redshift observations that indicate a vacuum energy density may be dominating the large–scale dynamics of the universe at the present epoch [82].

A cosmological constant may arise in a number of different contexts and we consider a general action of the form

$$S = \int d^4x \sqrt{-g} \left\{ e^{-\Phi} \left[ R + (\nabla \Phi)^2 - 6 (\nabla \beta)^2 - \frac{1}{2} e^{2\Phi} (\nabla \sigma)^2 - 2\Lambda \right] - \Lambda_R \right\}. \quad (4.4)$$

The constant, $\Lambda$, is determined by the central charge deficit of the string theory and may be viewed as a cosmological constant in the gravitational sector of the theory. In principle, it may take arbitrary values if the string is coupled to an appropriate conformal field theory. Such a term may also have an origin in terms of the reduction of higher degree form–fields [83]. The constant, $\Lambda_R$, represents a phenomenological cosmological constant in the matter sector. Since it does not couple directly to the dilaton field, it may be viewed in a stringy context as a Ramond–Ramond (RR) degree of freedom (a 0–form) [84]. Such a cosmological constant may also be interpreted as the potential energy of a scalar field that is held in a false vacuum state.

We shall include the combined effects of the axion, modulus and dilaton fields, thereby extending previous qualitative analyses where one or more of these terms was neglected [85, 86, 87, 88]. A full stability analysis can be performed for all models by rewriting the FE in terms of a set of compactified variables. As usual, units in which $c = 8\pi G = 1$ will be utilized throughout.

### 4.1 Cosmological Field Equations

When $\Lambda_R = 0$, the spatially flat FRW cosmological FE derived from action (4.4) are given by

$$2\ddot{\alpha} - 2\dot{\alpha}\dot{\phi} - \dot{\sigma}^2 e^{2\phi+6\alpha} = 0 \quad (4.5)$$
$$2\ddot{\phi} - \dot{\phi}^2 - 3\dot{\alpha}^2 - 6\dot{\beta}^2 + \frac{1}{2} \dot{\sigma}^2 e^{2\phi+6\alpha} + 2\Lambda = 0 \quad (4.6)$$
$$\ddot{\beta} - \dot{\beta}\dot{\phi} = 0 \quad (4.7)$$
$$\ddot{\sigma} + \dot{\sigma} (\dot{\phi} + 6\dot{\alpha}) = 0, \quad (4.8)$$

where $\phi \equiv \Phi - 3\alpha$ defines the ‘shifted’ dilaton field, $R(t) \equiv e^\alpha$ is the scale factor of the universe and a dot denotes differentiation with respect to cosmic time, t. The
generalized Friedmann constraint equation is

$$3\dot{\alpha}^2 - \dot{\varphi}^2 + 6\dot{\beta}^2 + \frac{1}{2}\dot{\sigma}^2 e^{2\varphi + 6\alpha} + 2\Lambda = 0.$$  \hfill (4.9)

A number of exact solutions to Eqs. (4.5)–(4.9) are known when one or more of the degrees of freedom are trivial; these solutions lie in the invariant sets of the full phase space. The ‘dilaton–vacuum’ solutions, where only the dilaton field is dynamically important, are given in [89]; there is a curvature singularity in these solutions at $t = 0$. In the pre–big bang inflationary scenario, the pre–big bang phase corresponds to the range $t < 0$ and the post–big bang phase to the solution for $t > 0$. The ‘dilaton–moduli–vacuum’ solutions have $\dot{\sigma} = \Lambda = 0$. The general solution with $\Lambda = 0$ is the ‘dilaton–moduli–axion’ solution [80]; this cosmology asymptotically approaches a dilaton–moduli–vacuum solution in the limits of high and low spacetime curvature. The axion field induces a smooth transition between these two power-law solutions. The solutions where only the axion field is trivial and $\Lambda > 0$ are specific cases of the ‘rolling radii’ solutions found by Mueller [90]. The corresponding solutions for $\Lambda < 0$ are related by a redefinition. Finally, there exists the ‘linear dilaton–vacuum’ solution where $\Lambda > 0$ [91]. This solution is static and the dilaton evolves linearly with time.

### 4.2 Qualitative Analysis of the NS–NS Fields

For an arbitrary central charge deficit, the FE (4.5)–(4.9) may be written as an autonomous system of ODEs:

$$\dot{h} = \psi^2 + h\psi - 3h^2 - N - 2\Lambda \hfill (4.10)$$

$$\dot{\psi} = 3h^2 + N \hfill (4.11)$$

$$\dot{N} = 2N\psi \hfill (4.12)$$

$$\dot{\rho} = -6h\rho \hfill (4.13)$$

$$3h^2 - \psi^2 + N + \frac{1}{2}\rho + 2\Lambda = 0, \hfill (4.14)$$

where we have defined the new variables

$$N \equiv 6\dot{\beta}^2, \quad \rho \equiv \dot{\sigma}^2 e^{2\varphi + 6\alpha}, \quad \psi \equiv \dot{\varphi}, \quad h \equiv \dot{\alpha},$$

and $h = \theta/3$, where $\theta$ is the expansion scalar defined earlier. The variable $\rho$ may be interpreted as the effective energy density of the pseudo–scalar axion field [88]. It follows from Eq. (4.11) that $\psi$ is a monotonically increasing function of time and this implies that the singular points of the system of ODEs must be located either at zero or infinite values of $\psi$. In addition, due to the existence of a monotone function, it follows that there are no periodic or recurrent orbits in the corresponding phase
The sets \( \Lambda = 0 \) and \( \rho = 0 \) are invariant sets. In particular, the exact solution for \( \Lambda = 0 \) divides the phase space and the orbits do not cross from positive to negative \( \Lambda \).

We must consider the cases where \( \Lambda < 0 \) and \( \Lambda > 0 \) separately. In the case where the central charge deficit is negative, \( \Lambda < 0 \), it proves convenient to employ the generalized Friedmann constraint equation (4.14) to eliminate the modulus field. We may compactify the phase space by normalizing with \( \sqrt{\psi^2 - 2\Lambda} \) and we define a new time variable by

\[
\frac{d}{dT} = \frac{1}{\sqrt{\psi^2 - 2\Lambda}} \frac{d}{dt}.
\]  

(4.16)

The governing equations reduce to a three-dimensional system of autonomous ODEs. The singular points all lie on one of the two lines of non-isolated singular points (or one-dimensional singular sets) \( L_\pm \). On the line \( L_+ \) the singular points are either saddles or local sinks, and on the line \( L_- \) the singular points are local sources or saddles. The dynamics is very simple due to the existence of (two) monotonically increasing functions. A full analysis is given in [93]. Henceforward, let us consider the case \( \Lambda > 0 \).

4.2.1 Models with Positive Central Charge Deficit

In the case where the central charge deficit is positive, \( \Lambda > 0 \), we choose the normalization

\[
\epsilon \equiv \left( 3h^2 + \frac{1}{2} \rho + N + 2\Lambda \right)^{1/2}.
\]  

(4.17)

The generalized Friedmann constraint equation (4.14) now takes the simple form

\[
\frac{\psi^2}{\epsilon^2} = 1
\]  

(4.18)

and may be employed to eliminate \( \psi \). Since by definition \( \epsilon \geq 0 \), specifying one of the roots \( \psi/\epsilon = \pm 1 \) corresponds to choosing the sign of \( \psi \). However, it follows from the definition in Eq. (4.15) that changing the sign of \( \psi \) is related to a time reversal of the dynamics. In what follows, we shall consider the case \( \psi/\epsilon = +1 \); the case \( \psi/\epsilon = -1 \) is qualitatively similar.

Introducing the new normalized variables

\[
\mu \equiv \frac{\sqrt{3}h}{\epsilon}, \quad \nu \equiv \frac{\rho}{2\epsilon^2}, \quad \lambda \equiv \frac{N}{\epsilon^2}
\]  

(4.19)

and a new dynamical variable

\[
\frac{d}{dT} \equiv \frac{1}{\sqrt{3}\epsilon} \frac{d}{dt}
\]  

(4.20)
transforms Eqs. (4.10)–(4.13) to the three–dimensional autonomous system:
\[
\begin{align*}
\frac{d\mu}{dT} &= \nu + \frac{\mu}{\sqrt{3}} [1 - \mu^2 - \lambda] \quad (4.21) \\
\frac{d\nu}{dT} &= -2\nu \left[ \mu + \frac{1}{\sqrt{3}} (\lambda + \mu^2) \right] \quad (4.22) \\
\frac{d\lambda}{dT} &= \frac{2}{\sqrt{3}} \lambda (1 - \mu^2 - \lambda). \quad (4.23)
\end{align*}
\]

The phase space variables are bounded, \(0 \leq \{\mu^2, \nu, \lambda\} \leq 1\), and satisfy \(\mu^2 + \nu + \lambda \leq 1\). The sets \(\nu = 0\) and \(\lambda = 0\) are invariant sets corresponding to \(\rho = 0\) and \(\dot{\beta} = 0\), respectively. In addition, \(\mu^2 + \nu + \lambda = 1\) is an invariant set corresponding to \(\Lambda = 0\). We note that the right-hand side of Eq. (4.23) is positive-definite and this simplifies the dynamics considerably.

The singular points of the system (4.21)-(4.23) consist of the isolated singular point
\[
C : \mu = \nu = \lambda = 0, \quad (4.24)
\]
and the line of non-isolated singular points
\[
V : \nu = 0, \lambda = 1 - \mu^2 \quad (\mu \text{ arbitrary}). \quad (4.25)
\]

The eigenvalues associated with \(C\) are \(\lambda_1 = 1/\sqrt{3}\), \(\lambda_2 = 2/\sqrt{3}\) and \(\lambda_3 = 0\). Since \(C\) is an isolated singular point, it is therefore non-hyperbolic. However, a simple analysis shows that it is a global source. The eigenvalues associated with \(V\) are:
\[
\lambda_1 = -2 \left( \mu + \frac{1}{\sqrt{3}} \right), \quad \lambda_2 = -\frac{2}{\sqrt{3}} \quad (4.26)
\]
and the third eigenvalue is zero since \(V\) is a one–dimensional set. Therefore, on \(V\) the singular points are saddles for \(\mu \in [-1, -1/\sqrt{3}]\) and local sinks for \(\mu \in (-1/\sqrt{3}, 1]\).

It is also instructive to consider the dynamics on the boundary corresponding to \(\lambda = 0\), since the case \(N = 0\) is of physical interest in its own right as a four–dimensional model. In this case the ODEs reduce to the two-dimensional system:
\[
\begin{align*}
\frac{d\mu}{dT} &= \nu + \frac{\mu}{\sqrt{3}} (1 - \mu^2) \quad (4.27) \\
\frac{d\nu}{dT} &= -2\nu \left[ 1 + \frac{1}{\sqrt{3}} \mu \right]. \quad (4.28)
\end{align*}
\]

The singular points and their corresponding eigenvalues are:
\[
C : \quad \mu = \nu = 0; \quad \lambda_1 = \frac{1}{\sqrt{3}}, \lambda_2 = 0 \quad (4.29)
\]
\[ S : \mu = -1, \nu = 0; \quad \lambda_1 = -\frac{2}{\sqrt{3}}, \lambda_2 = 2 \left( 1 - \frac{1}{\sqrt{3}} \right) \]  
\[ A : \mu = 1, \nu = 0; \quad \lambda_1 = -\frac{2}{\sqrt{3}}, \lambda_2 = -2 \left( 1 + \frac{1}{\sqrt{3}} \right) \]  

Point \( C \) is a non-hyperbolic singular point; however, by changing to polar coordinates we find that \( C \) is a repeller with an invariant ray \( \theta = \tan^{-1}(-\sqrt{3}) \). The saddle \( S \) and the attractor \( A \) lie on the line \( V \).

### 4.2.2 Discussion

Let us briefly summarize the dynamics in the case \( \Lambda < 0 \), which was studied in [93]. When the modulus field is frozen the universe contracts from a singular initial state. The orbits are past-asymptotic to a dilaton–vacuum solution, and so the axion is negligible and the kinetic energy of the dilaton is dominant. As the collapse proceeds, however, the axion becomes dynamically more important and eventually induces a bounce. In the case of vanishing \( \Lambda \), these equations imply that the future attractor would correspond to a dilaton–vacuum solution. However, the combined effect of the axion and the central charge deficit is to cause the universe to evolve towards a singularity with \( \dot{\alpha} \to +\infty \) in a finite time. This behaviour is different from that found when no axion field is present in which there is no bounce [86, 90]. The inclusion of a modulus field leads to a line of sources and a line of sinks. The axion field is dynamically negligible in the neighbourhood of the singular points. Moreover, a bouncing cosmology is no longer inevitable and there exist solutions that expand to infinity in a finite time. The solutions are asymptotic to the dilaton–moduli–vacuum solutions near the lines \( L_\pm \).

In the case of a positive \( \Lambda \) discussed here, the isolated singular point \( C \) corresponds to the ‘linear dilaton–vacuum’ solution [86, 91]. When the modulus is frozen, all trajectories evolve away from \( C \) towards the point \( A \) and approach a superinflationary dilaton–vacuum solution defined over \( t < 0 \). Some of the orbits evolving away from \( C \) represent contracting cosmologies and the effect of the axion is to reverse the collapse in all these cases. For the rolling modulus solutions, the orbits tend to dilaton–moduli–vacuum solutions as they approach the attractors (the sinks on \( V \)). As in the case of a negative central charge deficit, the critical value \( \mu^2 = 1/3 \) corresponds to the case where \( \dot{\alpha}^2 = \dot{\beta}^2 \), representing the isotropic, ten–dimensional cosmology \( (\dot{\alpha} = \dot{\beta}) \) and its dual solution \( (\dot{\alpha} = -\dot{\beta}) \). In the latter solution, the ten–dimensional dilaton field, \( \hat{\Phi} \equiv \Phi + 6\beta \), is constant. The other boundary of \( V \) is the point \( A \) representing the case where the kinetic energy of the modulus field vanishes. We note that the qualitative behaviour of models with \( \psi < 0 \) is similar.
4.3 Qualitative Analysis of the RR Sector

When a cosmological constant is introduced into the matter sector of Eq. (4.4) and the central charge deficit vanishes (i.e., $\Lambda = 0$), the FE are given by

\begin{align*}
\ddot{\alpha} &= \dot{\alpha}\dot{\varphi} + \dot{\varphi}^2 - 3\dot{\alpha}^2 - 6\dot{\beta}^2 - \frac{3}{2}\Lambda_R e^{\varphi + 3\alpha} \tag{4.32} \\
\dot{\varphi} &= 3\dot{\alpha}^2 + 6\dot{\beta}^2 + \frac{1}{2}\Lambda_R e^{\varphi + 3\alpha} \tag{4.33} \\
\dot{\sigma} &= -(\dot{\varphi} + 6\dot{\alpha})\dot{\sigma} \tag{4.34} \\
\ddot{\beta} &= \dot{\beta}\dot{\varphi} \tag{4.35}
\end{align*}

and the generalized Friedmann constraint equation takes the form

\[ 3\dot{\alpha}^2 - \dot{\varphi}^2 + 6\dot{\beta}^2 + \frac{1}{2}\sigma^2 e^{2\varphi + 6\alpha} + \Lambda_R e^{\varphi + 3\alpha} = 0. \tag{4.36} \]

Eqs. (4.32)–(4.36) may be simplified by introducing the new time coordinate

\[ \frac{d}{dt} \equiv e^{-(\varphi + 3\alpha)/2} \frac{d}{d\tilde{t}} \tag{4.37} \]

and employing the generalized Friedmann constraint equation (4.36) to eliminate the axion field. The remaining FE are then given by

\begin{align*}
\alpha'' &= \varphi'^2 - \frac{9}{2}\alpha'^2 + \frac{1}{2}\alpha'\varphi' - 6\beta'^2 - \frac{3}{2}\Lambda_R \tag{4.38} \\
\varphi'' &= 3\alpha'^2 + 6\beta'^2 - \frac{1}{2}\varphi'^2 - \frac{3}{2}\alpha'\varphi' + \frac{1}{2}\Lambda_R \tag{4.39} \\
\beta'' &= \frac{1}{2}\beta' (\varphi' - 3\alpha') \tag{4.40}
\end{align*}

where in this section a prime denotes differentiation with respect to $\tilde{t}$.

4.3.1 Positive Cosmological Constant

When $\Lambda_R > 0$, we express Eqs. (4.38)–(4.40) as an autonomous system by defining

\[ h \equiv \alpha', \quad \psi \equiv \varphi', \quad N \equiv \beta'. \tag{4.41} \]

Eq. (4.36) then implies that

\[ \psi^2 \geq 3h^2 + 6N^2 + \Lambda_R \geq 0 \tag{4.42} \]
and consequently we may normalize using $\psi$. We therefore define

$$
x \equiv \frac{\sqrt{3}h}{\psi},
$$

$$
y \equiv \frac{6N^2}{\psi^2},
$$

$$
z \equiv \frac{\Lambda_R}{\psi^2},
$$

$$
\frac{d}{d\Theta} \equiv \frac{1}{\psi} \frac{d}{dt},
$$

and assume that $\psi > 0$ ($\psi < 0$ is again related to time-reversal).

The resulting three–dimensional autonomous system is:

$$
\frac{dx}{d\Theta} = (x + \sqrt{3})[1 - x^2 - y - z] + \frac{1}{2} z[x - \sqrt{3}]
$$

$$
\frac{dy}{d\Theta} = 2y \left\{ [1 - x^2 - y - z] + \frac{1}{2} z \right\}
$$

$$
\frac{dz}{d\Theta} = 2z \left\{ [1 - x^2 - y - z] - \frac{1}{2} (1 - z - \sqrt{3}x) \right\}.
$$

It follows from the definitions (4.43)–(4.45) that the phase space is bounded with $0 \leq \{x^2, y, z\} \leq 1$, subject to the constraint $1 - x^2 - y - z \geq 0$. The invariant set $1 - x^2 - y - z = 0$ corresponds to a zero axion field. The dynamics of the system (4.47)–(4.49) is determined primarily by the dynamics in the invariant sets $y = 0$ and $z = 0$. These correspond to a zero modulus field and a zero $\Lambda_R$, respectively. The dynamics is also determined by the fact that the right-hand side of Eq. (4.48) is positive–definite so that $y$ is a monotonically increasing function. This guarantees that there are no closed or recurrent orbits in the three-dimensional phase space.

### 4.3.2 Four–Dimensional Model

In the invariant set $y = 0$, where the modulus field is trivial, the system (4.47)-(4.49) reduces to the following plane system:

$$
\frac{dx}{d\Theta} = (x + \sqrt{3})[1 - x^2 - z] + \frac{1}{2} z[x - \sqrt{3}]
$$

$$
\frac{dz}{d\Theta} = 2z \left\{ [1 - x^2 - z] - \frac{1}{2} (1 - z - \sqrt{3}x) \right\}.
$$

The singular points and their associated eigenvalues are given by

$S_1$: $x = -1, z = 0$; \quad $\lambda_1 = 2(\sqrt{3} - 1)$, \quad $\lambda_2 = -(1 + \sqrt{3})$

$S_2$: $x = 1, z = 0$; \quad $\lambda_1 = -2(\sqrt{3} + 1)$, \quad $\lambda_2 = (\sqrt{3} - 1)$

$F$: $x = -\frac{1}{3\sqrt{3}}, z = \frac{16}{27}$; \quad $\lambda_{1,2} = \frac{1}{3} \pm \frac{i}{9} \sqrt{231}$. 


The points $S_1$ and $S_2$ are saddles and $F$ is a repelling focus.

In the invariant set $1 - x^2 - z = 0$, corresponding to the case of a zero axion field, Eqs. (4.50) and (4.51) reduce to the single ODE
\[
\frac{dx}{d\Theta} = \frac{1}{2} \left( 1 - x^2 \right) \left( x - \sqrt{3} \right),
\]
which can be integrated to yield an exact solution (in terms of $\Theta$ time).

4.3.3 Ten–dimensional Model

In the full system (4.47)-(4.49) with a non–trivial modulus field, there exists the isolated singular point $F : x = -1/(3\sqrt{3}), y = 0, z = 16/27$ with the associated eigenvalues $\lambda_{1,2}$ and $\lambda_3 = 4/3$. This implies that $F$ is a global source. All of the remaining singular points belong to the one–dimensional set
\[
W : y = 1 - x^2, z = 0 \ (x \text{ arbitrary})
\]
and the associated eigenvalues are given by
\[
\lambda_1 = -2\sqrt{3} \left( x + \frac{1}{\sqrt{3}} \right), \quad \lambda_2 = \sqrt{3} \left( x - \frac{1}{\sqrt{3}} \right), \quad \lambda_3 = 0.
\]
$W$ lies in the invariant set $z = 0$ on the boundary $y = 1 - x^2$. Points on $W$ with $x \in (-1/\sqrt{3}, 1/\sqrt{3})$ are local sinks, while the remaining points are saddles in the full three-dimensional phase space. (In the invariant set $z = 0$ singular points with $x \in [-1, -1/\sqrt{3}]$ are repelling and those with $x \in (-1/\sqrt{3}, 1]$ are attracting).

We note that there exists an exact solution of Eqs. (4.47)-(4.49) with non–trivial modulus field with
\[
x = -\frac{1}{3\sqrt{3}} = \text{constant},
\]
and
\[
y = -\frac{13}{8} \left( z - \frac{16}{27} \right),
\]
whence
\[
\frac{dz}{d\Theta} = \frac{9}{4} z \left( z - \frac{16}{27} \right),
\]
which can be integrated in terms of $\Theta$ time explicitly.

Finally, we could consider the case $\Lambda_R < 0$. The generalized Friedmann constraint equation (4.36) implies that
\[
\psi^2 - \Lambda_R \geq 3h^2 \geq 0,
\]
and we could therefore normalize using $\sqrt{\psi^2 - \Lambda_R}$. The resulting dynamical system can be analysed in a manner similar to that above (see [96]).
The mathematical structure of the dynamics is much richer when there is a positive RR charge. The dynamics of the system (4.50)–(4.51) is of interest from a mathematical point of view due to the existence of the cyclic behaviour. The orbits are future asymptotic to a heteroclinic cycle, consisting of the two saddle singular points \( S_1 \) and \( S_2 \) and the single (boundary) orbits in the invariant sets \( z = 0 \) and \( 1 - x^2 - z = 0 \) joining \( S_1 \) and \( S_2 \). The former set corresponds to the zero \( \Lambda_R \) solution and the latter to the solution with constant axion field. In a given ‘cycle’ an orbit spends a long time close to \( S_1 \) and then moves quickly to \( S_2 \) shadowing the orbit in the invariant set \( z = 0 \). It is then again quasi-stationary and remains close to the singular point \( S_2 \) before quickly moving back to \( S_1 \) shadowing the orbit in the invariant set \( 1 - x^2 - z = 0 \).

We stress that the motion is not periodic, and on each successive cycle a given orbit spends more and more time in the neighbourhood of the singular points \( S_1 \) and \( S_2 \). The exact solution corresponding to the singular point \( F \) is power-law and is defined over the range \( -\infty < t < 0 \), and represents a cosmology that collapses monotonically to zero volume at \( t = 0 \) in which the curvature and coupling are both singular [96].

The cyclical nature of the orbits can be physically understood by reinterpreting the axion field in terms of a membrane. Since the axion is constant on the surfaces of homogeneity, the field strength of the two–form potential must be directly proportional to the volume form of the three–space. If the spatial topology of the universe is that of an isotropic three–torus, the axion field can be formally interpreted as a membrane wrapped around this torus [94]. All orbits begin at \( F \). As the universe collapses, the membrane resists being squashed into a singular point. This forces the universe to bounce into an expansionary phase. The cosmological constant then dominates the axion field as the latter’s energy density decreases. The subsequent effect of the cosmological constant can be determined by viewing Eq. (4.4) in terms of a Brans–Dicke action, where the coupling parameter between the dilaton and graviton is given by \( \omega = -1 \) [95]; the late–time attractor then corresponds to a dilaton–vacuum solution. In effect, therefore, the cosmological constant resists the expansion and ultimately causes the universe to recollapse and asymptotically approach the saddle point \( S_1 \). On the other hand, the collapse causes the axion field to become relevant once more and a further bounce ensues. The process is then repeated with the universe undergoing a series of bounces. The orbits move progressively closer towards the two saddles, \( S_{1,2} \), and spend increasingly more time near to these points. This behaviour is related to the fact that the kinetic energy of the shifted dilaton field increases monotonically with time, since Eq. (4.33) implies that \( \ddot{\varphi} > 0 \). It would be interesting to consider the implications of this behaviour for the pre–big bang inflationary scenario [89].

When a modulus field is included \( (y \neq 0) \), \( F \) still represents the only source in the system. The orbits follow cyclical trajectories in the neighbourhood of the invariant set \( y = 0 \) and they spiral outwards monotonically around the orbit which
corresponds to the exact solution given by Eqs. (4.58)-(4.60), with $dy/d\Theta > 0$ from Eq. (4.48). After a finite (but arbitrarily large) number of cycles the kinetic energy of the modulus field becomes more important until a critical point is reached where it dominates the axion and cosmological constant. The orbits then asymptote toward the dilaton–moduli–vacuum solutions corresponding to sources on the line $W$.

A finite number of modulus fields and shear modes can be introduced into the model by defining the variable $N^2$ in Eqs. (4.41) and (4.44) via $N^2 = \sum_{i=1}^{n} N_i^2$. Orbits with non-trivial modulus field or shear term ‘shadow’ the orbits in the invariant set $y=0$ at early times and undertake cycles between the saddles on the singular set $W$ close to $S_1$ and $S_2$. These saddles may be interpreted as Kasner–like solutions [7]; the orbits thus experience a finite number of cycles in which the solutions interpolate between different Kasner-like states. This is perhaps reminiscent of the mixmaster behaviour that occurs in the Bianchi type VIII and IX cosmologies [7, 97]. Mixmaster oscillations also occur in less general (i.e., lower-dimensional) Bianchi models with a magnetic field [12] or Yang-Mills fields [98]. It is interesting to note in the string context that mixmaster behaviour also occurs in scalar-tensor theories of gravity in general and in Brans-Dicke theory in particular [99]. Unlike the mixmaster oscillations, however, the orbits eventually spiral away from $y=0$, although there are orbits that experience a finite but arbitrarily large number of oscillations.

Finally, we observe that all of the exact solutions corresponding to the singular points of the governing autonomous systems of ODEs are self-similar since in each case the scale factor is a power-law function of cosmic time. Therefore, exact self-similar solutions play an important rôle in determining the asymptotic behaviour in string cosmologies. However, we note that not all of the string solutions are asymptotically self-similar due to the existence of the heteroclinic cycle in the RR sector with trivial modulus field.

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