Abstract

The cosmology of the Randall-Sundrum scenario for a positive tension brane in a 5-D Universe with localized gravity has been studied previously. In the radiation-dominated Universe, it was suggested that there are two solutions for the cosmic scale factor $a(t)$: the standard solution $a \sim t^{1/2}$, and a solution $a \sim t^{1/4}$, which is incompatible with standard big bang nucleosynthesis. In this note, we reconsider expansion of the Universe in this scenario. We derive and solve a first order, linear differential equation for $H^2$, the square of the expansion rate of the Universe, as a function of $a$. The differences between our equation for $H^2$ and the relationship found in standard cosmology are (i) there is a term proportional to density squared, which is small when the density is small compared to the brane tension, and (ii) there is a contribution which acts like a relativistic fluid. We show that this second contribution is due to gravitational wave degrees of freedom in the bulk (presumably a massless 5-D graviton mode which is trapped on the brane). Thus, we find that there need not be any conflict between cosmology of the Randall-Sundrum scenario and the standard model of cosmology. As an illustration, we discuss how reheating at the end of inflation matches smoothly onto the correct Big Bang behavior, $a(t) \sim t^{1/2}$.
I. INTRODUCTION

Recently Randall and Sundrum [1] presented a new static solution to the 5-D (classical) Einstein equations in which spacetime is flat on a 3-brane with positive tension provided that the bulk has an appropriate negative cosmological constant. Even if the fifth dimension is uncompactified, standard 4-D gravity (specifically, Newton’s force law) is reproduced on the brane. In contrast to the compactified case [2], this follows because the near-brane geometry traps the massless graviton.

To see if such a scenario is viable phenomenologically, one application to check is the evolution of the early Universe [5,6]. It was pointed out [3] that a 5-D Universe with branes may have a rather unconventional, and perhaps unacceptable, cosmology. The inclusion of matter inside the brane is suggested naturally by the brane world picture. So it is important to see if the standard expanding Universe can be recovered by extending the static solution to a time-dependent one when matter/radiation inside the brane is included. As shown in Ref [4], the standard matter-dominated expanding Universe is recovered for large enough brane tension. However, Ref [4] proposed two possible behaviors for the cosmic scale factor, \( a(t) \), during the radiation-dominated Universe: (i) \( a(t) \sim t^{1/4} \) (first found in Ref [3]), or (ii) \( a(t) \sim t^{1/2} \) (found in Ref [4]). That there might be two different powerlaw solutions is not surprising, since a non-linear second order differential equation may have more than one such solution, and initial conditions will determine the precise time evolution of \( a(t) \) (which may not be exactly a powerlaw, but might be close to either of the candidate powerlaws at different times). On the other hand, to match the known observations of the expanding Universe, at least back to the time of electron-positron annihilation and nucleosynthesis, the expansion rate of the Universe should be approximately \( H^2 = \frac{8\pi G \rho}{3} \), its value in standard Big Bang cosmology.

In this note, we clarify the situation with respect to the cosmology of the Randall-Sundrum scenario. First, we elucidate why the evolution equation that one obtains for the scale factor [Eq. (24) below] is second order in time rather than first order in time as in standard cosmology. The reason is that a piece of the gravitational wave dynamics in the bulk is coupled to the brane dynamics, and thus there is an extra free parameter in the cosmological equations, describing the amount of effective 4-D energy density due to the 5-D gravitational waves. We derive this in Sec. III below.

Second, we derive and solve a first-order differential equation for \( H^2 \) as a function of \( \log a \). This equation shows that the 5-D gravitational waves act like a relativistic fluid, and thus become unimportant at late times, so that \( H^2 \to \frac{8\pi G \rho}{3} \) at late times. As an illustration, we start with an inflationary epoch [7,5,8], and follow the Universe as it evolves into the radiation-dominated phase. Our final result for \( H^2 \) shows that, invariably, standard cosmology is recovered at late times in the Randall-Sundrum scenario.

II. DERIVATION OF THE BRANE DYNAMICS

In this section we derive the equations governing the dynamics of matter and geometry on the brane, using a local expansion of the metric near the brane. In Sec. III below we will analyze the global structure of the 5-D Einstein equations in order to clarify why one can
derive a description of the brane dynamics that is (almost) independent of the dynamics in the bulk.

We consider a 5-D spacetime with coordinates $x^A = (x^0, x^1, x^2, x^3, x^4) = (t, x^1, x^2, x^3, y)$, and we assume that there is a single brane located at $y = 0$. We assume the following form for the 5-D metric

$$ds^2 = \exp[2\beta(y, t)](-dt^2 + dy^2) + \exp[2\alpha(y, t)]\delta_{ij}dx^i dx^j. \quad (1)$$

Thus, the metric only depends on $t$ and $y$, and is flat in ordinary 3-D space (labelled by Latin indices which run over 1,2,3). For simplicity, we also restrict attention to the case where $\alpha$ and $\beta$ are even functions of $y$. For this metric the non-zero components of the Einstein tensor, as shown by Binétruy et al. [3], are

$$G_{00} = 3 \left[ \dot{\alpha}^2 + \dot{\beta}\ddot{\alpha} - \alpha'' - 2\dot{\alpha}'^2 + \alpha\beta' \right], \quad (2)$$

$$G_{ij} = \delta_{ij}e^{2(\alpha - \beta)} \left[ -2\ddot{\alpha} - 3\dot{\alpha}^2 - \ddot{\beta} + 2\alpha'' + 3\dot{\alpha}'^2 + \beta'' \right], \quad (3)$$

$$G_{44} = G_{yy} = 3 \left[ -\ddot{\alpha} - 2\dot{\alpha}^2 + \dot{\beta}\alpha' + \alpha\beta' \right], \quad (4)$$

and

$$G_{04} = 3 \left[ \beta'\dot{\alpha} + \alpha'\dot{\beta} - \dot{\alpha}' - \dot{\alpha}\alpha' \right], \quad (5)$$

where dots denote derivatives with respect to $t$ and primes with respect to $y$.

We assume that $\alpha$ and $\beta$ are smooth functions of $|y|$ and of $t$, i.e., $\alpha(y, t) = \hat{\alpha}(|y|, t)$ and $\beta(y, t) = \hat{\beta}(|y|, t)$, where the functions $\hat{\alpha}(\xi, t)$ and $\hat{\beta}(\xi, t)$ are smooth in a neighborhood of $\xi = 0$. Then the derivative $\partial\alpha/\partial y$ will generically be discontinuous across the brane, as in Ref. [1]. We define

$$\alpha_1(t) = \lim_{y \to 0^+} \frac{\partial\alpha(y, t)}{\partial y} = \frac{\partial \hat{\alpha}(\xi, t)}{\partial \xi} \bigg|_{\xi=0}. \quad (6)$$

and

$$\beta_1(t) = \lim_{y \to 0^+} \frac{\partial\beta(y, t)}{\partial y} = \frac{\partial \hat{\beta}(\xi, t)}{\partial \xi} \bigg|_{\xi=0}. \quad (7)$$

$^1$Binétruy et al. [3] used the metric

$$ds^2 = -n^2(y, t)dt^2 + b^2(y, t)dy^2 + a^2(y, t)\delta_{ij}dx^i dx^j,$$

but did not employ the freedom to redefine the coordinates $y$ and $t$ to put the metric in the simpler form (1) used here. For any 2-D space, such as is formed by $t$ and $y$, the metric has only three independent components, and can be expressed in a conformally flat form with one independent component using the coordinate freedom.
It follows that
\[ \alpha_{yy}(y, t) = \hat{\alpha}_{\xi\xi}(|y|, t) + \alpha_1(t)\delta(y). \] (8)
together with a similar equation for \( \beta_{yy} \).

We now substitute the relation (8) into the Einstein tensor components (2)—(5), and insert into the 5-D Einstein equations \( G_{AB} = \kappa^2 T_{AB} \) where the energy momentum tensor is
\[ T_{AB} = \text{diag}([-\rho, p, p, p, 0]) \exp(-\beta) + (\Lambda_b, \Lambda_b, \Lambda_b, \Lambda_b, \Lambda_b). \] (9)

Here \( \rho \) and \( p \) are the density and pressure of the matter on the brane, and \( \Lambda_b \) is the bulk cosmological constant. The result is, first, two equations obtained by equating the coefficients of the \( \delta(y) \) in the \( G_{00} \) and \( G_{ij} \) equations:
\[ \alpha_1 = -\frac{\kappa^2 \rho}{6} \exp[\beta(y = 0)] \] (10)
\[ \beta_1 = \frac{\kappa^2(2\rho + 3p)}{6} \exp[\beta(y = 0)]. \] (11)

Second, there are the smooth pieces of the equations. Since we can restrict attention to \( y > 0 \) for these pieces of the equations (by the evenness assumption), we can drop the distinction between \( \hat{\alpha} \) and \( \alpha \), and between \( \hat{\xi} = |y| \) and \( y \), so we replace terms like \( \hat{\alpha}_{\xi\xi}(|y|, t) \) [cf. Eq. (8) above] by \( \alpha_{yy}(y, t) \). The resulting equations are, for \( y > 0 \),
\[ \ddot{\alpha} + \dot{\alpha} \dot{\beta} - \alpha'' - 2\alpha'^2 + \alpha'\beta' = -\frac{\kappa^2}{3} \Lambda_b e^{2\beta}, \] (12)
\[ -2\ddot{\alpha} - 3\dot{\alpha}^2 - \ddot{\beta} + 2\alpha'' + 3\alpha'^2 + \beta'' = \kappa^2 \Lambda_b e^{2\beta}, \] (13)
\[ -\ddot{\alpha} - 2\ddot{\alpha} + \dot{\alpha} \dot{\beta} + \alpha'^2 + \alpha'\beta' = \frac{\kappa^2}{3} \Lambda_b e^{2\beta}, \] (14)
and
\[ \beta' \dot{\alpha} + \alpha' \dot{\beta} - \dot{\alpha}' - \hat{\alpha} \dot{\alpha}' = 0. \] (15)

\(^{2}\)The representation of the brane stress-energy tensor as a \( \delta \)-function in \( y \) involves the tacit assumption that the thickness of the brane is smaller than \( |\beta_1|^{-1} \) or \( |\alpha_1|^{-1} \).
A. Power Series Expansion for the Metric Near the Brane

The equations (10) – (15) define the coupled dynamics of the brane and bulk degrees of freedom. In this section we derive a description of the dynamics of the brane by itself, by using a power series expansion for the metric near the brane.

We assume that, near $y = 0$,

$$\alpha(y, t) = \alpha_0(t) + \alpha_1(t)|y| + \frac{1}{2} \alpha_2(t)y^2 + \cdots$$

$$\beta(y, t) = \beta_0(t) + \beta_1(t)|y| + \frac{1}{2} \beta_2(t)y^2 + \cdots.$$ (16)

This assumption is compatible with our definitions (6) and (7) of $\alpha_1$ and $\beta_1$. If we insert this expansion into Einstein’s equations we obtain once again the relations (10) and (11) on the brane. Equating powers of the smooth pieces (12) – (15) of the equations of motion yields the equations yields, first, from the piece of the $G_{04}$ equation (15) that is $\propto y^0$,

$$\dot{\rho} + 3\dot{\alpha}_0(\rho + p) = 0,$$ (17)

and five others that follow from the remaining terms,

$$\ddot{\alpha}_0 + \beta_0\dot{\alpha}_0 - 2\alpha_1 + \alpha_1\beta_1 - \alpha_2 = -\frac{\kappa^2\Lambda_b\exp(2\beta_0)}{3}$$ (18)

$$-2\ddot{\alpha}_0 - 3\dot{\alpha}_0^2 - \ddot{\beta}_0 + 3\alpha_1^2 + 2\alpha_2 + \beta_2 = \kappa^2\Lambda_b\exp(2\beta_0)$$ (19)

$$-\dot{\alpha}_2 + \alpha_2(\dot{\beta}_0 - \dot{\alpha}_0) + \beta_2\dot{\alpha}_0 + \beta_1\dot{\alpha}_1 + \dot{\beta}_1\alpha_1 - \dot{\beta}_1\alpha_1 = 0$$ (20)

$$-\ddot{\alpha}_0 - 2\dot{\alpha}_0^2 + \dot{\alpha}_0\dot{\beta}_0 + \alpha_1(\dot{\alpha}_1 + \dot{\beta}_1) = \frac{\kappa^2\Lambda_b\exp(2\beta_0)}{3}$$ (21)

$$\alpha_2(2\alpha_1 + \beta_1) + \alpha_1\beta_2 - \ddot{\alpha}_1 - 4\dot{\alpha}_0\dot{\alpha}_1 + \dot{\alpha}_0\dot{\beta}_1 + \dot{\beta}_0\dot{\alpha}_1 = \frac{2\kappa^2\Lambda_b\beta_1\exp(2\beta_0)}{3}.$$ (22)

Equation (17) is simply the usual conservation of energy equation, and its derivation from the $G_{04}$ component of the 5-D Einstein equations was already given by Binetruy et al. [3]. Using Eq. (11) in Eq. (21) we find

$$-\ddot{\alpha}_0 - 2\dot{\alpha}_0^2 + \dot{\alpha}_0\dot{\beta}_0 = \left(\frac{\kappa^2\Lambda_b}{3} + \frac{\kappa^4\rho(\rho + 3p)}{36}\right)\exp(2\beta_0);$$ (23)

defining a new time variable $\hat{t}$ by $d\hat{t} = \exp(\beta_0)dt$, we obtain the equation already found by Binetruy et al. [3] and Csaki et al. [4],

$$-\frac{d^2\alpha_0}{d\hat{t}^2} - 2\left(\frac{d\alpha_0}{d\hat{t}}\right)^2 = \frac{\kappa^2\Lambda_b}{3} + \frac{\kappa^4\rho(\rho + 3p)}{36}. $$ (24)
Note that $\hat{t}$ is just the conventional cosmological proper time, that is, the proper time as measured by comoving observers on the brane. If we assume that $\rho = \sigma + \rho_m$ and $p = -\sigma + p_m$, where $\sigma$ represents the contribution from the brane tension, and $\rho_m$ and $p_m$ are the density and pressure due to the matter, then this equation reduces to

$$-\frac{d^2\alpha_0}{d\hat{t}^2} - 2\left(\frac{d\alpha_0}{d\hat{t}}\right)^2 = \frac{\kappa^2 \Lambda_b}{3} - \frac{\kappa^4 \sigma^2}{18} + \frac{\kappa^4 \sigma (3p_m - \rho_m)}{36} + \frac{\kappa^4 \rho_m (\rho_m + 3p_m)}{36}. \quad (25)$$

We shall find exact solutions of Eq. (25) below.

To complete the solution, note that Eqs. (18), (19) and (22) are algebraic equations for $\alpha_2$ and $\beta_2$. Equation (18) implies that

$$\alpha_2 = \dot{\alpha}_0^2 + \alpha_0 \dot{\beta}_0 - 2\alpha_1^2 + \alpha_1 \beta_1 + \frac{\kappa^2 \Lambda_b \exp(2\beta_0)}{3},$$

and

$$\alpha_2 = \dot{\alpha}_0^2 + \alpha_0 \dot{\beta}_0 + \left(\frac{\kappa^2 \Lambda_b}{3} - \frac{\kappa^4 \rho (4p + 3p)}{36}\right) \exp(2\beta_0), \quad (26)$$

where Eq. (11) was used to get the second expression. Using this result for $\alpha_2$ in Eq. (19) gives

$$\beta_2 = \frac{\kappa^2 \Lambda_b \exp(2\beta_0)}{3} + \dot{\alpha}_0^2 + 2\alpha_0 + \dot{\beta}_0 + \alpha_1^2 - 2\alpha_1 \beta_1 = -3\dot{\alpha}_0^2 + \dot{\beta}_0 + \left(\frac{\kappa^4 \rho^2}{12} - \frac{\kappa^2 \Lambda_b}{3}\right) \exp(2\beta_0), \quad (27)$$

where we used eqs. (11) and (21) in the last line. It is easy to check that once we have determined $\alpha_2$ and $\beta_2$, the remaining equations (20) and (22) are both satisfied identically.

The function $\beta_0(t)$ is not determined from this power series expansion near the brane. Thus, we conclude that it is not specified by physics on the brane itself. Note that, from the point of view of the 4-D induced metric on the brane, $\beta_0(t)$ is an un-observable, pure gauge piece of the metric. However, it is not a pure gauge piece of the full 5-D metric. We now turn to an analysis of the bulk degrees of freedom, which determine the function $\beta_0$.

### III. BULK DYNAMICS AND BRANE-BULK DECOUPLING

To understand the coupled brane-bulk dynamics better, it is useful to switch to null coordinates $v = t + y$, $u = t - y$. Then the smooth pieces (12) – (15) of the equations of motion which apply in the region $y > 0$ can be written as

$$C_1 \equiv \alpha_{,uu} + \alpha_{,u}^2 - 2\alpha_{,u} \beta_{,u} = 0, \quad (28)$$

$$C_2 \equiv \alpha_{,vv} + \alpha_{,v}^2 - 2\alpha_{,v} \beta_{,v} = 0, \quad (29)$$

$$E_1 \equiv \alpha_{,uv} + 3\alpha_{,u} \alpha_{,v} + \frac{\kappa^2}{6} \Lambda_b e^{2\beta} = 0, \quad (30)$$

and
\[ E_2 \equiv \beta_{uv} - 3\alpha_{u}\alpha_{v} - \frac{\kappa^2}{12}\Lambda_{b} e^{2\beta} = 0. \] (31)

In the absence of any branes, the initial value formulation of these equations is of a standard type. Equations (28) and (29) can be viewed as constraint equations that must be satisfied by initial data specified on a initial data surface which is the union of the null hypersurface \( u = u_0, v > v_0 \) together with the null hypersurface \( v = v_0, u > u_0 \). On each portion of such an initial null surface there is one free function to specify, and the other function is determined (up to constants) by the constraint. Thus, there is one gravitational wave polarization component that can propagate in the bulk [given the assumed symmetries in our starting metric (1)]. One can check that the evolution equations \( E_1 = E_2 = 0 \) preserve the constraints \( C_1 \) and \( C_2 \).

Turn now to the question of what is appropriate initial data for the brane-bulk system when there is a brane present at \( y = 0 \). The answer is that the appropriate initial data consists of the density \( \rho(0) \) at time \( t = 0 \) on the brane, together with a specification of \( \alpha \) and \( \beta \) along the null surface \( \Sigma \) given by \( y = t \) for \( y \geq 0 \) (see Fig. 1), where the constraint \( C_2 = 0 \) is satisfied along \( \Sigma \). This follows from the fact that \( \alpha \) and \( \beta \) are completely determined in the region \( y \geq 0, t \geq y \) by the evolution equations \( E_1 = E_2 = 0 \), given their values on the null surface \( \Sigma \) together with the values of \( \alpha_1 \) and \( \beta_1 \) on the brane at \( y = 0 \). An explicit formula for \( \alpha \), which can be obtained from \( E_1 = 0 \), is

\[
\alpha(u, v) = \alpha(0, v) + \alpha(0, u) - \alpha(0, 0) - \int_0^u \! d\bar{u} \alpha_1(\bar{u})
+ \int_0^u \! d\bar{u} \int_0^{\bar{u}} \! d\bar{v} f(\bar{u}, v)
+ \int_0^u \! d\bar{u} \int_0^{\bar{u}} \! d\bar{v} f(\bar{u}, \bar{v}),
\] (32)

where \( f \equiv \alpha_{,uv} = -3\alpha_{,u}\alpha_{,v} - \kappa^2 \Lambda_{b} e^{2\beta}/6 \). There is a similar formula for \( \beta \). The values of \( \alpha \) and \( \beta \) so obtained obey the constraint \( C_2 = 0 \), since \( C_2 = 0 \) on \( \Sigma \) and \( C_2,v \) is linear in \( C_2, E_1 \) and \( E_2 \). In addition they obey \( C_1 = 0 \) as long as \( C_1 = 0 \) on the brane at \( y = 0 \), since \( C_1,v \) is linear in \( C_1, E_1 \) and \( E_2 \). The requirement that \( C_1 = 0 \) on the brane is equivalent to (given that \( C_2 = 0 \) on the brane) the \( G_{04} \) component (15) of the Einstein equations evaluated on the brane, which as we saw above, is equivalent to the energy conservation equation (17). Thus, to summarize, the values of \( \alpha \) and \( \beta \) in the bulk are uniquely determined by their values on the initial surface \( \Sigma \), and by the values of \( \alpha_1 \) and \( \beta_1 \) on the brane \( y = 0 \).
FIG. 1. An illustration of the initial data necessary to determine a solution of the 5-D Einstein equations, when one assumes that all variables are reflection symmetric through the brane. The line $y = 0$ is the brane. At the point $P$ given by $y = t = 0$, one needs to specify the initial value of the energy density $\rho(t = 0)$. Along the null hypersurface $\Sigma$ given by $y = t$ or $u = 0$, one needs to specify the metric potentials $\alpha(v)$ and $\beta(v)$, where $v = t + y$. These potentials must satisfy the constraint equation (29). Then, the bulk variables $\alpha$ and $\beta$ are determined in the region $y \geq 0$, $t \geq y$, and on the brane the density $\rho(t)$ and pressure $p(t)$ are determined for $t > 0$.

Now the functions $\alpha_1$ and $\beta_1$ are determined from the values of $\rho$ and $p$ on the brane by Eqs. (10) and (11), and also $\rho$ and $p$ are determined by the equation of state $p = p(\rho)$, the initial value $\rho(0)$ of $\rho(t)$, the function $\alpha_0(t) = \alpha(t, y = 0)$ and $\beta_0(t) = \beta(t, y = 0)$, and the conservation equation (17). Thus, we see that specifying $\rho(0)$ at $t = 0$ on the brane, together with $\alpha$ and $\beta$ on $\Sigma$ satisfying the constraint (29), determine uniquely a solution of the coupled equations (28)–(31) [or equivalently (12)–(15)] for the bulk variables together with Eqs. (10), (11) and (17) for the brane variables.

Consider now the nature of the coupling between the bulk and brane degrees of freedom. Despite the fact that the bulk and brane degrees of freedom are coupled, we saw in Sec. II A above that it is possible to derive the second-order differential equation (24) for the evolution of $\alpha_0(t)$. This equation can be solved together with the energy conservation equation (17) once an equation of state $p = p(\rho)$ is specified. The essential difference from conventional cosmology is that the equation for $\alpha_0(t)$ is second order rather than first order, so that $\dot{\alpha}_0(t = 0)$ is freely specifiable instead of being determined by the mass density. We can now understand the physical reason why $\dot{\alpha}_0(t = 0)$ is freely specifiable, by considering the initial
data that needs to be specified in the bulk. Let us write the gravitational wave initial data on \( \Sigma \) as

\[
\alpha(u = 0, v) = \alpha(t = v, y = v) = h(v)
\]

and

\[
\beta(u = 0, v) = \beta(t = v, y = v) = g(v);
\]

the functions \( h \) and \( g \) must satisfy \( h'' + h'g' = 2h'g' \) from (29). Using the coordinate transformation \( v = t + y, \ u = t - y \) we find that

\[
\frac{\alpha}{\alpha}(t = 0, y) = \frac{\alpha}{\alpha}(t = v, y = v) = h(v)
\]

and

\[
\beta(t = v, y = v) = g(v);
\]

the functions \( h \) and \( g \) must satisfy \( h'' + h'g' = 2h'g' \) from (29). Using the coordinate transformation \( v = t + y, \ u = t - y \) we find that \( 2\alpha, v = \alpha, t + \alpha, y \). Applying this at \( t = y = 0 \) yields

\[
\dot{\alpha}_0(t = 0) = 2h'(0) - \alpha_1(0) = 2h'(0) - \frac{1}{6}\kappa^2 \rho(0)e^{\beta_0(0)}.
\]

Hence, the degree of freedom parameterized by \( \alpha_0(0) \) is a piece of the bulk gravitational wave degrees of freedom. Below we will show that this extra freedom acts like a free, relativistic fluid, and thus it is natural to suppose that it is due to a trapped, massless 5-D graviton mode.

The function \( \beta_0 \) is not determined by the brane dynamics, as discussed above, so it must instead be determined by the gravitational wave initial data \( g \) and \( h \).

### IV. An Equation for \( H^2 \)

This leaves us with the task of solving Eq. (25). The easiest way to do this is to change the equation into a first order equation for \( H^2 = (d\alpha_0/d\dot{\alpha})^2 \) as a function of \( \alpha_0 \). This may be accomplished by noting that

\[
\frac{d^2\alpha_0}{dt^2} = \frac{d(d\alpha_0/d\dot{\alpha})}{d\alpha_0} \frac{d\alpha_0}{d\dot{\alpha}} = H \frac{dH}{d\alpha_0} = \frac{1}{2} \frac{dH^2}{d\alpha_0};
\]

then Eq. (25) becomes

\[
-\frac{1}{2} \frac{dH^2}{d\alpha_0} - 2H^2 = -\exp(-4\alpha_0) \frac{d[H^2 \exp(4\alpha_0)]}{d\alpha_0} = \frac{\kappa^2 \Lambda}{3} - \frac{\kappa^4 \sigma^2}{18} + \frac{\kappa^4 \sigma(3\rho - \rho_m)}{36} + \frac{\kappa^4 \rho_m(\rho_m + 3\rho_m)}{36},
\]

or,

\[
\frac{d[H^2 \exp(4\alpha_0)]}{d\alpha_0} = \exp(4\alpha_0) \left[ \frac{\kappa^4 \sigma^2}{9} - \frac{2\kappa^2 \Lambda}{3} + \frac{\kappa^4 \sigma(3\rho - \rho_m)}{18} - \frac{\kappa^4 \rho_m(\rho_m + 3\rho_m)}{18} \right].
\]

Note that Eq. (38) represents a linear, first-order equation for \( H^2 \), if we regard the right hand side as a source term that is (implicitly) a function of \( \alpha_0 \). Implicit in our change of variables from \( t \) to \( \alpha_0 \) is the restriction to phases of the evolution of the Universe in which the two variables are related to one another monotonically. For oscillating Universes, or
Universes in which the scale factor $a(t) = \exp[\alpha_0(t)]$ may have a contracting phase as well as one or more expanding ones, we can derive Eq. (38) for each of these phases separately.

To solve Eq. (38) generally, let us first rewrite the equation of energy conservation, Eq. (17), as

$$\frac{d\rho_m}{d\alpha_0} + 3(\rho_m + p_m) = 0; \quad (39)$$

this equation implies

$$p_m = -\rho_m - \frac{1}{3} \frac{d\rho_m}{d\alpha_0}. \quad (40)$$

Substituting this result for $p_m$ into equation Eq. (38) yields

$$\frac{d[H^2 \exp(4\alpha_0)]}{d\alpha_0} = \exp(4\alpha_0) \left[ \frac{\kappa^4 \sigma^2}{9} - \frac{2\kappa^2 \Lambda_b}{3} + \frac{\kappa^4 \sigma}{18} \left(4\rho_m + \frac{d\rho_m}{d\alpha_0}\right) + \frac{\kappa^4}{36} \left(4\rho_m^2 + \frac{d\rho_m^2}{d\alpha_0}\right) \right] = \frac{d}{d\alpha_0} \left[ \exp(4\alpha_0) \left( \frac{\kappa^4 \sigma^2}{36} - \frac{\kappa^2 \Lambda_b}{6} + \frac{\kappa^4 \sigma \rho_m}{18} + \frac{\kappa^4 \rho_m^2}{36} \right) \right]; \quad (41)$$

we can easily read off the solution

$$H^2 = \frac{\kappa^4 \sigma^2}{36} - \frac{\kappa^2 \Lambda_b}{6} + \frac{\kappa^4 \sigma \rho_m}{18} + \frac{\kappa^4 \rho_m^2}{36} + K \exp(-4\alpha_0), \quad (42)$$

where $K$ is a constant of integration. In the case of the Randall-Sundrum static solution [1], we have $\kappa^2 \sigma^2 = 6\Lambda_b$, $\kappa^4 \sigma = 48\pi G$, and $\rho_m = K = 0$, giving $H = 0$ and $a(t) = \exp[\alpha_0(t)]$ constant.

Just as in standard cosmology based on 4-D General Relativity the evolution of the Universe can be reduced to the solution of an equation for $H^2$ and an equation for energy conservation. However, the equation for $H^2$ has a different structure than in standard cosmology for three reasons. First, there is an explicit term that results from the brane tension and negative cosmological constant in the bulk; in the Randall-Sundrum scenario, these may be chosen to cancel exactly. Second, in addition to a term proportional to the matter density $\rho_m$, there is a term that is proportional to $\rho_m^2$. This high-density “correction”, which is reminiscent of the small-scale deviation from Newton’s law found in this scenario, becomes unimportant once $\rho_m \ll \sigma$. Third, there is the term $K \exp(-4\alpha_0)$ that arises purely from initial conditions. This is a qualitatively new feature of the Randall-Sundrum scenario.

In cosmology based on 4-D General Relativity, $H^2$ is completely determined by the energy density of the Universe (presuming a spatially flat model, as was done above). But in the reduction from five dimensions to four dimensions done here, we find that $H^2$ can be specified freely at some initial time. The additional term that results decays exactly as any relativistic matter density would. In Sec. III above we identified this term as the effect of 5-D gravitational waves (propagating transverse to the brane) on the 4-D dynamics.

In specific cosmological scenarios, the additional term can be important or negligible, depending on the magnitude of $K$. More concretely, let us imagine that the Universe begins with an early, inflationary phase of expansion, during which the energy density and pressure are dominated by an inflaton field, and that once the inflaton approaches its potential
minimum, its energy dissipates into relativistic particles. For example, in one model for inflation and reheating (e.g. [8])

\[
\begin{align*}
\rho_m &= \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V(\phi) + \rho_r \\
p_m &= \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 - V(\phi) + \frac{\rho_r}{3},
\end{align*}
\]

(43)

where the inflaton field \(\phi\) obeys the equation

\[
\frac{d^2\phi}{dt^2} + (3H + \Gamma_\phi) \frac{d\phi}{dt} + \frac{dV(\phi)}{d\phi} = 0,
\]

(44)

and the radiation energy density \(\rho_r\) is determined from

\[
\frac{d\rho_r}{dt} + 4H\rho_r = \Gamma_\phi \left( \frac{d\phi}{dt} \right)^2.
\]

(45)

Here \(\Gamma_\phi\) is a decay rate that leads to the production of relativistic particles, primarily from oscillations of the inflaton field about its minimum. It is straightforward to set up and solve these equations along with Eq. (25) numerically, and we have done so. To begin the integration, we must specify not only initial values of \(\phi, \frac{d\phi}{dt}\) and \(\rho_r\), but also the starting value of \(H\). Our (limited) exploration\(^3\) of numerical solutions shows that, as expected, Eq. (42) is satisfied (to the accuracy of our numerical integrations) during both the inflationary epoch and the radiation-dominated era that follows reheating, even though our solutions were based on Eq. (25) (with independent variable \(\alpha_0\) instead of \(t\), not Eq. (42).

Using Eq. (42) along with eqs. (43), (44) and (45), we can gain insight into how inflation might be affected by the new features of our relationship for \(H^2\). We assume that the Randall-Sundrum condition, \(\kappa^2 \sigma^2 / 6\Lambda_b = 1\), holds. Let us first consider the effect of choosing the initial value of \(H\), which translates (with suitable definition of \(\alpha_0 = 0\)) into choosing a value of \(K\). To keep matters as simple as possible, let us assume that \(V_0\), the value of \(V(\phi)\) in its flat portion (where we presume the inflaton starts), is smaller than \(\sigma\), so we can neglect the nonlinear term in Eq. (42) for \(H^2\). Then if the initial value of \(K \sim 8\pi GV_0/3\), the starting value of \(H^2\) in inflation theory based on 4-D General Relativity, deviations in \(H^2\) from its standard cosmological value will damp away exponentially as the inflaton rolls toward its minimum. Given enough expansion, \(K \exp(-4\alpha_0)\) becomes negligible compared with \(8\pi GV_0/3\) by the end of inflation, and reheating results in a radiation density that dominates the \(K \exp(-4\alpha_0)\) term in Eq. (42) thereafter. If the initial value of \(H\) is far larger than in standard cosmology, the inflaton hardly rolls at all until the Universe expands enough that \(K \exp(-4\alpha)\) becomes comparable to \(8\pi GV_0/3\); the subsequent evolution is identical to

\[^3\text{We chose } d\phi/d\hat{t} = 0 \text{ and } \rho_r = 0 \text{ initially in all of our numerical integrations, and also focussed on cases where the Randall-Sundrum condition, } \kappa^2 \sigma^2 / 6\Lambda_b = 1 \text{ is satisfied. A simple effective potential of the form } V(\phi) = V_0[\exp(-\epsilon) - \exp(-\sqrt{\epsilon^2 + \phi^2/\phi_0^2})] \text{ was assumed, with } \epsilon \text{ chosen to be small but nonzero to guard against pathologies in } dV(\phi)/d\phi \text{ near } \phi = 0. \text{ Although } V_0/\sigma \text{ is a parameter that can be chosen arbitrarily, so far we have considered } V_0/\sigma \leq 1.\]
what would transpire if we had started with $H^2 \sim 8\pi GV_0/3$. If the initial value of $H$ is well below its standard value initially (but the Universe is still expanding) then the total number of e-foldings during inflation is diminished, but can still be large enough that $K \exp(-4\alpha_0)$ becomes unimportant by the time reheating occurs.

The nonlinear term in Eq. (42) can be important if $V_0 \gg \sigma$. In that case, the Universe expands faster than in standard cosmology, but $H$ is still time-independent while $\phi$ is rolling down the flat part of $V(\phi)$, so much of the standard picture of inflation can be carried over intact. The number of e-foldings of $a(\tilde{t})$ between the time when the observable portion of the Universe crossed the horizon and the end of inflation is $N_H \sim HT_0/V^{1/4}H_0$, where $T_0$ and $H_0$ are the present temperature and Hubble parameter. Since $N_H \propto H$, larger $H$ means more e-foldings, and, generically, larger primordial density fluctuations, which could be problematic. When reheating is complete, it is still possible that $\rho_r > \sigma$ for awhile, until expansion can reverse the inequality. During the phase in which $H^2 \approx \kappa^4 \rho_m^2/36$, the scale factor behaves as $a(t) = \exp[\alpha_0(t)] \sim t^{1/4}$ as found in Refs. [3,4], but once $H^2 \rightarrow \kappa^4 \sigma \rho_m/18$, the scale factor behaves as $a(t) \sim t^{1/2}$. [The term $K \exp[-4\alpha_0(t)]$ can also give rise to the $a(t) \sim t^{1/4}$ behavior, but we assume that $K \exp[-4\alpha_0(t)]$ becomes unimportant by the time reheating occurs, as discussed above]. As long as $\sigma$ is large compared with $\sim$(MeV)$^4$, the nonlinear term in Eq. (42) has no effect on cosmological nucleosynthesis.

V. CONCLUSIONS AND REMARKS

We have re-examined the evolution of homogeneous, isotropic cosmologies in the Randall-Sundrum brane world scenario, building on the earlier work of Binétruy et. al. [3] and Csaki et. al. [4]. We derived the explicit form of the equations that governed the dynamics of both the bulk and brane degrees of freedom [Eqs. (10), (11), (17) and (28)—(31) above], and deduced the necessary initial data that must be specified in order to determine a unique evolution of the bulk and brane variables. We showed that the brane dynamics decouple almost completely from the dynamics of the bulk degrees of freedom (gravitational waves), by deriving a generalization [Eq. (42) above] of the standard relation between the Hubble parameter $H$ and energy density $\rho_m$. This equation contains a term (an effective 4-D energy density) that acts like a free, relativistic species, and is the only effect of the 5-D gravitational waves on the bulk dynamics. It is natural to conjecture that this term describes the effect of a massless 5-D graviton mode that is trapped on the brane, but this needs to be explored further.

Another puzzling aspect of the Randall-Sundrum scenario is the almost complete decoupling between the bulk 5-D gravitational waves and the dynamics on the 4-D brane. A signature of this decoupling is that the function $\beta_0(t)$ cancels out of the 4-D dynamical equations and is not even determined when the 5-D equations are expanded to second order in distance from the brane. As far as the 4-D dynamics are concerned, $\beta_0$ is equivalent to a pure gauge mode. But, in the full 5-D context, $\beta_0$ is a physical degree of freedom which is determined by the solution for the 5-D gravitational waves propagating in the bulk.

A second signature of the decoupling is that one can change the initial data for the 5-D gravitational waves far from the brane without changing the constant $K$, and thus without influencing the dynamics of the 4-D spacetime on the brane. Thus it appears that, even if one aimed a pulse of gravitational radiation towards the brane, the 4-D spacetime...
is unaffected. We conjecture that this decoupling is analogous to the fact that to observe gravitational waves in 4-D one must measure the relative displacement of a pair of separated test particles; such waves cannot be detected from observations of a single test particle. In a brane of zero thickness, there are no observers displaced in the $y$ direction, which is the direction in which the gravitational wave forces associated with $\beta(y,t)$ act. However, for a brane of finite thickness, one would expect that these gravitational waves will cause pulsations in the brane which would manifest themselves as excitations that carry additional 4-D energy density. The extent to which these pulsations are excited presumably depends on the amplitude of the incident gravitational waves and the rigidity and thickness of the brane.

The cosmological equation we find for $H$ contains extra terms compared to 4-D general relativity (a term proportional to density squared and the term due to 5-D gravitational waves) but, generically, these become unimportant at late times. In particular, we analyzed how reheating at the end of inflation smoothly matches onto the standard behavior $a(t) \propto t^{1/2}$ of the radiation dominated epoch, and not the behavior $a(t) \propto t^{1/4}$ found in Ref. [3]. Thus, the Randall-Sundrum scenario is compatible with standard cosmology.

Finally, we remark that to solve the hierarchy problem, it was suggested [9] that we live in a probe brane (or "TeV" brane), which is at a distance $y_0$ away from the Planck brane, where the graviton is trapped. In this case, the tension of the TeV brane is substantially smaller than that of the Planck brane. In the cosmological setting, depending on the details, the $\rho_m^2$ term contribution to $H^2$ may no longer be negligible on the "TeV" brane, and the standard cosmology will be modified accordingly. It will be interesting to see if cosmological constraints can be consistent with this solution to the hierarchy problem.

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4 An important difference between the 5-D gravitational waves described by $\alpha$ and $\beta$ and conventional 4-D gravitational waves is that the propagation direction of the former (the $y$ direction) is not perpendicular to the direction of the gravitational wave forces. This is due to the form of the metric assumed in Eq. (1), and is analogous to near-zone, longitudinal, propagating disturbances in the gravitational field.
REFERENCES