Quantum Field Theory of Topological Defects as Inhomogeneous Condensates

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1. Introduction

Topological defects play an important rôle in many physical systems ranging from cosmology to condensed matter. Thus they link apparently unrelated areas characterized by very different energy and time scales.

The issue of spontaneous defect formation during symmetry breaking phase transitions has recently attracted much attention [1]. As originally pointed out by Kibble [2] and more recently by Zurek [3], different regions of a system may be unable to correlate during the quench time which characterizes the transition and, as a result, some parts of space may remain trapped in the original (symmetric) phase, giving rise to topological defects.

Although the Kibble–Zurek mechanism gives a reasonable estimate of the defect density as a function of the quench time (as confirmed, for instance, by recent experiments on superfluid Helium [4]), this picture is essentially phenomenological. It is clear that a full understanding of the process of defect formation requires a full quantum field theoretical formulation of the problem. There has recently been much progress in this direction [5], and here we give some novel results based on the approach which we are currently developing. A more systematic account will be presented in two forthcoming papers [6,7].

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We are inspired by the work done in the 70's by Umezawa et al. [8], who showed how solitons can arise in QFT as result of a localized (inhomogeneous) condensation of particles. In this picture the extended objects have an inherently quantum origin. The corresponding classical soliton solutions are then obtained in the \( \hbar \to 0 \) limit.

Following these ideas we construct a QFT of topological defects using the Closed–Time–Path (CTP) formalism [9]. The latter is a vital ingredient of our construction, as it has a natural extension to finite temperature and to non–equilibrium situations (which is relevant for example in realistic phase transitions).

The paper is organized as follows: in Section 2, we give a brief account of CTP formalism and of some fundamentals of QFT which are essential to what follows. In Section 3, we consider a simple model, namely \( \lambda \psi^4 \) in \( 2D \) (i.e.\( 1+1 \)), showing how the kink solution can be constructed at zero and finite temperature. In Section 4, we discuss the vortex solution for \( 4D \) complex \( \lambda \psi^4 \) theory. We conclude in Section 5.

2. The Haag expansion in Closed–Time–Path formalism

Let us consider the dynamics of an Heisenberg field described by

\[
\dot{\psi}(x) = i[H, \psi(x)] \quad , \quad \dot{\Pi}(x) = i[H, \Pi(x)] ,
\]

where \( \Pi \) is the momentum conjugate to \( \psi \) and \( H \) is the full (renormalized) Hamiltonian in the Heisenberg picture. Assuming that the Heisenberg and interaction pictures coincide at some time \( t_i \), we can write the formal solution of Eqs.(1) as (“in” denotes quantities in the interaction picture)

\[
\psi(x) = Z^{\psi}_{\chi} e^{-i\int_{t_i}^{t_0} \Pi(x) \Lambda(t)} \quad , \quad \Pi(x) = Z^{\Pi}_{\Pi} e^{-i\int_{t_i}^{t_0} \psi(x) \Lambda(t)}
\]

\[
\Lambda(t) = e^{i(t-t_i)H_0} e^{-i(t-t_i)H} \quad , \quad U(t_2; t_1) = \Lambda(t_2) \Lambda^{-1}(t_1) = T \left\{ \exp \left[ -i \int_{t_i}^{t_2} dx \mathcal{H}^{I}_{\Pi}(x) \right] \right\}
\]

where \( \mathcal{H}^{I}_{\Pi} \) is the interacting Hamiltonian, \( T \) is the time ordering and \( Z_{\psi}, Z_{\Pi} \) are the wave-function renormalizations (usually \( \Pi \propto \bar{\psi} \), and so \( Z_{\psi} = Z_{\Pi} \)). Eqs.(2) must be understood in a weak sense: if not, we would get the canonical commutator between \( \psi \) and \( \Pi \) equal to \( iZ_{\psi} \delta^3(x-y) \). This would imply \( Z_{\psi} = 1 \), whilst the Källen-Lehmann representation requires \( Z_{\psi} < 1 \). The solution of this problem is well known, the Hilbert spaces for \( \psi \) and \( \psi_{in} \) are unitarily inequivalent, and the wave function renormalizations \( Z_{\psi} \) and/or \( Z_{\Pi} \) then “indicate” how much the unitarity is violated [10]. Thus, the choice of the \( \psi_{in} \) and of the associated Hilbert space is not unique: selecting a particular set of \( \psi_{in} \) corresponds to defining a particular physical situation, i.e. initial–time data for the operator equation (1).

We will use this feature to construct a QFT which “contains” topological defects.

If derivatives of fields are not present in \( \mathcal{L}_I \) (and thus \( \mathcal{H}_I = -\mathcal{L}_I \)) we can write

\[
\psi(x) = Z^{\psi}_{\chi} U(t_i ; t) \psi_{in}(x) \psi^{-1}(t_i ; t) = Z^{\psi}_{\chi} T C \left\{ \psi_{in}(x) \exp \left[ i \int_C d^4x \mathcal{L}^{I}_{\Pi}(x) \right] \right\}.
\]

Here \( C \) denotes a closed–time (Schwinger) contour (see Fig.1), running from \( t_i \) to a later time \( t_f \) and back again. \( T_C \) denotes the corresponding time–path ordering symbol.

In the limit \( t_i \to -\infty \), \( \psi_{in} \) becomes the usual in–(or asymptotic) field. Since \( t_f \) is arbitrary, it is useful to set \( t_f = +\infty \). Eq.(4) is the so called Haag expansion, for the Heisenberg field \( \psi \) [9].

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Generalization of Eq.(4) to an arbitrary polynomial $P$ of $\psi$ reads

$$T \{ P[\psi] \} = Z_\psi T_C \left\{ P[\psi_{in}] \exp \left[ i \int_C d^4x \mathcal{L}^I_{in}(x) \right] \right\}.$$  \hspace{1cm} (5)

3. Kinks in two–dimensional $\lambda\psi^4$ theory

In this Section we apply the formal considerations developed above to a specific model, namely the 2D $\lambda\psi^4$ theory both at zero and finite temperature. Let us consider the following Lagrangian (we adopt the Minkowski signature (+ − )):

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \psi)^2 - \frac{1}{2} \mu^2 \psi^2 - \frac{\lambda}{4} \psi^4.$$  \hspace{1cm} (6)

The Heisenberg equation of the motion for the field $\psi$ is

$$(\partial^2 + \mu^2)\psi(x) = -\lambda \psi^3(x).$$  \hspace{1cm} (7)

For $\mu^2 < 0$, this model admits at a classical level kink solutions. Defining

$$\psi(x) = v + \rho(x) \hspace{0.5cm} \text{with} \hspace{0.5cm} v = \langle 0 | \psi(x) | 0 \rangle, \quad -2\mu^2 = m^2, \quad -\mu^2 = \lambda v^2, \quad g = \sqrt{2\lambda},$$  \hspace{1cm} (8)

we obtain

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \rho)^2 - \frac{1}{2} m^2 \rho^2 - \frac{g^2}{8} \rho^4 - \frac{1}{2} mg \rho^3 + \frac{m^2}{16\lambda}$$  \hspace{1cm} (9)

$$(\partial^2 + m^2)\rho(x) = -\frac{3}{2} mg \rho^3(x) - \frac{1}{2} g^2 \rho^3(x).$$  \hspace{1cm} (10)

Note that $m^2 > 0$. The asymptotic field (at $t \to -\infty$) now satisfies

$$(\partial^2 + m^2)\rho_{in}(x) = 0.$$  \hspace{1cm} (11)

The Haag expansion for the field $\psi$ reads (we suppress the renormalization factor $Z_\psi$)

$$\psi(x) = v + T_C \left\{ \rho_{in}(x) \exp \left[ -\frac{i}{2} \int_C d^4y \mathcal{L}^I_{in}(y) \right] \right\}.$$  \hspace{1cm} (12)
where $\mathcal{L}_m = \frac{g^2}{4} \rho_m^4 + mg \rho_m^3$. We now consider the following (canonical) transformation:

$$\rho_m(x) \to \rho_m^f = \rho_m(x) + f(x) \quad , \quad (\partial^2 + m^2)f(x) = 0 , \quad (13)$$

with $f$ being a c-number function. It is known [11] that such a canonical transformation induces an inhomogeneous condensation of $\rho_m$ quanta.

As the shifted field $\rho_m^f$ fulfills the same asymptotic equations as the unshifted field $\rho_m$ we may equally well use it as initial-time data for $\psi$. In this case the Haag expansion reads

$$\psi^f(x) = v + T_C \left\{ \rho_m^f(x) \exp \left\{ -\frac{i}{2} \int_C d^2 y \mathcal{L}_m^{f f} (y) \right\} \right\} . \quad (14)$$

The superscript $f$ in $\psi^f(x)$ and $\mathcal{L}_m^{f f}$ indicates that we work with asymptotic field $\rho_m^f$. Using the (operatorial) Wick’s theorem [10], we rewrite (14) in the following way

$$\psi^f(x) = v + \left[ \frac{\delta}{i \delta J(x)} + f(x) \right] \exp \left\{ -\frac{i}{2} \int_C d^2 y \mathcal{L}_m^{f f} \left[ \frac{\delta}{i \delta J}, y \right] \right\} \times : \exp \left\{ i \int_C d^2 y J(y) \rho_m(y) \right\} : \exp \left\{ -\frac{1}{2} \int_C d^2 y d^2 z \Delta_C(y; z) J(z) \right\} \bigg|_{J=0} , \quad (15)$$

where $\Delta_C(x; y) = \langle T_C [\rho_m(x) \rho_m(y)] \rangle$ ($\langle \ldots \rangle \equiv \langle 0 | \ldots | 0 \rangle$ where $| 0 \rangle$ is the vacuum for the $\rho_m$ field). We now take the vacuum expectation value (vev) of Eq.(15), use the relation $F \left[ \frac{\delta}{i \delta J} \right] G[J] = G \left[ \frac{\delta}{i \delta K} \right] F[K] e^{i K J} \bigg|_{K=0}$, perform the change of variables $K \to K + f$ and set to zero the term with $J$ (there are no derivatives with respect to it). The result is [6]

$$\langle \psi^f(x) \rangle = v + \exp[\hbar a] K(x) \exp \left\{ \frac{1}{\hbar} b \right\} \bigg|_{K=f} , \quad (16)$$

where we have reintroduced $\hbar$ and defined

$$a \equiv -\frac{1}{2} \int_C d^2 z d^2 y \Delta_C(z; y) \frac{\partial^2}{\delta K(\delta K)} , \quad b \equiv -\frac{i}{2} \int_C d^2 z \left[ \frac{g^2}{4} K^4(z) + mg K^3(z) \right] . \quad (17)$$

After some manipulations we arrive at the following form for the order parameter [6]

$$\langle \psi^f(x) \rangle = v + C[K](x) \bigg|_{K=f} + D[K](x) \bigg|_{K=f} , \quad (18)$$

$$C[K](x) = -\int_C d^2 y \Delta_C(x; y) \frac{\delta}{\delta K(y)} \exp[\hbar a] \exp \left\{ \frac{1}{\hbar} b \right\} , \quad (19)$$

$$D[K](x) = K(x) \exp[\hbar a] \exp \left\{ \frac{1}{\hbar} b \right\} . \quad (20)$$

We remark that the order parameter Eq.(18) still contains all quantum corrections.

**Classical kinks at zero temperature**

Let us now deal with the Born, or classical, approximation of (18). For this purpose we keep only finite parts in $C$ and $D$ terms in the $\hbar \to 0$ limit. We then get:
\[ C(h \to 0)_{\text{finite}} = \text{Res}_{h=0} [C(h)] = -i \int_{-\infty}^{\infty} d^2y \, G_R(x, y) \frac{\delta}{\delta K(y)} \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} a^n b^{n+1}, \quad (21) \]

\[ D(h \to 0)_{\text{finite}} = \text{Res}_{h=0} [D(h)] = K(x) \sum_{n=0}^{\infty} \frac{1}{(n!)^2} a^n b^n. \quad (22) \]

where \( iG_R(x, y) = \theta(x_0 - y_0) \Delta(x, y) \) is the retarded Green’s function of the free theory and \( \Delta(x; y) \equiv \{[\rho_m(x), \rho_m(y)]\} \) is the Pauli–Jordan function:

\[ \Delta(x_0, x_1; 0, 0) = \int \frac{d^2k}{(2\pi)} \delta(k^2 - m^2) \varepsilon(k_0) e^{-ikx} = -\frac{i}{2} \theta(x_0 - |x_1|) J_0(m \sqrt{x_0^2 - x_1^2}). \quad (23) \]

Finally we may write [6]

\[ \langle \psi_0^f(x) \rangle = v + \sum_{n=1}^{\infty} Q_n[K](x) \bigg|_{K=f}, \quad (24) \]

where \( Q_1 = f \) and \((i, j, k = 1, 2\ldots \) and \( n > 1 \). An empty sum is zero

\[ Q_n(x) = -\int_{-\infty}^{\infty} d^2y \, G_R(x, y) \left[ \frac{3}{2} \sum_{i+j=n} Q_i(y) Q_j(y) + \frac{1}{2} \sum_{i+j+k=n} Q_i(y) Q_j(y) Q_k(y) \right], \quad (25) \]

This recurrence relation belongs to the class of the so called functional equations of Cauchy–Marley’s type [12,13] whose fundamental solution cannot be expressed (apart from a very narrow class of kernels) in terms of elementary functions. This means that we cannot hope to resolve (25) in terms of general \( K \) (or \( f \)). We can however obtain analytical solutions of the classical version of Eq.(7) - the analytical kinks. These may be obtained if we realize that the convolution of the 2D retarded Green’s function \( G_R(x) \) with an exponential is proportional to the very same exponential. Thus, if \( Q_1(x) = f(x) \) is an exponential, Fourier non–transformable, solution of Eq.(13), we have \( Q_n(x) \propto (Q_1(x))^n \) and

\[ Q_n(x) = A_n f^n(x) = A_n e^{\pm mn \gamma(x_1 - x_0 u)}, \quad (26) \]

where \( \gamma = (1 - u^2)^{-\frac{1}{2}} \). Plugging this form into (25) we arrive at the following equation

\[ A_n = \frac{1}{(n^2 - 1)} \left\{ \frac{3}{2} \sum_{i+j=n} A_i A_j + \frac{1}{2v^2} \sum_{i+j+k=n} A_i A_j A_k \right\}. \quad (27) \]

A solution of (27) is \( A_n = 2v \left( \frac{s}{2\pi} \right)^n \), with \( s \) being a real constant. The order parameter is

\[ \langle \psi_0^f(x) \rangle = v + 2v \sum_{n=1}^{\infty} \left( \frac{sf(x)}{2v} \right)^n. \quad (28) \]

Thus, provided \( f(x) \) is an exponential solution of the linear equation (13), the solution (28) fulfills the (classical) Euler–Lagrange equation of motion:

\[ (\partial^2 + \mu^2) \langle \psi_0^f(x) \rangle = -\lambda \langle \psi_0^f(x) \rangle^3, \quad (29) \]
which is nothing but the vev of Eq.(7) in the Born approximation. For instance, if we choose
\( f(x) = e^{-m\gamma(x_1-x_0u)} \) with \( s = -2ve^{m\gamma s} \), we obtain the standard kink solution [14–16]
\[
\langle \psi_0^f(x) \rangle = v \text{th}\left[ \frac{m}{2} \gamma (x_1 - a) - x_0 u \right],
\]
describing a constantly moving kink of a permanent profile with a center localized at \( a + ux_0 \).

**Classical kinks at finite temperature**

Let us now concentrate on the finite temperature case. It is important to understand what happens with the topological defects if the system is immersed in a heat bath at temperature \( T \).

The crucial observation at finite temperature is that the operatorial Wick’s theorem still holds [17] and thus Eq.(15) retains its validity provided (\( \ll \ldots \gg \) denotes thermal average)
\[
\Delta_C(x,y) = \langle 0|T_C(\rho(m(x)\rho(m(y)))|0 \rangle \to \Delta_C(x,y,T) = \ll T_C(\rho(m(x)\rho(m(y))) \gg ,
\]
together with : \ldots : \to N(\ldots). The thermal normal ordering \( N(\ldots) \) is defined in such a way [17] that \( \ll N(\ldots) \gg = 0 \). This is of a great importance as all the previous formal considerations go through also for finite \( T \).

At finite temperature the question of \( \hbar \) appearance is more delicate than in the zero-temperature case. The whole complication is hidden in the thermal propagator \( \Delta_C(x,y;T) \). To understand this, let us make \( \hbar \) explicit. The free thermal propagator in spectral or Mills’s representation [18,19] then reads
\[
i\Delta_C(x,y;T) = \hbar \int \frac{d^2k}{(2\pi)^2} e^{-ik(x-y)} \rho(k) \theta_C(x_0 - y_0) + f_b(\hbar k_0/T)]
= i\Delta_C(x,y;0) + i\Delta_C^T(x,y),
\]
where the spectral density \( \rho(k) = (2\pi)\varepsilon(k_0) \delta(k^2 - m^2) \) with \( \varepsilon(k_0) = \theta(k_0) - \theta(-k_0) \). The contour step function \( \theta_C(x_0 - y_0) \) is 1 if \( y_0 \) precedes \( x_0 \) along the contour \( C \). The Bose–Einstein distribution \( f_b(x) = (e^x - 1)^{-1} \). To calculate \( \text{Res}_{\hbar \to 0} \) we need to perform a Laurent expansion of \( f_b \) around \( \hbar \), i.e.
\[
f_b + \frac{1}{2} = \frac{T}{\hbar k_0} + \frac{1}{12} \frac{\hbar k_0}{T} + \ldots ,
\]
which converges for \( \hbar |k_0| < 2\pi T \). Taking a regulator \( \Lambda \sim \frac{\hbar}{T} \) in \( k_0 \) integration, i.e considering only soft modes we get [7] (see also ref. [20])
\[
\ll \psi^f(x) \gg = v(T) \text{th}\left[ \frac{m(T)}{2} \gamma (x_1 - a) - x_0 u \right],
\]
where \( v^2(T) = \frac{1}{\lambda} (|\mu|^2 - 3\lambda \ll \rho^2_{\text{in}} : \gg) \) and \( m(T) = \sqrt{2\lambda} v(T) \). Thus, at a “critical” temperature defined by equation: \( |\mu|^2 = 3\lambda \ll \rho^2_{\text{in}} : \gg \), the kink disappears.

It is an interesting question to ask whether for higher dimensional systems, this temperature is related to the critical temperature of the system.
4. Vortices in four--dimensional \( \lambda \psi^4 \) theory

In this Section we sketch the treatment of the 4D complex \( \lambda \psi^4 \) theory in presence of vortices. For a more thorough discussion see ref. [7]. We show how the vortex solution can be obtained by “shifting” both the fields of the massless (Goldstone) mode and of the massive (unstable) mode: the appearance of a topological charge is controlled by the shift of the Goldstone field only. We consider the Ginzburg–Landau type Lagrangian for a charged scalar field:

\[
\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi - \mu^2 \psi^\dagger \psi - \frac{\lambda}{4} |\psi^\dagger \psi|^2 .
\]  

(35)

The (unrenormalized) equations of motion for the Heisenberg operator \( \psi \) read

\[
\left( \Box + \mu^2 + \frac{\lambda}{2} |\psi(x)|^2 \right) \psi(x) = 0 .
\]  

(36)

We assume symmetry breaking, i.e. \( \mu^2 < 0 \) and \( \langle \psi \rangle = v = \sqrt{-2\mu^2/\lambda} \). We parametrize the field as \( \psi(x) \equiv (\rho(x) + v)e^{i\chi(x)} \), where both the fields \( \rho \) and \( \chi \) are hermitian and have zero vev. We also put \( g = \sqrt{\lambda} \) and \( m^2 = \lambda v^2 > 0 \). The equations of motion for \( \rho \) and \( \chi \) read

\[
\left[ \Box - (\partial_\mu \chi)^2 + m^2 \right] \rho + \frac{3}{2} mg \rho^2 + \frac{1}{2} g \rho^3 = v (\partial_\mu \chi)^2 \\
\partial_\mu \left[ (\rho + v)^2 \partial^\mu \chi \right] = 0 .
\]  

(37)

(38)

We now choose as the relevant asymptotic fields the ones described by the equations

\[
\left[ \Box - (\partial_\mu \chi)^2 + m^2 \right] (\rho_{in} + v) = 0 \\
\partial_\mu \left[ (\rho_{in} + v)^2 \partial^\mu \chi_{in} \right] = 0 ,
\]  

(39)

(40)

i.e. \( (\Box + m^2) \psi_{in} = 0 \), where we have used \( \psi_{in} \equiv (\rho_{in} + v)e^{i\chi_{in}} \). This means that the interaction Lagrangian density is

\[
\mathcal{L}^I = (m^2 - \mu^2)\psi^\dagger \psi - \frac{\lambda}{4} |\psi^\dagger \psi|^2 = 2m^2 v \rho - gm \rho^3 - \frac{g^2}{4} \rho^4 + \frac{5}{4} m^2 v^2 .
\]  

(41)

The corresponding Haag expansion for the Heisenberg field operator \( \psi \) is then

\[
\psi(x) \equiv (\rho(x) + v)e^{i\chi(x)} = T_C \left\{ (\rho_{in}(x) + v)e^{i\chi_{in}(x)} \exp \left[ -i \int_C d^4 y \mathcal{L}^I_{in}(y) \right] \right\} .
\]  

(42)

As before, the solutions of the asymptotic equations (39) and (40) are not unique. Indeed, we may define the following shifted fields:

\[
\rho_{in}(x) \rightarrow \rho_{in}^f(x) = \rho_{in}(x) + f(x) \\
\chi_{in}(x) \rightarrow \chi_{in}^g(x) = \chi_{in}(x) + g(x) ,
\]  

(43)

(44)

satisfying the same equations as the ones for the unshifted fields. The c–number functions \( h \equiv f + v \) and \( g \) solve the coupled equations:
\[
\begin{align*}
\Box - (\partial_\mu \chi_{in}(x))^2 - (\partial_\mu g(x))^2 + m^2 \right) h(x) &= m^2 v \\
\partial_\mu \left[ (\rho_{in}^2 + h^2(x)) \partial^\mu g(x) \right] &= 0.
\end{align*}
\]

The Haag expansion for \( \psi \) in terms of the new asymptotic fields reads

\[
\psi^{f,g}(x) = T_C \left\{ (\rho_{in}^f(x) + v) e^{i\chi_{in}(x)} \exp \left[ -i \int_C d^4 y \mathcal{L}_{in}^{lf}(y) \right] \right\}.
\]

By construction the field \( \psi^{f,g}(x) \) satisfies the same Heisenberg equations as the field \( \psi(x) \). A particular choice of solutions of Eqs.(45), (46) can lead to the description of a particular physical situation, e.g. of a system with topological defects. Observe that in (47) the whole dependence on \( g \) factorises out of \( T_C \).

Let us now consider the vev of the Heisenberg operator (47), i.e. the order parameter:

\[
\langle \psi^{f,g}(x) \rangle \equiv e^{ig(x)} F_{f}^{g}(x) = e^{ig(x)} \langle 0 | T_C \left\{ (\rho_{in}^f(x) + v) e^{i\chi_{in}(x)} \exp \left[ -i \int_C d^4 y \mathcal{L}_{in}^{lf}(y) \right] \right\} | 0 \rangle.
\]

In the classical approximation, the order parameter satisfies

\[
\begin{align*}
\Box - (\partial_\mu g(x))^2 + m^2 + \lambda F_{f}^2(x) \right) F_{f}(x) &= 0 \\
\partial_\mu \left[ F_{f}^2(x) \partial^\mu g(x) \right] &= 0.
\end{align*}
\]

This is nothing but the well known vortex equations [21]. A particular solution – a static vortex along the third axis – is obtained taking \( F_{f}(x) \) time independent with a radial dependence only. Eq.(50) then reduces to the Laplace equation having the polar angle as solution:

\[
g(x) = n \theta(x) = n \tan^{-1} \left( \frac{x_2}{x_1} \right),
\]

where the integer \( n \) guarantees the single valuedness of the order parameter \( \langle \psi^{f,g}(x) \rangle \). \( F_{f}(r) \) then fulfils the following (static) equation:

\[
\left[ \partial_r^2 + \frac{1}{r} \partial_r - \frac{n^2}{r^2} + m^2 \right] F_{f}(r) = \lambda F_{f}^3(r).
\]

At this point we can use the fact that the function \( g \) appearing in the vortex equations (49), (50) and in the equations for the shift functions (45), (46) is the same: we pick up the solution of the static vortex equation and plug it into the shift equations (45), (46) to determine\(^1\) the other shift function, \( f \). Let us denote this solution by \( \tilde{f} \). Then the expression for the Heisenberg operator in the presence of a static vortex at \( x = 0 \) will be

\[^1\text{In doing this, we use the renormalized equations, thus the terms with the vev will drop. The notation is kept in (45), (46) since in general these terms are present (e.g. at finite temperature as thermal averages).}\]
\( \psi^{\text{vort.}}(x) = T_C \left\{ \left( \rho_{in}(x) + \tilde{f}(r(x)) + v \right) e^{i \chi_{in}(x) + i n \theta(x)} \exp \left[ -i \int_C d^4 y \mathcal{L}_{in}^L(\tilde{J})(y) \right] \right\} \). \quad (53)

This expression can be used as a starting point for further analysis [7]. Here we only consider the simplest approximation, i.e. we completely neglect the radial dependence by setting \( f = 0 \). Then the following relation holds:

\[ \psi^\theta(x) \simeq e^{i n \theta(x)} \psi(x) , \quad (54) \]

from which also follows the relation between Green’s functions with and without vortices ( \( x \) and \( y \) are far from the vortex core):

\[ G^\theta(x, y) \equiv \langle 0 | T(\psi^\theta(x) \psi^{\dagger \theta}(y)) | 0 \rangle \simeq e^{i n[\theta(x) - \theta(y)]} G(x - y) . \quad (55) \]

It is interesting to note that a similar relation can be derived for a systems of (charged) particles in a Bohm–Aharonov potential. In this case, one has [22]:

\[ G_A(x, y) = e^{-ie[\Omega(x) - \Omega(y)]} G_0(x, y) , \quad (56) \]

where the Bohm–Aharonov potential is that induced by a magnetic flux = \( n \) times the elementary magnetic flux:

\[ A = \nabla \Omega \quad , \quad \oint ds \cdot A = -2\pi n/e . \quad (57) \]

In the case of a vortex line of strength \( n \) we also have, beside Eq.(55):

\[ J = n \nabla \theta \quad , \quad \oint ds \cdot J = 2\pi n . \quad (58) \]

Thus the “duality” correspondence reads \( n \theta(x) \leftrightarrow -e \Omega(x) \) and \( J \leftrightarrow -e A \).

5. Conclusions

We have developed a field theoretical (operator) formalism suitable for the description of quantum systems containing (topological) defects. The use of the Closed–Time–Path formulation is crucial in our approach, since it allows to treat systems both at zero temperature or in thermal equilibrium as well as systems out of equilibrium.

We have applied our method to \( \lambda \psi^4 \) theory in 2D and 4D cases. In 2D case, kink solutions were studied at zero and at finite temperature. In 4D case, vortex solution was shown to arise from a inhomogeneous condensation of Goldstone modes; an analogy with Bohm–Aharonov effect was discussed.

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