Product Integral Formalism
and
Non-Abelian Stokes Theorem

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Abstract
We make use of the properties of product integrals to obtain a surface product integral representation for the Wilson loop operator. The result can be interpreted as the non-abelian version of Stokes theorem.

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I Introduction

The main purpose of this work is to make use of product integrals to give two unambiguous proofs of the non-abelian version of Stokes theorem. The product integral formalism has been used extensively in the theory of differential equations and of matrix valued functions [1]. In the latter context, it has a built-in feature for keeping track of the order of the matrix valued functions involved. As a result, product integrals are ideally suited for the description of path ordered quantities such as holonomies. Moreover, since the theory is well developed independently of particular applications, we can be confident that the properties of such path ordered quantities which we establish using this method are correct and unambiguous.

Among the important advantages of the product integral representation of the path dependent exponential of a matrix valued function is that in such a framework the Banach space structure of the corresponding matrix valued functions is already

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built into the formalism. In particular, for a closed path enclosing an orientable 2-
surface, this will permit a surface representation of such operators. Based on the
central role of Stokes theorem in physics and in mathematics, it is not surprising that
the non-abelian version of this theorem has attracted a good deal of attention in the
physics literature [2]- [12]. The central features of the earlier attempts[2]- [8] have
been reviewed and improved upon in a recent work [9]. Other recent works on non-
abelian Stokes theorem [10, 11, 12] focus on specific problems such as confinement [11],
zig-zag symmetry [12] suggested by Polyakov [13], etc. With one exception [9], the
authors of these works seem to have been unaware of a 1927 work in the mathematical
literature by Schlessinger [14], which bears strongly on the content of this theorem.
Schlesinger’s work dealt with integrals of matrix valued functions and their ordering
problems. This amounts to establishing the non-abelian Stokes theorem in two (target
space) dimensions. By an appropriate extension and reinterpretation of his results,
we show using the product integral approach that this theorem is valid in any target
space dimension.

This work is organized as follows: To make this manuscript self-contained, we
review in Section II the main features of product integration [1] and state without
proof a number of theorems which will be used in the proof of the non-abelian Stokes
theorem. In Section III, we deal with path ordered exponentials of matrix valued
functions which can be expressed as product integrals and turn to the proof of the
non-abelian Stokes theorem for orientable surfaces. In section IV, we give a variant
of this proof. In section V, we explicitly demonstrate the gauge covariance of the
results obtained in sections III and IV. Finally, section is devoted to some additional
remarks.

II Some properties of product integrals

One of the initial motivations for the introduction of product integrals was [1] to solve
differential equations of the type
\[ Y'(s) = A(s)Y(s). \]  
(1)

In this expression, \( Y(s) \) is an n-dimensional vector, \( A(s) \) is a matrix valued function,
and prime indicates differentiation. So, for two real numbers \( a \) and \( b \), the problem
is to obtain \( Y(b) \) given \( Y(a) \). To deal with this problem, we make a partition \( P = \{ s_0, s_1, \ldots, s_n \} \) of the interval \([a, b]\). Let \( \Delta s_k = s_k - s_{k-1} \) for \( k = 1, \ldots, n \), and set
\( a = s_0, b = s_n \). Then, solving the differential equation in each subinterval, we can
write approximately [1]
\[ Y(b) \approx \prod_{k=1}^{n} e^{A(s_k)\Delta s_k}Y(a) \equiv \Pi_p(A)Y(a). \]  
(2)
Since $A(s)$ is matrix valued, the order in this product is important. Let $\mu(P)$ be the length of the longest $\Delta s_k$ in the partition $P$. Then, as $\mu(P) \to 0$, we get

$$Y(b) = \lim_{\mu(P) \to 0} \Pi_P(A)Y(a).$$  \hspace{1cm} (3)

The limit is clearly valid for all $Y(a)$.

The limit of the ordered product on the right hand side of Eq.(3) is the fundamental expression in the definition of a product integral [1]. It is formally defined, in an obvious notation, as

$$\prod_a^x e^{A(s)ds} = \lim_{\mu(P) \to 0} \Pi_P(A) \equiv F(x, a).$$  \hspace{1cm} (4)

It is easy to see that $F(x, a)$ satisfies the differential equation

$$\frac{d}{dx}F(x, a) = A(x)F(x, a)$$  \hspace{1cm} (5)

with $F(a, a) = 1$. The corresponding integral equation is

$$F(x, a) = 1 + \int_a^x ds \ A(s)F(s, a).$$  \hspace{1cm} (6)

Clearly, $F(a, a) = 1$, and $F(x, a)$ is unique.

Consider now some of the properties of the product integral matrices. For each $x \epsilon [a, b]$ the product integral is non-singular, and its determinant is given by

$$\det \left( \prod_a^x e^{A(s)ds} \right) = e^{\int_a^x \text{tr}A(s)ds}.$$  \hspace{1cm} (7)

In analogy with the additive property of ordinary integrals, product integrals have the multiplicative property, or the composition rule,

$$\prod_z^x e^{A(s)ds} = \prod_y^x e^{A(s)ds} \prod_z^y e^{A(s)ds}.$$  \hspace{1cm} (8)

Where $x, y, z \epsilon [a, b]$ and $z \leq y \leq x$. The result is independent of the choice of $y$ and any further decomposition of the products on the right hand side.

Derivatives with respect to the end points are given by

$$\frac{\partial}{\partial x} \prod_y^x e^{A(s)ds} = A(x) \prod_y^x e^{A(s)ds},$$  \hspace{1cm} (9)

and

$$\frac{\partial}{\partial y} \prod_y^x e^{A(s)ds} = -\prod_y^x e^{A(s)ds} A(y).$$  \hspace{1cm} (10)
One of the fundamental features associated with a connection is the notion of parallel transport. To see how it can be formulated in product integral formalism, consider a map \( P : [a, b] \to \mathbf{C}_{n \times n} \), which is continuously differentiable. Then \( P(x) \) is an indefinite product integral if for a given \( A(s) \)

\[
P(x) = \prod_a^x e^{A(s) \, ds} \, P(a). \tag{11}
\]

Next, we define an operation known as \( L \) operation which is like the logarithmic derivative operation on non-singular functions. Let

\[
LP(x) = P'(x) \, P^{-1}(x), \tag{12}
\]

where prime indicates differentiation. Then, from Eq. (11) it follows that

\[
(LP)(x) = A(x)
\]

One of the byproducts of this operation is that

\[
L(PQ)(x) = LP(x) + P(x)(LQ(x)) \, P^{-1}(x). \tag{13}
\]

The \( L \) operation is a crucial ingredient in establishing the analog of the fundamental theorem of calculus for product integrals. With the map \( P \) as defined above, this theorem states that

\[
\prod_a^x e^{LP(s) \, ds} = P(x) \, P^{-1}(a). \tag{14}
\]

From the results given above, it follows that \( P \) is a solution of the initial value problem

\[
P'(x) = (LP)(x) \, P(x). \tag{15}
\]

With the unique solution given by Eq. (11), this establishes the fundamental theorem of product integration. Just as in ordinary integration, the knowledge of simple product integrals can be used to evaluate more complicated product integrals. For example, one can prove the \textit{sum rule} for product integrals:

\[
\prod_a^x e^{[A(s) + B(s)] \, ds} = P(x) \prod_a^x e^{P^{-1}(s) \, B(s) \, P(s) \, ds}. \tag{16}
\]

Finally, we state two other important properties of product integrals which will be used in the sequel. One is the \textit{similarity theorem} which states that

\[
P(x) \prod_a^x e^{B(s) \, ds} \, P^{-1}(a) = \prod_a^x e^{[LP(s) + P(s) \, B(s) \, P^{-1}(s)] \, ds}. \tag{17}
\]

The other property is differentiation with respect to a parameter. Let

\[
P(x, y; \lambda) = \prod_y^x e^{A(s; \lambda) \, ds}, \tag{18}
\]

where \( \lambda \) is a parameter. Then the differentiation rule with respect to this parameter is given by

\[
\frac{\partial}{\partial \lambda} P(x, y; \lambda) = \int_y^x ds P(x, s; \lambda) \frac{\partial}{\partial \lambda} A(s; \lambda) \, P(s, y; \lambda). \tag{19}
\]
III The Non-abelian Stokes Theorem

To provide the background for using the product integral formalism of Section II to prove the non-abelian Stokes theorem, we begin with a statement of the problem as it arises in a physical context. Let $M$ be an $n$-dimensional manifold representing the space-time (target space). Let $A$ be a (connection) 1-form on $M$. When $M$ is a differentiable manifold, we can choose a local basis $dx^\mu$, $\mu = 1, \ldots, n$, and express $A$ in terms of its components:

$$A(x) = A_\mu(x) \, dx^\mu.$$  

We take $A$ to have values in the Lie-algebra, or a representation thereof, of a Lie group. Then, with $T_k$, $k = 1, \ldots, m$, representing the generators of the Lie group, the components of $A$ can be written as

$$A_\mu(x) = A^{k}_\mu(x) \, T_k.$$  

With these preliminaries, we can express the path ordered phase factor of the non-abelian gauge theories [15, 16, 17, 18] in the form

$$W_{ab}(C) = P \exp \int_C A.$$  

where $P$ indicates path ordering, and $C$ is a path in $M$. When the path $C$ is closed, the corresponding holonomy operator becomes:

$$W(C) = P \exp \oint_C A.$$  

(20)

The path $C$ in $M$ can be described in terms of an intrinsic parameter $\sigma$, so that for points $x^\mu$ of $M$ which lie on the path $C$, we have $x^\mu = x^\mu(\sigma)$. One can then write

$$A_\mu(x(\sigma))dx^\mu = A(\sigma)d\sigma,$$

where

$$A(\sigma) \equiv A^\mu(x(\sigma)) \frac{dx^\mu(\sigma)}{d\sigma}.$$  

It is the quantity $A(\sigma)$, and the variations thereof, which we will identify with the matrix valued functions of the product integral formalism.

Let us next consider the loop operator. For simplicity, we assume that $M$ has trivial first homology group with integer coefficients, i.e., $H_1(M, \mathbb{Z}) = 0$. This insures that the loop may be taken to be the boundary of a two dimensional surface $\Sigma$ in $M$. More explicitly, we take the 2-surface to be an orientable submanifold of $M$. It will be convenient to describe the properties of the 2-surface in terms of its intrinsic parameters $\sigma$ and $\tau$ or $\sigma^a$, $a = 0, 1$. So, for the points of the manifold $M$, which lie on $\Sigma$, we have $x = x(\sigma, \tau)$. The components of the 1-form $A$ on $\Sigma$ can be obtained by means of the vielbeins (by the standard pull-back construction):

$$v^\mu_a = \partial_a \, x^\mu(\sigma).$$
Thus, we get
\[ A_a = v_a^\mu A_\mu. \]
The curvature 2-form \( F \) of the connection \( A \) is given by
\[ F = dA + A \wedge A = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu. \]
The components of \( F \) on \( \Sigma \) can again be obtained by means of the vielbeins:
\[ F_{ab} = v_a^\mu v_b^\nu F_{\mu\nu}. \]

We want to express the loop operator in terms of the product integral definition in a specific way. To achieve this, we begin with the definition of the path ordered phase factor in terms of a product integral. Consider the continuous map \( A : [s_0, s_1] \to \mathbb{C}_{n \times n} \) where \([s_0, s_1]\) is a real interval. Then, we define the non-abelian phase factor given above in terms of a product integral as follows:
\[ P e^{\int_{s_0}^{s_1} A(s)ds} \equiv \prod_{s_0}^{s_1} e^{A(s)ds}. \]
In particular, anticipating that we will identify the closed path \( C \) over which the Wilson loop is defined with the boundary of a 2-surface, it is convenient to work from the beginning with the matrix valued functions \( A(\sigma, \tau) \). This means that our expression for the path ordered phase factor will depend on a parameter. That is, let
\[ A : [\sigma_0, \sigma_1] \times [\tau_0, \tau_1] \to \mathbb{C}_{n \times n}, \]
where \([\sigma_0, \sigma_1]\) and \([\tau_0, \tau_1]\) are real intervals on the two surface \( \Sigma \) and hence in \( M \). Then, we define the path ordered phase factor
\[ P(\sigma, \sigma_0; \tau) = \prod_{\sigma_0}^{\sigma} e^{A_1(\sigma':\tau')d\sigma'} \equiv P e^{\int_{\sigma_0}^{\sigma} A_1(\sigma';\tau')d\sigma'}. \]
In this expression, \( P \) indicates path ordering with respect to \( \sigma \), as defined by the product integral, while \( \tau \) is a parameter. To be able to describe such an operator for a closed path, we similarly define the path dependent operator
\[ Q(\sigma; \tau, \tau_0) = \prod_{\tau_0}^{\tau} e^{A_0(\sigma;\tau')d\tau'} \equiv P e^{\int_{\tau_0}^{\tau} A_0(\sigma;\tau')d\tau'}. \]
In this case, the path ordering is with respect to \( \tau \), and \( \sigma \) is a parameter.

To prove the non-abelian version of the Stokes theorem, we want to make use of product integration techniques to express the holonomy loop operator as an integral over a two dimensional surface bounded by the corresponding loop. In terms of the intrinsic coordinates of such a surface, we can write this loop operator in the form
\[ W(C) = P e^{\oint A_a ds^a}, \]
where, as mentioned above,
\[ \sigma^a = (\tau, \sigma); \quad a = (0, 1). \tag{25} \]

The expression for the loop operator depends only on the homotopy class of paths in \( M \) to which the closed path \( C \) belongs. We can, therefore, parameterize the path \( C \) in any convenient manner consistent with its homotopy class. In particular, we can break up the path into piece wise continuous segments along which either \( \sigma \) or \( \tau \) remains constant. The composition rule for product integrals given by Eq. (8) ensures that this break up of the closed loop into a number of segments does not depend on the intermediate points on the closed path, which are used for this purpose. So, we write the closed loop operator as
\[ W = W_4 W_3 W_2 W_1, \tag{26} \]

In this expression, \( W_k, k = 1, \ldots, 4, \) are Wilson lines such that \( \tau = \text{const.} \) along \( W_1 \) and \( W_3, \) and \( \sigma = \text{const.} \) along \( W_2 \) and \( W_4. \) We emphasize that the expressions \( \sigma = \text{const.} \) and \( \tau = \text{const.} \) represent arbitrary curves.

To see the advantage of parameterizing the closed path in this manner, consider the exponent of Eq. (24):
\[ A_a d\sigma^a = A_0 d\tau + A_1 d\sigma. \tag{27} \]

Along each segment, one or the other of the terms on the right hand side vanishes. For example, along the segment \([\sigma_0, \sigma],\) we have \( \tau' = \tau_0 = \text{const.} \). Recalling Eqs. (22) and (23), we get for the segments \( W_1 \) and \( W_2, \) respectively,
\[ W_1 = \prod_{\sigma_0}^{\sigma} e^{A_1(\sigma'; \tau_0) d\sigma'} \equiv \mathcal{P} e^{\int_{\sigma_0}^{\sigma} A_1(\sigma'; \tau_0) d\sigma'} = P(\sigma, \sigma_0; \tau_0), \tag{28} \]

and
\[ W_2 = \prod_{\tau_0}^{\tau} e^{A_0(\sigma; \tau') d\tau'} \equiv \mathcal{P} e^{\int_{\tau_0}^{\tau} A_0(\sigma; \tau') d\tau'} = Q(\sigma; \tau_0). \tag{29} \]

When the 2-surface \( \Sigma \) requires more than one coordinate patch to cover it, the connections in different coordinate patches must be related to each other in their overlap region by transition functions [16]. Then, the decomposition given in Eq. (26) must be suitably augmented to take this complication into account. The product integral representation of the path ordered phase factor and the composition rule for product integrals given by Eq. (8) will still make it possible to describe the corresponding loop operator as a composite product integral. For definiteness, we will confine ourselves to the representation given by Eq. (26).

It is convenient for later purposes to define two composite Wilson line operators \( U \) and \( T \) according to
\[ U(\sigma, \tau) = Q(\sigma; \tau_0) P(\sigma, \sigma_0; \tau), \tag{30} \]
\[ T(\sigma; \tau) = P(\sigma, \sigma_0; \tau) Q(\sigma_0; \tau, \tau_0). \] (31)

Using the first of these, we have

\[ W_2 W_1 = U(\sigma, \tau). \] (32)

Similarly, we have for \( W_3 \) and \( W_4 \)

\[ W_3 = P^{-1}(\sigma, \sigma_0; \tau), \] (33)

and

\[ W_4 = Q^{-1}(\sigma_0; \tau, \tau_0). \] (34)

From the Eq. (31), it follows that

\[ W_4 W_3 = T^{-1}(\sigma, \tau). \] (35)

Appealing again to Eq. (8) for the composition of product integrals, it is clear that this expression for the Wilson loop operator is independent of the choice of the point \((\sigma, \tau)\). In terms of the quantities \(T\) and \(U\), the closed loop operator will take the compact form

\[ W = T^{-1}(\sigma; \tau) U(\sigma; \tau). \] (36)

As a first step in the proof of the non-abelian Stokes theorem, we obtain the action of the \(L\)-derivative operator on \(W\):

\[ L_\tau W = L_\tau \left[ T^{-1}(\sigma, \tau) Q(\sigma; \tau, \tau_0) P(\sigma, \sigma_0; \tau_0) \right]. \] (37)

Using the definition of the \(L\)-operation given by Eq. (12), noting that \(P(\sigma, \sigma_0; \tau_0)\) is independent of \(\tau\), and carrying out the \(L\) operations on the right hand side (RHS), we get

\[ L_\tau W = L_\tau T^{-1}(\sigma, \tau) + T^{-1}(\sigma, \tau) \left[ L_\tau Q(\sigma; \tau, \tau_0) + Q(\sigma; \tau, \tau_0)(L_\tau P(\sigma, \sigma_0; \tau_0))Q^{-1}(\sigma; \tau, \tau_0) \right] T(\sigma, \tau). \] (38)

Simplifying this expression by means of Eqs. (12) and (13), we end up with

\[ L_\tau W = T^{-1}(\sigma, \tau) [A_0(\sigma, \tau) - L_\tau T(\sigma, \tau)] T(\sigma, \tau). \] (39)

Next, we prove the analog of Eq. (14), which applies to an elementary product integral, for the composite loop operator defined by Eqs. (26) and (36).

**Theorem 1** The loop operator given by Eq. (36) can be expressed in the form

\[ W = \prod_{\tau_0}^{\tau} e^{T^{-1}(\sigma, \tau') [A_0(\sigma, \tau') - L_\tau T(\sigma, \tau')]} T(\sigma, \tau') d\tau'. \] (40)
To prove this theorem, first we note from the definition of the $L$ operation that the right hand side (RHS) of this equation can be written as

$$RHS = \prod_{\tau_0} e^{[T^{-1}(\sigma, \tau') A_0(\sigma, \tau') - T^{-1}(\sigma, \tau') \frac{\partial}{\partial \tau'} T(\sigma, \tau')] \frac{\partial}{\partial \tau} T(\sigma, \tau')] d\tau'}. \quad (41)$$

Noting that $-T^{-1} \partial_\tau T = L_\tau T$, we can use the similarity theorem given by Eq. (17) to obtain

$$RHS = T^{-1}(\sigma; \tau) \prod_{\tau_0} e^{A_0(\sigma, \tau') d\tau'} T(\sigma; \tau_0). \quad (42)$$

Moreover, making use of the defining Eq. (23), we get

$$RHS = T^{-1}(\sigma; \tau) Q(\sigma; \tau, \tau_0) P(\sigma, \sigma_0; \tau_0) Q(\sigma_0; \tau_0) = T^{-1}(\sigma; \tau) U(\sigma; \tau). \quad (43)$$

The last line is clearly the expression for $W$ given by Eq. (36).

Finally, we want to express the quantity $W$ in yet another form which we state as:

**Theorem 2** The loop operator defined in Eq. (36) can be expressed as a surface integral of the field strength:

$$W = \prod_{\tau_0} e^{\int_{\sigma_0}^{\sigma} T^{-1}(\sigma', \tau') F_{01}(\sigma', \tau') T(\sigma', \tau') d\sigma' d\tau'}. \quad (44)$$

where $F_{01}$ is the 0-1 component of the non-Abelian field strength.

To prove this theorem, we note that

$$\frac{\partial}{\partial \sigma} [T^{-1}(\sigma, \tau) A_0(\sigma, \tau) T(\sigma, \tau)] = T^{-1}(\sigma, \tau) [\partial_\sigma A_0(\sigma, \tau) + [A_0(\sigma, \tau), A_1(\sigma, \tau)]] T(\sigma, \tau). \quad (45)$$

Moreover,

$$\frac{\partial}{\partial \sigma} \{T^{-1}(\sigma, \tau) (L_\tau T(\sigma, \tau)) T(\sigma, \tau)\} = T^{-1}(\sigma, \tau) \partial_\tau A_1(\sigma, \tau) T(\sigma, \tau). \quad (46)$$

It then follows that

$$\frac{\partial}{\partial \sigma} \{T^{-1}(\sigma, \tau) [A_0(\sigma, \tau) - L_\tau T(\sigma, \tau)]\} T(\sigma, \tau)\}
= T^{-1}(\sigma, \tau) [\frac{\partial}{\partial \sigma} A_0(\sigma, \tau) - \frac{\partial}{\partial \tau} A_1(\sigma, \tau) + [A_0(\sigma, \tau), A_1(\sigma, \tau)]] T(\sigma, \tau)
= T^{-1}(\sigma, \tau) F_{01}(\sigma, \tau) T(\sigma, \tau). \quad (47)$$

The last step follows from the definition of the field strength in terms of the connection given above

$$F_{01} = \frac{\partial}{\partial \sigma} A_0(\sigma, \tau) - \frac{\partial}{\partial \tau} A_1(\sigma, \tau) + [A_0(\sigma, \tau), A_1(\sigma, \tau)]. \quad (48)$$
Integrating Eq. (47) with respect to $\sigma$, we get
\[
T^{-1}(\sigma, \tau)[A_0(\sigma, \tau) - L_\tau T(\sigma, \tau)]T(\sigma, \tau) = \int_{\sigma_0}^\sigma T^{-1}(\sigma', \tau')F_{01}(\sigma', \tau')T(\sigma', \tau')d\sigma'd\tau'.
\] (49)

We thus arrive at the surface integral representation of the loop operator [19]:
\[
W = \prod_{\tau_0}^\tau e^{\frac{1}{2}\int_\Sigma d\sigma^a T^{-1}(\sigma, \tau)F_{ab}(\sigma, \tau)T(\sigma, \tau)},
\] (50)

We note that in this expression the ordering of the operators is defined with respect to $\tau$ whereas $\sigma$ is a parameter. Recalling the antisymmetry of the components of the field strength, we can rewrite this expression in terms of path ordered exponentials familiar from the physics literature:
\[
W = \mathcal{P}_\tau e^{\frac{1}{2}\int_\Sigma d\sigma^a T^{-1}(\sigma, \tau)F_{ab}(\sigma, \tau)T(\sigma, \tau)},
\] (51)
where $d\sigma^a$ is the area element of the 2-surface. Despite appearances, it must be remembered that $\sigma$ and $\tau$ play very different roles in this expression.

IV A Second Proof

To illustrate the power and the flexibility of the product integral formalism, we give here a variant of the previous proof for the non-Abelian Stokes theorem. This time the proof makes essential use of the non-trivial relation (19) for product integrals.

We start with the form of $W$ given in Eq. (36) and take its derivatives with respect to $\tau$:
\[
\frac{\partial W}{\partial \tau} = \partial_\tau Q^{-1}(\sigma_0; \tau, \tau_0)P^{-1}(\sigma, \sigma_0; \tau)Q(\sigma, \sigma_0; \tau_0)P(\sigma, \sigma_0; \tau_0) + \frac{1}{2}\int_{\sigma_0}^\sigma d\sigma' \partial_\tau A_1(\sigma', \tau) \partial_\tau P(\sigma, \sigma_0; \tau)A_1(\sigma, \sigma_0; \tau_0)P(\sigma, \sigma_0; \tau_0). \tag{52}
\]

Here, we have made use of the fact that $P(\sigma, \sigma_0; \tau_0)$ is independent of $\tau$. Applying Eq. (12) to $W$ and using Eq. (31), we get
\[
L_\tau W = \frac{\partial W}{\partial \tau} W^{-1} = T^{-1}(\sigma, \tau) [A_0(\sigma, \tau) - P(\sigma, \sigma_0; \tau)A_0(\sigma_0, \tau)P^{-1}(\sigma, \sigma_0; \tau) - \partial_\tau P(\sigma, \sigma_0; \tau)P^{-1}(\sigma, \sigma_0; \tau)]T(\sigma, \tau). \tag{53}
\]

Now we can use Eq. (19) to evaluate the derivative of the product integral with respect to the parameter $\tau$:
\[
\partial_\tau P(\sigma, \sigma_0; \tau) = \int_{\sigma_0}^\sigma d\sigma' P(\sigma, \sigma'; \tau) \partial_\tau A_1(\sigma', \tau) P(\sigma', \sigma_0; \tau). \tag{54}
\]
Then, after some simple manipulations using the defining equations for the various terms in Eq. (53), we get:

\[
T^{-1}(\sigma; \tau)\partial_\tau P(\tau)P^{-1}(\tau)T(\sigma; \tau) = \int_{\sigma_0}^\sigma d\sigma'T^{-1}(\sigma'; \tau)\partial_\tau A_1(\sigma'; \tau)T(\sigma'; \tau). \tag{55}
\]

Using Eq. (9) and the fact that \( P(\sigma_0, \sigma_0; \tau) = 1 \), we can write the rest of Eq. (53) as an integral too:

\[
T^{-1}(\sigma; \tau)[A_0(\sigma; \tau) - P(\sigma, \sigma_0; \tau)A_0(\sigma_0; \tau)P^{-1}(\sigma, \sigma_0; \tau)]T(\sigma; \tau) =
\int_{\sigma_0}^\sigma d\sigma' P^{-1}(\sigma', \sigma_0; \tau)(\partial_\tau A_0(\sigma', \tau) + [A_0(\sigma', \tau), A_1(\sigma', \tau)])P(\sigma', \sigma_0; \tau). \tag{56}
\]

Combining Eqs. (53), (55), and (56), we obtain:

\[
L_\tau W = \frac{\partial W}{\partial \tau}W^{-1} = \int_{\sigma_0}^\sigma d\sigma'T^{-1}(\sigma', \tau)F_{01}(\sigma', \tau)T(\sigma', \tau). \tag{57}
\]

Finally, recalling Eq. (14), we are immediately led to Eq. (50) which was obtained by the previous method of proof.

There are two reasons for the relative simplicity of this proof over the one which was given in the previous section. One is due to the use of differentiation with respect to a parameter according the Eq. (19). The other is due to the use of Eq. (14) for the composite operator \( W \). In the first proof, the use of this theorem for \( W \) was not assumed. Its justification for using it in the second proof lies in the composition law for product integrals given by Eq. (8).

V Gauge Covariance of the result

As a consistency check, we must show that the surface representation given by Eq. (50) is gauge covariant. To this end, it will be recalled that under a gauge transformation, the components of the connection, i.e. the gauge potentials, transform according to [20]

\[
A_\mu(x) \rightarrow g(x)A_\mu(x)g^{-1}(x) - g(x)\partial_\mu g(x)^{-1}. \tag{58}
\]

The components of the field strength (curvature) transform covariantly:

\[
F_{\mu\nu}(x) \rightarrow g(x)F_{\mu\nu}(x)g^{-1}(x). \tag{59}
\]

From these, it follows that [20]

\[
\mathcal{P}e^{\int_a^b A_\mu(x)dx^\mu} \rightarrow g(b) \left( \mathcal{P}e^{\int_a^b A_\mu(x)dx^\mu} \right) g^{-1}(a), \tag{60}
\]

\[
\]
Equivalently, from its definition (22) in terms of product integrals, it is easy to show that the gauge transform of the quantity $P(\sigma, \sigma_0; \tau)$ has the form

$$g(\sigma; \tau)g^{-1}(\sigma_0; \tau)\prod_{\sigma_0} e^{g(\sigma_0; \tau) A_1(\sigma'; \tau')g^{-1}(\sigma_0; \tau)}. \quad (61)$$

To show the gauge covariance of the surface representation, we need to know how the operator $T(\sigma, \tau)$ transforms under gauge transformations. To this end, we note that the Wilson line $Q(\sigma; \tau, \tau_0)$ given by Eq. (23) transforms as

$$Q(\sigma; \tau, \tau_0) = \prod_{\tau_0} e^{A_0(\sigma; \tau')}d\tau' \longrightarrow g(\sigma; \tau)Q(\sigma; \tau, \tau_0)g^{-1}(\sigma_0; \tau_0). \quad (62)$$

Then, the gauge transform of the composite operator $T(\sigma, \tau)$ given by Eq. (31) follows immediately:

$$T(\sigma; \tau) = P(\sigma, \sigma_0; \tau) Q(\sigma_0; \tau, \tau_0) \longrightarrow g(\sigma; \tau)T(\sigma; \tau)g^{-1}(\sigma_0; \tau_0). \quad (63)$$

From the above results, it is straightforward to show that the surface integral representation of Wilson loop transforms as

$$W \longrightarrow \prod_{\tau_0} e^{\int_{\tau_0}^{\tau} T^{-1}(\sigma; \tau') F_{01}(\sigma; \tau') T(\sigma; \tau')dt'}g^{-1}(\sigma_0; \tau_0). \quad (64)$$

It follows from the composition rule (8) that the constant factors in the exponent factorize, so that under gauge transformations the surface representation of transforms covariantly, i.e.,

$$W \longrightarrow g(\sigma_0; \tau_0) \prod_{\tau_0} e^{\int_{\tau_0}^{\tau} T^{-1}(\sigma; \tau') F_{01}(\sigma; \tau') T(\sigma; \tau')dt'}g^{-1}(\sigma_0; \tau_0). \quad (65)$$

We view this result as a nontrivial confirmation of our proofs.

**VI Concluding Remarks**

We have provided two proofs of the non-abelian Stokes theorem using the product integral method. An immediate question which comes to mind is whether there is a supersymmetric generalization of this theorem. Given the important developments in supersymmetric gauge theories in recent years, this question is not merely of academic interest. To explore this possibility using the product integral method, it is necessary to generalize this method to encompass Grassmann valued operators. It turns out that such a generalization is indeed possible [21]. Further developments of this subject will be reported in a forthcoming work.
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