A9

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

CERN LIBRARIES, GENEVA

SCAN-9910061

SISSA 53/99/FM

TURAEV-VIRO INVARIANT AND 3N-J SYMBOLS

Gaspere Carbone

International School for Advanced Studies.

Via Beirut 2-4, 34013 Trieste, Italy

E-mail: carbone@sissa.it

Abstract.

We propose in this work a new method to construct the Turaev-Viro state sums, using a diagrammatic presentation describing surgery operation as Heegaard-splittings. The resulting invariants can be connected with suitable 3nj-symbols, and we evaluate them for the lens spaces.

1 Introduction

In their seminal paper, Ponzano and Regge [1], pointed out that it is possible to reproduce the partition function for 3-dimensional gravity by using the semiclassical limit of 6j-symbols. A coherent algebraic and geometric approach explaining the rationale behind Ponzano-Regge model come with Turaev-Viro in [2]. Their construction draws together the deep intuition of Ponzano-Regge and the newly developed representation theory of quantum groups into the construction of a topological invariant, $|M|_{TV}$ of a closed 3-manifold M .

The Ponzano-Regge and Turaev-Viro approach focus on PL-geometry and group representation theory; however one can equivalently discuss the quantization of

3d-gravity from the vantage point offered by Chern-Simons Theory (Witten in [3], [4]). This latter approach has been formalized by Reshetikhin-Turaev in [5] and further developed by Melvin-Kirby in [6]. By Using surgery operations their analysis provides a new topological invariant of closed orientable 3-manifolds, $\tau(M)$ related to the Turaev-Viro invariant by

$$|M|_{TV} = \tau(M)\tau(\bar{M}),$$

where $\tau(\bar{M})$ denotes the invariant evaluated on M with the opposite orientation. The issue of the relation between $|M|_{TV}$ and $\tau(M)$ has been extensively discussed by Turaev in [7], [8], by Roberts in [9], [10], [11], by Beliakova and Durhuus in [12] using the spin-networks formalism and by Mizoguchi and Tada in [13], considering the perturbative development of the quantum 6j-symbols around the parameter $q = 1$.

These two facets of 3-d quantum gravity suggest that a formulation of the theory which involves both the T-V and R-T approach can provide a deeper understanding of the geometry involved. In this paper we develop the necessary theory for such a unification. The high-point is perhaps a rather elementary derivation of the T-V invariant.

Section II sets the stage of the paper, focusing on the characterization of links in the T-V formalism. We will construct the associated link invariant which turns out to be the corresponding Homfly polynomial [14], [12]. Such an extension of the theory permits to introduce the notion of observable in the T-V context. We also explain the Heegaard splitting definition of the T-V invariant.

Section III reviews the basic of Dehn surgery, in particular the construction of 3-manifolds and the corresponding action of the Rolfsen moves.

The final section focuses on rewriting the surgery process in terms of the Heegaard splitting, by considering the action of the surgery maps on the handlebodies defining the manifold. We will define a simple diagrammatic presentation of 3-manifolds. This latter procedure will give us the explicit expression of the invariant for all 3-dimensional lens spaces, and will allow us to connect it to 3nj-symbols.

To my father

2 Invariants of links in the framework of the Turaev model

In the original formulation given by Turaev and Viro in [2] links were absent, a gap filled only later on by Turaev himself in [14]. Our starting point is the notion of fat graph in a compact 3-manifold N . By this we mean a finite graph whose vertices and edges are extended to small 2-disks and *narrow* bands respectively. We will consider here only 3-valent fat graphs equipped with colors given by the assignement to edges of non negative integer or half-integer lying between 0 and $(r-2)/2$, where r is the deformation parameter of the quantum group associated to the theory. In our case we consider $SL_q(2, C)$ with $q = \exp \frac{2\pi i}{r}$ [14]. Note that the model without links will turn out to reproduce the usual form of the Turaev-Viro invariant.

Fix a commutative ring K and call K^* the subgroup of invertible element of K .

Let I be a given set, $i \rightarrow \omega_i$ given functions, $I \rightarrow K^*$ and an element ω of K^* . A G -tuple will be admissible if his triplets are admissible, *adm*, and a triplet is admissible if, in the explicitly realization of this context using a quantum group, its elements satisfy the triangular inequality. To every six-tuple we associate the symbol $\left| \begin{smallmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{smallmatrix} \right| \in K$.

The initial data defined in this way satisfy the conditions:

a) $\forall j_1, j_2, \dots, j_6 \in I$ such that $(j_1 j_3 j_4)$, $(j_2 j_4 j_5)$, $(j_1 j_3 j_6)$ and $(j_2 j_5 j_6)$ are admissible triples, we have

$$\sum_j \omega_j^2 \left| \begin{smallmatrix} j_2 & j_1 & j \\ j_3 & j_5 & j_4 \end{smallmatrix} \right| \left| \begin{smallmatrix} j_3 & j_4 & j_6 \\ j_2 & j_5 & j \end{smallmatrix} \right| = \delta_{j_4 j_6}; \quad (1)$$

b) $\forall a, b, c, e, f, j_1, j_2, j_3, j_{23} \in I$, such that $(j_{23} a e j_1 f b)$ e $(j_3 j_2 j_{23} b f c)$ are admissible, we have

$$\sum_j \omega_j^2 \left| \begin{smallmatrix} j_2 & a & j \\ j_1 & c & b \end{smallmatrix} \right| \left| \begin{smallmatrix} j_3 & j & e \\ j_1 & f & c \end{smallmatrix} \right| \left| \begin{smallmatrix} j_3 & j_2 & j_{23} \\ a & e & j \end{smallmatrix} \right| = \left| \begin{smallmatrix} j_{23} & a & e \\ j_1 & f & b \end{smallmatrix} \right| \left| \begin{smallmatrix} j_3 & j_2 & j_{23} \\ b & f & c \end{smallmatrix} \right|; \quad (2)$$

c) $\forall j \in I$ we have

$$\omega^2 = \omega_j^{-2} \sum_{k,l:(jkl) \in amm.} \omega_k^2 \omega_l^2; \quad (3)$$

To such initial data we add another function $q_i \in K^*$, which satisfies the relation

$$\sum_{j_{13} \in I} \omega_{j_{13}}^2 q_{j_{13}} \left| \begin{smallmatrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{smallmatrix} \right| \left| \begin{smallmatrix} j_2 & j_3 & j_{23} \\ j_1 & j & j_{13} \end{smallmatrix} \right| = q_j q_{j_1} q_{j_2} q_{j_3} q_{j_{12}}^{-1} q_{j_{23}}^{-1} \left| \begin{smallmatrix} j_3 & j_2 & j_{23} \\ j_1 & j & j_{123} \end{smallmatrix} \right|. \quad (4)$$

From this latter, the following important relations are obtained

$$\left| \begin{smallmatrix} j_3 & g & e \\ j_1 & d & c \end{smallmatrix} \right| = \left| \begin{smallmatrix} j_1 & c & g \\ j_3 & e & d \end{smallmatrix} \right| = \sum_{j_{13} \in I} \omega_{j_{13}}^2 q_e q_c q_{j_1} q_{j_3} q_b^{-1} q_g^{-1} q_{j_{13}}^{-1} \left| \begin{smallmatrix} j_1 & j_3 & j_{13} \\ c & e & d \end{smallmatrix} \right| \left| \begin{smallmatrix} c & j_1 & g \\ j_3 & e & j_{13} \end{smallmatrix} \right|, \quad (5)$$

$$\sum_{j \in I} \omega_j^2 (q_a q_j q_b^{-2})^\varepsilon \begin{vmatrix} i & b & j \\ i & b & a \end{vmatrix} = q_i^{2\varepsilon} \quad (6)$$

where $\varepsilon = \pm 1$, and

$$\sum_{\substack{k \\ (i, j, k) \in adm}} \omega_k^2 = \omega_i^2 \omega_j^2. \quad (7)$$

If we consider the quantum initial data [14], namely if we explicitly realize the model using a quantum group, the functions, defined before, will assume the value:

$$\begin{aligned} \omega^2 &= -2r/(q^{1/2} - q^{-1/2})^2 \\ \omega_j &= (-1)^{2i} [2j + 1]^{1/2} \\ \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} &= (\sqrt{-1})^{-2(\sum j_i)} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}_q^{RW} \end{aligned} \quad (8)$$

$$q_i = \exp(\pi \sqrt{-1} (i - i(i+1)r^{-1})).$$

where $[k] = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}}$ and $\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}_q^{RW}$ is the quantum Racah-Wigner symbols.

The Turaev-Viro invariants can be generalized to the form $|M, F|_\mu$, with F a certain union of components of ∂M ,

$$|M, F|_\mu = \omega^{-2\alpha + \beta'} \prod_{e''} \omega_\mu(e'') \prod_e \omega_{\mu(e)}^2 \prod_T |T|_\mu \quad (9)$$

where α is the number of vertices of M , β' the number of vertices of $\partial M \setminus F$, e runs over the edges of M , which do not lie in ∂M , e'' runs over the edges of $\partial M \setminus F$ and T runs over all 3-simplices of M .

A coloring of a 3-valent graph φ is a function which associates an element of the set I with each edge of φ . This assignment is such that, for any vertex of φ incident to 3 (resp. 2) edges of φ , the colors of these edges form an admissible triple (resp. are equal to each other). Each fat graph Γ has a core $c(\Gamma)$ which is an ordinary graph consisting of edges and vertices.

If F is any compact surface, a fat graph in the cylinder $F \times [-1, 1]$ may be represented by graph diagrams on F containing only double transversal crossings of edges (provided with an additional structure showing the undercrossings and overcrossings). Let now F be an oriented compact surface and let φ and ψ two colored 3-valent graphs embedded in F . Let Γ be an oriented colored fat graph in $F \times [-1, 1]$ and D its graph diagram on F . We may assume that φ , ψ and D lie in general position so that all crossings of $\varphi \cup \psi \cup D$ are double transversal crossings of edges. We derive a graph diagram from $\varphi \cup \psi \cup D$ assuming that φ lies everywhere over $\psi \cup D$, and ψ lies everywhere under $\varphi \cup D$. Denote the resulting graph diagram on F by σ and denote by Σ the graph in F obtained from σ ignoring the over/under crossing information. The set of vertices of Σ may be split into five subsets:

- 1) the 2-valent vertices of φ , ψ , D ;
- 2) the 3-valent vertices of φ , ψ , D ;
- 3) crossings of φ with ψ ;
- 4) crossings of D with φ or ψ ;
- 5) self-crossings of D .

A *region* of D with respect to φ and ψ is a connected component of $F \setminus \Sigma$ and an *area-coloring* of D is an arbitrary mapping from the set of the regions of D into

the set I . An *area-coloring* η of D is called admissible if for each edge e of Σ , the color of e together with the η -colors of the two regions of D adjacent to e form an admissible triple. Denote the set of admissible area-colorings by $adm(D)$. With each $\eta \in adm(D)$ we will associate an element $|D|_\eta$ of the ring K . For a region y of D we set

$$|y|_\eta = \omega_{\eta(y)}^{2\chi(y)} \quad (10)$$

where $\eta(y)$ and $\chi(y)$ are respectively the η -color and the Euler characteristic of y .

With respect to the previous classification we have five possible ways of characterizing a vertex $a \in \Sigma$, namely:

1) $|a|_\eta = 1$;

Fig. 1: Σ vertices configurations.

2)

$$|a|_\eta = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \quad (11)$$

where i, j, k denote the colors of the three edges of Σ incident in a and l, m, n are the η -colors of the opposite regions;

3) the same expression used as in 2) where l is the color of the upper branch and

i that of lower branch; moreover j, k, m, n are the η -colors of the four regions of D incident in a ;

4)

$$|a|_\eta = q_k^{1/2} q_n^{1/2} q_j^{-1/2} q_m^{-1/2} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}. \quad (12)$$

5)

$$|a|_\eta = q_k q_n q_j^{-1} q_m^{-1} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}. \quad (13)$$

Finally, let us define the following quantity:

$$\langle \varphi | D | \Psi \rangle_\eta = \prod_y |y|_\eta \prod_a |a|_\eta \quad (14)$$

where y runs over all regions of D and a runs over all vertices of Σ . The state sum

$$\langle \varphi | \Gamma | \Psi \rangle = \sum_{\eta \in \text{am}(D)} \langle \varphi | D | \Psi \rangle_\eta \in K \quad (15)$$

turns out to be an invariant both under ambient isotopies of φ and ψ in F , and under isotopies of Γ in $F \times [-1, 1]$ [14]. If F is a disjoint union of n surfaces $F_1, \dots, F_n$ then we can extend (14) and (15) according to:

$$\langle \varphi | \Gamma | \Psi \rangle = \prod_{k=1}^n \langle \varphi_k | \Gamma_k | \Psi_k \rangle \quad (16)$$

where φ_k, ψ_k are the components of φ, ψ lying on F_k , and Γ_k is the part of Γ lying in $F_k \times [-1, 1]$, ($\forall k = 1, 2, \dots, n$) [14].

We can define now the invariants of links on a generic manifold. Let N be a compact 3-manifold with a triangulated boundary ∂N and let Γ be a 3-valent

colored fat graph lying in $IntN$. We set $F = \partial U$, where U is an oriented closed regular neighborhood of Γ in N . Consider a non-singular normal vector field on the surface of Γ which, together with the fixed orientation of this surface, determines uniquely the orientation of U . Shifting Γ along this vector field we get a parallel copy Γ' of Γ lying on F . U is a handlebody consisting of 3-balls and solid cylinders. Choose in each of these cylinders a meridian disk which lies transversal with respect to the corresponding band of Γ . Let $\psi_1, \dots, \psi_m$ be the boundaries of discs obtained in this way, where m is the number of edges of Γ . The former loops can be considered as graphs with one vertex and one edge. We color them with a sequence $J = (j_1, \dots, j_m) \in I^m$ and set $\omega_J = \prod_{k=1}^m \omega_{j_k}^2$. Let M be the compact 3-manifold $N \setminus U$ bounded by $\partial M = F \cup \partial N$. We provide M with an arbitrary triangulation, which extends the given triangulation of ∂N ; also F is equipped with the induced triangulation and let s be the number of vertices of Γ . Then for each $\lambda \in col(\partial N)$ we define a relative invariant of the pair N, Γ , with respect to λ :

$$\langle N, \Gamma | \lambda \rangle = \omega^{2-2s} \sum_{\mu \in col(K), \mu|_{\partial N} = \lambda, J \in I^m} \omega_J |M, F|_{\mu} \langle \gamma_F^{\mu_F} | \Gamma' | \Psi_J \rangle \quad (17)$$

where γ_F is the dual graph of the 1-skeleton of the triangulation of F and $\mu_F = \mu|_F$.

2.1 Heegaard-splitting

In this subsection we introduce invariants on Heegaard diagrams of a closed 3-manifolds [14]. Recall that an Heegaard surface in a closed 3-manifold N is a closed connected oriented surface $F \subset N$ which splits N into the union of two handlebodies U and V , bounded by F . We distinguish these handlebodies

assuming that the orientation of F , together with the normal vector field on F directed outwards U , defines uniquely the orientation of N . Let $\varphi_1, \dots, \varphi_g$ (resp. $\psi_1, \dots, \psi_g$) be the boundaries of a system of meridian disks of V (resp. of U), where g is the genus of F . The surface F , together with these sets of loops, is a Heegaard diagram of N . We will treat the loops $\varphi_1, \dots, \psi_g$ as graphs on F , each one of them having just one vertex and one edge. Denote by ψ_J the colored graph on F obtained from $\psi_1 \cup \dots \cup \psi_g$ by assigning the coloring $j_1, \dots, j_g$ to the edges of $\psi_1, \dots, \psi_g$ respectively; a similar definition holds true for φ_H . Therefore if Γ is a colored 3-valent fat graph lying in $F \times [-1, 1] \subset N$ we obtain the invariant

$$\langle N, \Gamma' \rangle = \omega^{-2} \sum_{\substack{J = (j_1 \dots j_g) \in I^g \\ H = (h_1 \dots h_g) \in I^g}} \prod_{i=1}^r \omega_{j_i}^2 \prod_{k=1}^g \omega_{h_k}^2 \langle \varphi_k | \Gamma' | \psi_j \rangle. \quad (18)$$

where Γ' is defined as before.

3 Dehn surgery

A manifold M can be understood as the union of several components glued together by some given identification of the points on their boundaries. If these components are glued in a different way, one may find a new manifold M' . In this case, we say that M' can be obtained from M by means of surgery. As is well known, any closed, orientable and connected 3-manifold can be obtained by surgery from the 3-sphere S^3 [15]. In this section we give a brief review of the surgery operations in S^3 . In particular, we concentrate on Dehn surgery performed along knots or links in S^3 . Recall that the points of $T^2 = S^1 \times S^1$, if we consider the complex representation of S^1 , have like coordinates $(e^{i\theta_1}, e^{i\theta_2})$

and the maps $e^{i\phi} \rightarrow (e^{i\phi}, 1)$ and $e^{i\phi} \rightarrow (1, e^{i\phi})$ define a longitude and a meridian respectively. The homotopy class $[f]$ of a map $f : S^1 \rightarrow T^2$, can be written in the longitude-meridian basis as $[f] = (a, b)$. The longitude class is $(1, 0)$ and the meridian class is $(0, 1)$. The class (a, b) of a knot in T^2 has necessarily a and b relatively prime. Let us recall that a homeomorphism h of a generic space X is called an ambient isotopy if h is the end map h_1 of an homotopy $h_t : X \rightarrow X$ such that h_0 is the identity and h_t is an homeomorphism $\forall t \in [0, 1]$. The group of orientation preserving self-homeomorphism of T^2 , modulo ambient isotopy, is generated by the longitudinal and meridional twists $h_l(a, b) = (a+b, b)$, $h_m(a, b) = (a, a+b)$; this group is isomorphic to $SL(2, \mathbb{Z})$. Two knots C_1 and C_2 in T^2 are ambient isotopic if and only if $[C_1] = \pm[C_2]$. In order to describe Dehn surgery operations, we need to consider solid tori. A solid torus is a 3-dimensional space V homeomorphic with $S^1 \times D^2$. A given homeomorphism $h : S^1 \times D^2 \rightarrow V$ is called a framing of V . Given a tubular neighborhood N of a knot C and a framing h of N , the longitude $h(S^1 \times 1)$ of N defines a framing C_f of C , which is a preferred framing if the linking number of C and C_f is equal to zero. The longitude $\lambda = h(S^1 \times 1)$, defined by a preferred framing h of N , is oriented in the same way as C , and the meridian $\mu = h(1 \times \partial D^2)$ is oriented in such a way that its linking number with C is equal to $+1$. We say that the longitude λ and the meridian μ of N are the homotopy generators of a Rolfsen basis in ∂N and any class $[f] \in \pi_1(\partial N)$ is expressed in this basis as

$$[f] = a \cdot [\lambda] + b \cdot [\mu] = (a, b).$$

A self-homeomorphism τ_+ of N , which is an extension in N of a meridian twist $h_m : \partial N \rightarrow \partial N$, is called a right-handed meridian twist of N ; the inverse homeomorphism τ_- is called a left-handed twist of N . The maps $\tau_{\pm}$ act on a generic link L in $S^3 \setminus N$, where N is the interior of N , simply by twisting the band through a $\pm 2\pi$ rotation. In general, if there are n components in the link L , the action of τ_+ is described by the element $(\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$ of the braid group B_n , $\{\sigma_i\}$ being its generators. It is clear that $\tau_{\pm}$ do not modify the number of the components of a link L in the solid torus, but the linking numbers between the different components can be changed.

A Dehn surgery performed along a knot Z in S^3 can be described in the following way

- 1) first remove the interior N , of a tubular neighborhood N of the knot Z , from S^3 ;
- 2) consider $S^3 \setminus N$ and N as distinct spaces whose boundary $\partial(S^3 - N)$ and ∂N are tori;
- 3) glue back N and $S^3 - N$ by identifying the points on their boundaries with a given homeomorphism $h : \partial N \rightarrow \partial(S^3 - N)$.

The knot Z and the glueing homeomorphism h completely specify the surgery operation and the resulting manifold is denoted by

$$M = (S^3 - N) \bigcup_h N. \quad (19)$$

Actually, the manifold (19) depends (up to homeomorphism) only upon the homotopy class of $h(\mu)$ in $\partial(S^3 - N)$, where μ is a meridian of N . The surgery is characterized then by the knot Z and by a closed curve $Y \in \partial N$ represent-

ing $h(\mu)$. The convention introduced by Rolfsen in order to codify the surgery instruction consist in choosing:

$$Y = a \cdot [\lambda] + b \cdot [\mu] = (a, b),$$

where the generators λ and μ are the longitude and the meridian of a Rolfsen basis in ∂N . The ratio $r = b/a$ is called the surgery coefficient. In conclusion, the surgery instructions are specified simply by the knot Z in S^3 and by the rational surgery coefficient r . Clearly, the surgery operation of removing and gluing back a solid torus can be iterated. Therefore, a general surgery instruction consists in the assignement of an unoriented link L in S^3 with given surgery coefficients $\{r_i\}$ associated with its components $\{L_i\}$. For example, when L is a circle with surgery coefficient $r = b/a$, the resulting space is homeomorphic with the lens space $L(a, b)$.

Two manifolds associated with different surgery instructions are homeomorphic if and only if the two surgery instruction are related by a finite sequence of Rolfsen moves [15] [16]. A Rolfsen move of the first type state amounts to add or delete a component of the surgery link L with surgery coefficient $r = \infty$. A Rolfsen move of the second type describes the effects of an appropriate twist homeomorphism $\tau_{\pm}$ on the surgery instruction.

Let L be a surgery link such that one of its components, say L_1 , is a circle with surgery coefficient r_1 . This means that all the remaining components L_j , with $j \neq 1$, belong to the complement solid torus V_1 of L_1 in S^3 . Under a twist homeomorphism $\tau_{\pm}$ of V_1 , the component L_1 is not modified, whereas L_j are

changed according to $\tau_{\pm} : L_j \rightarrow L'_j$. Moreover the surgery coefficients become:

$$r'_1 = \frac{1}{1/r_1 \pm 1} \quad (20)$$

$$r'_j = r_j \pm [\chi(L_j, L_1)]^2 \text{ for } j \neq 1, \quad (21)$$

where $\chi(L_j, L_1)$ is the linking number of L_j and L_1 .

When the surgery coefficient r is an integer, one can take $a = 1$ and $b = r$; in this case, the curve Y is a longitude of N and can be interpreted as a framing of the surgery knot Z .

4 Reformulation of the Turaev–Viro invariants

The main purpose of this section is to rewrite the Surgery construction in terms of the Turaev formalism. We start from the T–V invariant expression in Heegaard splitting framework

$$\langle N \rangle = \omega^{-2} \sum_{\substack{J = (j_1 \dots j_g) \in I^g \\ H = (h_1 \dots h_g) \in I^g}} \prod_{i=1}^g \omega_{j_i}^2 \prod_{k=1}^g \omega_{h_k}^2 \langle \varphi_k | 0 | \psi_j \rangle, \quad (22)$$

where φ_k and ψ_j are the meridians of the two handlebodies generating the manifolds. Let us consider the case in which the surgery link consists of many distinct loops with generic framings. This is a sufficiently general situation which can be used to represents a great number of manifolds by exploiting the Kirby relations. In this situation $\langle \varphi_k | 0 | \psi_j \rangle$ becomes $\prod_i \langle \varphi_{k_i} | 0 | \psi_{j_i} \rangle$ where everything lives on a single torus, the regular neighborhood of each link component. It is necessary now to consider what happens to the torus in the Heegaard–splitting configuration as a consequence of the surgery operation. In the following we shall consider several

distinct cases:

4.1 $S^2 \times S^1$

We start with the case $r = 0$, where the surgery operation sends a meridian into a longitude on the surface; the curves that realize the quantity $\langle \varphi_k | 0 | \psi_j \rangle$ are a meridian of the torus (transformed) and a meridian of the complement respectively. The starting graph is showed in figure (2) where we consider the

Fig. 2: Graph associated to the surgery operation with $r = 0$ in which we consider the trasformed torus as handle-body.

transformed torus. If we had considered the initial torus instead, the meridian of the complement would have been the curve that sticks to the meridian of the torus under surgery operation. Since the stick operation forces us to associate longitude to meridian, the graph that we must consider is shown in figure (3): it represents the torus meridian and the meridian of the complement after the action of the surgery operation. The value of the invariant associated to this graph is

$$\omega^{-2} \sum \omega_i^2 \omega_j^2 \sum_{\substack{ab \\ (iab) \\ (jab)}} 1, \quad (23)$$

Fig. 3: Graph associated to the surgery operation with $r = 0$ in which we consider the torus before the transformation, namely the object associated to the S^3 splitting.

since the surfaces that the two links bound on the torus surface have Euler number equal to zero. We rewrite the two sums in (23) according to:

$$\begin{aligned}
 \sum_{ij} \omega_i^2 \omega_j^2 \sum_{\substack{ab \\ (iab) \in adm \\ (jab) \in adm}} 1 &= \sum_{ab} \sum_{\substack{ij \\ (iab) \in adm \\ (jba) \in adm}} \omega_i^2 \omega_j^2 = \\
 &= \sum_{ab} \sum_{\substack{i \\ (iab) \in adm}} \omega_i^2 \sum_{\substack{j \\ (jab) \in adm}} \omega_j^2 = \sum_{ab} \left(\sum_{\substack{i \\ (iab) \in adm}} \omega_i^2 \right)^2.
 \end{aligned} \tag{24}$$

Using the relation (7) we obtain:

$$\omega^{-2} \sum_{ab} \omega_a^4 \omega_b^4 = \omega^2$$

which is exactly the T-V invariant for $S^2 \times S^1$ [2].

4.2 S^3

As is well known S^3 can be obtained with a surgery operation along a link with framing equal to one: the meridian of the torus is sent into a longitude that

become knotted once along the surface. The graph associated to this situation is depicted in figure (4), where we can see the curve that goes trough one meridian and one longitude before closing. The corresponding expression for the invariants reads:

Fig. 4: Graph associated to the surgery operation with $r = 1$.

$$\omega^{-2} \sum \omega_i^2 \omega_j^2 \sum_a \omega_a^2 \begin{vmatrix} i & a & a \\ j & a & a \end{vmatrix}. \quad (25)$$

Now we consider $\langle S^3 \rangle$ in its expression coming from (17):

$$\sum_{\mu j} \omega_j^2 |MF|_{\mu} \langle \gamma_F^{\mu} | 0 | \psi_j \rangle = \langle S^3 \rangle, \quad (26)$$

where the term $\langle \gamma_F^{\mu} | 0 | \psi_j \rangle$ of $\langle S^3 \rangle$ corresponds exactly to:

$$\sum_a \omega_a^2 \begin{vmatrix} j & a & a \\ \mu & a & a \end{vmatrix} \quad (27)$$

and which represents the graph showed in figure (5).

thus, by sing the expression of $|MF|_{\mu}$, we get:

$$\langle S^3 \rangle = \omega^{-2} \sum_{\mu j} \omega_j^2 \omega_{\mu}^2 \sum_a \omega_a^2 \begin{vmatrix} j & a & a \\ \mu & a & a \end{vmatrix} \quad (28)$$

Fig. 5: Graph associated to S^3 in his definition of invariant.

which represents the desired result.

4.3 The lens space $L(2, 1)$

In the $L(2, 1)$ case the associated graph is depicted in the figure (6), where we can see the curve that goes trough one meridian and two longitudes before closing. To this graph we associate the quantity

Fig. 6: Graph associated to $L(2, 1)$

$$\omega^{-2} \sum \omega_a^2 \omega_b^2 \left| \begin{array}{ccc} i & a & b \\ j & a & b \end{array} \right| \left| \begin{array}{ccc} i & b & a \\ j & b & a \end{array} \right| \quad (29)$$

Carrying out the sum on i and using the first axiom on the structure of the initial data we obtain that the invariant, expressed in the Heegaard–splitting framework,

is equal to:

$$\omega^{-2} \sum_a \omega_a^2 \omega_b^2 \sum_{\substack{i \\ (iab)}} = \omega^{-2} \sum_i \sum_a \omega_a^2 \sum_{\substack{b \\ (iab)}} \omega_b^2. \quad (30)$$

Accordin to relation (7), we eventually obtain the formula

$$\omega^{-2} \sum_i \omega_i^2 \sum_a \omega_a^4, \quad (31)$$

which gives exactly the T-V invariant for $L(2, 1)$ [2].

4.4 The lens space $L(3, 1)$

Let us consider now $L(3, 1)$. In this case, the surgery link has a framing $r = 3$, so the meridian of the complement is sent into a curve which goes trough one meridian and tree longitudes before closing, so we obtain the graph of figure (7), and the associated invariant is expressed by the quantity:

Fig. 7: Graph associated to $L(3, 1)$.

$$\omega^{-2} \sum_a \omega_a^2 \omega_b^2 \omega_c^2 \left| \begin{array}{ccc} i & a & b \\ j & c & b \end{array} \right| \left| \begin{array}{ccc} i & c & b \\ j & c & a \end{array} \right| \left| \begin{array}{ccc} i & c & a \\ j & b & a \end{array} \right|. \quad (32)$$

Carrying out the sum on i and by using the second axiom we obtain

$$\omega^{-2} \sum_j \omega_j^2 \omega_a^2 \omega_b^2 \omega_c^2 \left| \begin{array}{ccc} j & j & j \\ c & a & b \end{array} \right| \left| \begin{array}{ccc} j & j & j \\ a & b & c \end{array} \right|. \quad (33)$$

Finally, upon summation over b and using the relation (1), we get as in [2]

$$\omega^{-2} \sum_j \sum_{\substack{ac \\ (jjj)}} \omega_a^2 \omega_c^2 = \sum_j \omega_j^2. \quad (34)$$

4.5 The lens space $L(n, 1)$

$L(n, 1)$ is obtained by a surgery along a link with framing $r = n$; the meridian of the complement solid torus is sent into a curve that goes through one meridian and n longitudes of the torus surface before closing, the associated graph is given by the figure (8). The expression of the invariant becomes

Fig. 8: Graph associated to the surgery operation with $r = n$.

$$|L(n, 1)| = \omega^{-2} \sum \omega_i^2 \omega_j^2 \prod_i \omega_{j_i}^2 \begin{vmatrix} i & j_1 & j_2 \\ j & j_n & j_2 \end{vmatrix} \dots \begin{vmatrix} i & j_n & j_1 \\ j & j_{n-1} & j_1 \end{vmatrix} \quad (35)$$

and using the relation defining the $3n-j$ symbols [17] we get

$$\begin{aligned} & \left[\begin{array}{cccccc} a_1 & a_2 & \dots & a_n \\ & b_1 & & b_2 & \dots & b_n \\ c_1 & c_2 & \dots & c_n \end{array} \right]_q = \sum_z (-1)^{S+nz} [2z+1] \left\{ \begin{array}{ccc} a_1 & c_1 & z \\ c_2 & a_2 & b_1 \end{array} \right\}_q \times \\ & \times \left\{ \begin{array}{ccc} a_2 & c_2 & z \\ c_3 & a_3 & b_2 \end{array} \right\}_q \dots \left\{ \begin{array}{ccc} a_n & c_n & z \\ c_1 & a_1 & b_n \end{array} \right\}_q \end{aligned} \quad (36)$$

where, S denotes the sum over all the $3n$ arguments and

$$\left\{ \begin{array}{ccc} a_1 & c_1 & z \\ c_2 & a_2 & b_1 \end{array} \right\}_q = (-1)^{(a_1+a_2+b_1+c_1+c_2+z)} \left| \begin{array}{ccc} a_1 & c_1 & z \\ c_2 & a_2 & b_1 \end{array} \right|.$$

We can rewrite the expression of the invariant in terms of $3nj$ -symbols

$$|L(n, 1)| = \omega^{-2} \sum [2j+1] \prod_i [2j_i+1] \left[\begin{array}{cccccc} j_1 & j_2 & \cdots & & j_n & \\ & j & & j & & \cdots & j \\ j_n & & j_1 & \cdots & & j_{n-1} & \end{array} \right]_q. \quad (37)$$

5 The lens space $L(p, q)$

We generalize now the procedure, introduced in the previous section, to the generic 3-dimensional lens space $L(p, q)$ where p, q are relatively prime, $(p, q) = 1$. We can write them $L(nq+f, q)$ with n integer and $(f, q) = 1$ with $q > f$. We have seen in section 3 that, in the surgery representation, the meridian of the tubular neighborhood of a surgery link is sent into a curve which goes through p meridians and q longitudes before closing. The graphs associated to these configurations are quite more complicated than the ones considered before. Let us consider first the graphs and the invariants in some particular and then their generic expression.

5.1 The lens space $L(2n+1, 2)$

Put $q = 2$ and consider as an example of this class $L(5, 2)$, with the associated graph shown in figure (9). In order to construct this graph we put three points on each horizontal line and five points on each vertical one at the same distance from each other, connect among themselves the first point of the horizontal lines, the second point of bottom horizontal line with the first point of vertical left line and the other in pairs with lines “parallel” to this one. It is easy to see that

Fig. 9: Graph associated to $L(5, 2)$.

the graph represents the intersection of two links, one of them going through two meridians and five longitudes before closing. Using the laws described above, the expression of the invariant is

$$\begin{aligned}
 |L(5, 2)| &= \omega^{-2} \sum \omega_i^2 \omega_j^2 \prod_i \omega_{j_i}^2 \left| \begin{array}{ccc} i & j_4 & j_5 \\ j & j_2 & j_1 \end{array} \right| \left| \begin{array}{ccc} i & j_5 & j_1 \\ j & j_3 & j_2 \end{array} \right| \times \\
 &\times \left| \begin{array}{ccc} i & j_1 & j_2 \\ j & j_4 & j_3 \end{array} \right| \left| \begin{array}{ccc} i & j_2 & j_3 \\ j & j_5 & j_4 \end{array} \right| \left| \begin{array}{ccc} i & j_3 & j_4 \\ j & j_1 & j_5 \end{array} \right|, \quad (38)
 \end{aligned}$$

which, using (36), can be rewritten in term of 15-j symbol

$$|L(5, 2)| = \omega^{-2} \sum [2j + 1] \prod_i [2j_i + 1] \left[\begin{array}{ccccc} j_4 & j_5 & j_1 & j_2 & j_3 \\ & j & j & j & j \\ j_1 & j_2 & j_3 & j_4 & j_5 \end{array} \right]_q. \quad (39)$$

Starting from the expression (38) and using some basic relations involving 6j symbols [18], it is possible to recover easily the value given in [19]. We will give, in the appendix, the explicitly computation for this example and for a similar one. From this analysis we can recover the following expression of the invariant for the class $L(2n + 1, 2)$, (see fig (10)),

Fig. 10: Graph associated to $L(2n + 1, 2)$.

$$|L(2 + 1n, 1)| = \omega^{-2} \sum [2j + 1] \prod_i [2j_i + 1] \times$$

$$\times \begin{bmatrix} j_{2n} & j_{2n+1} & j_1 & \cdots & j_{2n-1} \\ j & j & j & \cdots & j \\ j_1 & j_2 & j_3 & \cdots & j_{2n+1} \end{bmatrix}_q. \quad (40)$$

5.2 The lens spaces $L(3n + 1, 3)$ and $L(3n + 2, 3)$

For $q = 3$, we have two classes $L(3n + 1, 3)$ and $L(3n + 2, 3)$. As an example of the first we consider $L(7, 3)$, and for the second $L(8, 3)$. The graph associated to $L(7, 3)$ is shown in figure (11). The link goes through tree meridians and seven

Fig. 11: Graph associated to $L(7, 3)$.

longitudes before closing. The associated expression for the invariant is

$$|L(7, 3)| = \omega^{-2} \sum \omega_i^2 \omega_j^2 \prod_i \omega_{j_i}^2 \begin{vmatrix} i & j_5 & j_6 \\ j & j_2 & j_1 \end{vmatrix} \times \quad (41)$$

$$\times \begin{vmatrix} i & j_6 & j_7 \\ j & j_3 & j_2 \end{vmatrix} \begin{vmatrix} i & j_7 & j_1 \\ j & j_4 & j_2 \end{vmatrix} \cdots \begin{vmatrix} i & j_4 & j_5 \\ j & j_1 & j_7 \end{vmatrix}.$$

Note that, by using the relation (36), it can be rewritten in terms of 21j symbol

$$|L(7, 3)| = \omega^{-2} \sum [2j + 1] \prod_i [2j_i + 1] \times \quad (42)$$

$$\times \begin{bmatrix} j_5 & j_6 & j_7 & j_1 & \cdots & j_4 & \\ j & j & j & j & j & \cdots & j \\ j_1 & j_2 & j_3 & j_4 & \cdots & j_7 & \end{bmatrix}_q.$$

It is also possible to obtain the value given in [19], using a sequence of relation described in [18]. We are now in condition of constructing for $L(3n + 1, 3)$, the associated graph, (figure (12)), and the corresponding invariant given explicitly

Fig. 12: graph associated to $L(3n + 1, 3)$.

by the formula

$$|L(3n + 1, 3)| = \omega^{-2} \sum [2j + 1] \prod_i [2j_i + 1] \times \quad (43)$$

$$\times \begin{bmatrix} j_{3n-1} & j_{3n} & j_{3n+1} & j_1 & \cdots & j_{3n-2} & \\ j & j & j & j & j & \cdots & j \\ j_1 & j_2 & j_3 & j_4 & \cdots & j_{3n+1} & \end{bmatrix}_q.$$

Let us now consider $L(8, 3)$, the associated graph is shown in figure (13) and

Fig. 13: Grap associated to $L(8, 3)$.

constructed using the usual procedure.

The corresponding expression of the T-V invariant, can be written, by exploiting (36), in term of a 24j-symbols, namely

$$|L(8, 3)| = \omega^{-2} \sum [2j + 1] \prod_i [2j_i + 1] \times$$

$$\times \left[\begin{array}{ccccccccc} j_6 & j_6 & j_8 & j_1 & \cdots & j_5 \\ j & j & j & j & j & \cdots & j \\ j_1 & j_2 & j_3 & j_4 & \cdots & j_8 \end{array} \right]_q. \quad (44)$$

Also in this case it is possible to obtain the value given in [19].

Finally we can associate to the lens space $L(3n + 2, 3)$ the graph shown in figure (14) with the invariant expression given by

$$|L(3n + 2, 3)| = \omega^{-2} \sum [2j + 1] \prod_i [2j_i + 1] \times$$

$$\times \left[\begin{array}{ccccccccc} j_{3n} & j_{3n+1} & j_{3n+2} & j_1 & \cdots & j_{3n-1} \\ j & j & j & j & j & \cdots & j \\ j_1 & j_2 & j_3 & j_4 & \cdots & j_{3n+2} \end{array} \right]_q. \quad (45)$$

Fig. 14: Graph associated to $L(3n + 2, 3)$.

5.3 The lens space $L(nq + f, q)$

Drawing from these examples we are now able to construct the graph associated to a generic 3 dimensional lens space and his related invariant. The strategy is to put $qn + f$ point on each vertical line, at the same distance from each other, $q + 1$ points on each horizon line and then connect among themselves the first point of the horizontal lines; the first point of the vertical left line with the second one of the bottom horizontal line and the other in pair with “parallel” lines to this. The latter graph obtained is shown in figure (15) where $\bar{f}$ is the value of f such that $(q, f) = 1$, and the expression of the invariant can be written in terms of $3(nq + \bar{f})j$ symbols with entries position depending from q , in the form

$$|L(nq + \bar{f}, q)| = \omega^{-2} \sum [2j + 1] \prod_i [2j_i + 1] \times$$

$$\times \left[\begin{array}{cccccccc} j_{nq+\bar{f}-q+1} & j_{nq+\bar{f}-q+2} & \cdots & j_{nq+\bar{f}} & j_1 & \cdots & j_{nq+\bar{f}-q} & \\ j & j & j & j & j & \cdots & j & \\ j_1 & j_2 & \cdots & j_q & j_{q+1} & \cdots & j_{nq+\bar{f}} & j \end{array} \right]_q \quad (46)$$

Fig. 15: Graph associated to $L(nq + \bar{f}, q)$.

6 Conclusions

We have proposed a new diagrammatic approach, using the Heegaard–splitting, for evaluating the Turaev–Viro invariants of 3-manifolds. We have, explicitly, constructed the invariants for lens spaces in terms of the $3n-j$ symbols of the II kind. With this method one can construct explicitly the T-V invariant for 3-manifolds which are connected sum of lens spaces, namely manifolds with disconnected surgery simple links. Obviously, in order to obtain a complete construction of the invariant, for a generic 3-manifold, it is necessary to implement, in this context, the Rolfsen moves. Such a procedure, if constructively implemented, would give us the possibility to define the “observable” link invariants. We are working in this direction but, in such a case, it is more difficult to extract the information from the diagrams. It is perhaps worth stressing that we have put in evidence

the 2-dimensional nature of the Turaev–Viro theory obtaining the invariants from the configuration of two links on the torus surface. This fact, together with the possibility, [20], to obtaining the Turaev–Viro invariants using rational conformal field theory, may shed a new light on the nature of the T-V theory.

Acknowledgments

The author would like to thank M. Carfora and A. Marzuoli for their encouragement and their help to draft the manuscript.

7 Appendix

In this appendix we want to show how it is possible to recover from our approach the Ionicioiu–Williams results [19]. We discuss only the simplest examples: $L(4, 1)$ and $L(5, 2)$. Recall that the T-V invariant is given by a sum, over a coloring, of a particular configuration of $3nj$ symbols of II type; to perform the calculus we use the diagrammatic method [18]. The diagrammatic representation of this symbol is given in figure (16) and for the examples we are considering, the

Fig. 16: Diagrammatic representation of $3nj$ symbol of II kind.

diagrams as in figure (17).

For $L(4, 1)$ we can start using relation (10), pag. 455 in [18], which gives the sum in terms of three diagrams, like those showed in figure (18). On the first of them we can use relation (6), pag. 454 in [18], obtaining the expression of the invariant in term of four $6j$ symbols. We then perform the sum over j_3 ,

Fig. 17: Diagrammatic representation of $L(4,1)$ and $L(5,2)$.

Fig. 18: First step in $L(4,1)$ calculation.

involving only three $6j$ -symbols, using the B-E identity. We get, in this way, three $6j$ symbols, one involving only j, k , $\begin{vmatrix} j & j & k \\ j & j & k \end{vmatrix}$ and the other a square of $\begin{vmatrix} j & j & k \\ j_2 & j_4 & j_1 \end{vmatrix}$. Summing over j_i and using the orthogonal relation between $6j$ symbols and the definition of ω^2 , eventually we obtain the value given in [19].

For $L(5,2)$ we can use the relation (15), pag. 457 in [18] to obtain the invariant expressed in terms of four diagram (figure (19)). We use, for the first diagram, the

Fig. 19: First step in $L(5,2)$ calculation.

relation (6), pag. 454 in [18]; we get in this way a decomposition of the invariant in terms of six $6j$ symbols. Now we can perform the sum over j_5 , involving three $6j$ symbols, and, using the B-E identity, we obtain $\left| \begin{smallmatrix} j & x & x \\ j & j & y \end{smallmatrix} \right|$ and $\left| \begin{smallmatrix} j & x & x \\ j_1 & j_3 & j_4 \end{smallmatrix} \right|$. Carrying out the sum over j_2 , (which implies, using the orthogonal condition, $x = y$) and then summing over j_i , we obtain the value provided by Ionicioiu-Williams in [19].

Bibliography

- [1] G. Ponzano, T. Regge, *Semiclassical limit of Racah coefficients. In: Spectroscopic and group theoretical methods in physics*. North-Holland, Amsterdam, 1968.
- [2] V. G. Turaev, O. Y. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols. Topology. 31: 865; 1992.*
- [3] E. Witten, *2+1 dimensional gravity as an exactly soluble. Nucler Physics B311: 46; 1988/1989.*
- [4] E. Witten, *Topology-changing amplitudes in 2+1 dimensional gravity. Nucler Physics B323: 113; 1989.*
- [5] N. Yu. Reshetikhin, V. G. Turaev, *Invariant of 3-manifolds via link polynomials and quantum groups. Invent. Math. 103: 547; 1991.*
- [6] R. Kirby, P. Melvin, *Evaluations of the 3-manifolds invariants of Witten and Reshetikhin-Turaev for $sl(2, \mathbb{C})$. In: Geometry of low dimensional Manifolds. Durham 1989 (Lond. Math. Soc. Lect. Note Ser.) Cambridge: Cambridge University Press 1990.*

- [7] V. G. Turaev, *Topology of shadows*. preprint; 1992.
- [8] V. G. Turaev, *Quantum invariants of knots and 3-manifold. Expositions in Math.* **186**; 1994. W. de Gruyter, Berlino.
- [9] J. Roberts, *Skein theory and Turaev-Viro invariant*. preprint; 1993. To appear in *Topology*.
- [10] J. Roberts, *Refined state-sum invariants of 3 and 4-manifolds*. preprint; 1994.
- [11] R. Benedetti, C. Petronio, *On Roberts' Proof of the Turaev-Walker Theorem*. *Quaderni:sezione di geometria e algebra. Dip. Mat. Pisa*; 1994.
- [12] A. Beliakova, B. Durhuus, *Topological Quantum field theory and invariants of graphs for quantum group*. *Comm. Math. Phys.* **167**: 395; 1995.
- [13] S. Mizoguchi, T. Tada, *Three-dimensional gravity from the Turaev-Viro invariant*. *Phys. Rev. Lett.* **68** (12): 1795; 1992.
- [14] V. G. Turaev, *Quantum invariants of links and 3-valent graphs in 3-manifolds*. *Inst. Hautes Études Sci. Publ. Math.* **77**: 121; 1992.
- [15] D. Rolfsen, *Knots and links*. Publish or Perish, 1976.
- [16] R. Kirby, *A calculus for framed links in S^3* . *Invent. Math.* **45**: 35; 1978.
- [17] M. Nomura, *Various kind of relations for $3n-j$ symbols of quantum $SU_q(2)$* . *J. Phy. Soc. Jap.* **59**: 3851; 1990.
- [18] D. A. Varshalovich, A. N. Moskalev, V. K. Khersonskii, *Quantum theory of angular momentum*. World Scientific publishing co., 1988.

- [19] R. Ionicioiu, R. M. Williams, *Lens space and handlebodies in 3d quantum gravity*. Preprint gr-qc/9806027.
- [20] S. Mizoguchi, *Rational conformal field theory and multiwormhole partition function in three-dimensional gravity*. *Int. J. Mod. Phys. A***8**: 3909; 1993.