Scaling Behaviors of Branched Polymers

Hajime Aoki¹*, Satoshi Iso²†, Hikaru Kawai³‡ and Yoshihisa Kitazawa²§

1) Institute for Theoretical Physics, UCSB
   Santa Barbara, CA 93106, USA
2) High Energy Accelerator Research Organization (KEK),
   Tsukuba, Ibaraki 305, Japan
3) Department of Physics, Kyoto University,
   Kyoto 606-8502, Japan

Abstract

We study the thermodynamic behavior of branched polymers. We first study random walks in order to clarify the thermodynamic relation between the canonical ensemble and the grand canonical ensemble. We then show that correlation functions for branched polymers are given by those for $\phi^3$ theory with a single mass insertion, not those for the $\phi^3$ theory themselves. In particular, the two-point function behaves as $1/p^4$, not as $1/p^2$, in the scaling region. This behavior is consistent with the fact that the Hausdorff dimension of the branched polymer is four.

*e-mail address : haoki@itp.ucsb.edu, JSPS abroad research fellow
†e-mail address : satoshi.iso@kek.jp
‡e-mail address : hkawai@gauge.scphys.kyoto-u.ac.jp
§e-mail address : kitazawa@post.kek.jp
1 Introduction

Branched polymers are the simplest generalization of the random walk and have been studied extensively\cite{1}\cite{2}. It is of great importance not only in statistical physics but also in particle physics, in particular, for understanding the critical behavior of random surfaces and quantum gravity\cite{3}\cite{4}\cite{5}. Our recent interest in branched polymers arose in our attempt to formulate superstring theory nonperturbatively. In the paper \cite{6}, we have studied the dynamics of the type IIB matrix model in such a framework (\cite{7} and see \cite{8} for review).

In our matrix model approach, the eigenvalues of matrices are interpreted as space-time coordinates. In these investigations, we find the system of branched polymers in a simple approximation. Although it is far from the flat four dimensional manifold, branched polymers share the same (fractal) dimensionality four with our space-time. It might be the first indication that superstring can explain the dimensionality of our space-time.

In this paper, we comment on a field theoretic description of branched polymers. It is well-known that a system of random walks is described by a free scalar field theory if there is no effect of self-avoidance. Similarly it is widely believed that the system of branched polymers is described by a scalar field theory with a three-point coupling, that is, $\phi^3$ scalar field theory. We will, however, show that it is not so by treating the universal part of the partition function carefully. The system of branched polymers without self-avoidance can be exactly solvable by introducing the grand canonical ensemble and using the so called Schwinger-Dyson technique. In order to extract the correct large $N$ limit ($N$ is the system size) or the thermodynamic limit, we have to check that the grand canonical ensemble is dominated by the larger size system. In other words, we have to extract the universal part.

Our claim is that we need a single mass insertion in each $m$-point correlation function of the $\phi^3$ scalar field theory in order to describe $m$-point correlation functions in branched polymers. Mass insertion here means a change of a propagator in each $m$-point function from an ordinary one, $1/p^2$, to $1/p^4$. In particular, the two-point function behaves as $1/p^4$, not $1/p^2$. Let us count the number of points which lie within distance $R$ from a fixed point in $d(> 4)$ dimensions. In the random walk, it can be estimated as $R^2$ by using the two-point function. We obtain $R^4$ for branched polymers by using the $1/p^4$ type propagator. So our finding are consistent with the claim that branched polymers are four dimensional fractals. A multi-point correlation function is given by a sum of graphs of the corresponding correlation function for $\phi^3$ scalar field theory at tree level with a single mass insertion in each graph.

Our main results were announced in \cite{10}. In this paper we would like to give a fuller account of our results by providing more detailed derivations and explanations. The organization of this paper is the following. In section 2, we review random walks in order to clarify the relation between the canonical and grand canonical ensemble. In section 3, we investigate branched polymers. First we define a canonical ensemble for a system of branched polymers (sec. 3.1) and then introduce grand canonical ensembles (sec. 3.2). We emphasize that the definition of a grand canonical ensemble is not unique. In sec. 3.3, we solve Schwinger-Dyson equations for the conventional grand canonical ensemble and obtain the results which correspond to the correlation functions of a scalar $\phi^3$ theory. In sec. 3.4, we consider the thermodynamic limit of the correlation functions and obtain the correct universal behavior of them. In section 4, we give a physical interpretation why the propagator behaves $1/p^4$ in branched polymers. Section 5 is devoted to conclusions and discussions. We have two
appendices A and B which derive the partition function and the two point function in the canonical ensemble of branched polymers.

2 Random Walks

In this section, we give a brief introduction to random walks in order to clarify the thermodynamic relation between the canonical and grand canonical ensemble.

The canonical partition function with the system size $N$ is given by

$$Z_N = \int \prod_{i=1}^{N} d^d y^i \prod_{i=1}^{N-1} f(y^i - y^{i+1}) = V(\hat{f}(0))^{N-1}$$  \hfill (2.1)$$

where $V$ is the total volume of the system, and $f(x)$ is a function assigned to each bond and damps sufficiently fast at long distance compared to the typical length scale $a_0$. $\hat{f}(p)$ is its Fourier transform:

$$\hat{f}(p) = \int d^d x \ e^{ipx} f(x).$$  \hfill (2.2)$$

For example, we can take $f(x) = \exp(-(x/a_0)^2/2)$.

Correlation functions for density $\rho(x) = \sum_{i=1}^{N} \delta^d(x - y^i)$ can be easily calculated. One point function becomes

$$\langle \rho(x) \rangle_N = \frac{1}{Z_N} \int \prod_{i=1}^{N} d^d y^i \prod_{i=1}^{N-1} f(y^i - y^{i+1}) \sum_{i=1}^{N} \delta^d(x - y^i) = \frac{N}{V}. \hfill (2.3)$$

Two-point function is defined as

$$\langle \rho(x^1) \rho(x^2) \rangle_N = \frac{1}{Z_N} \int \prod_{i=1}^{N} d^d y^i \prod_{i=1}^{N-1} f(y^i - y^{i+1}) \sum_{i=1}^{N} \delta^d(x^1 - y^i) \sum_{j=1}^{N} \delta^d(x^2 - y^j)$$  \hfill (2.4)$$

and its Fourier transformation is given by

$$\hat{g}_N^{(2)}(p) = \int d^d x \ \langle \rho(x) \rho(0) \rangle_N e^{-ipx}$$

$$= \frac{1}{Z_N} \left( \sum_{i,j} (\hat{f}(p))^{i-j}(\hat{f}(0))^{N-|i-j|-1} \right)$$

$$= \frac{1}{V} \sum_{i,j} \left( \frac{\hat{f}(p)}{\hat{f}(0)} \right)^{|i-j|}$$

$$= \frac{2}{V} \sum_{s=0}^{N-1} (h(p))^s (N-s) - \frac{N}{V}$$

$$= \frac{2N}{V(1-h(p))} \left( 1 - \frac{h(p)(1-h(p)^N)}{N(1-h(p))} \right) - \frac{N}{V}$$

$$= \frac{2N}{VH(p)} \left( 1 - \frac{1 - e^{-NH(p)}}{NH(p)} \right) \left( 1 + O\left( \frac{1}{N}, H(p) \right) \right), \hfill (2.5)$$

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where
\[ h(p) \equiv \hat{f}(p)/\hat{f}(0) = \exp(-(a_0p)^2/2), \]  
\[ H(p) \equiv 1 - h(p) = (a_0p)^2 / 2 + \cdots. \]  

We can approximate the two-point function as
\[ \hat{g}^{(2)}_{\kappa_0}(p) = \frac{2N}{V} \frac{1}{H(p)} \left(1 - \frac{1}{NH(p)} + O\left(\frac{1}{(NH(p))^2}\right)\right) \]

in the following scaling region:
\[ \frac{1}{N} < H(p) \ll 1, \]

or
\[ N^{-1/2}/a_0 < p \ll 1/a_0. \]

The scaling region is between the ultraviolet cutoff scale \( a_0 \) and the infrared scale of the system extent, which is given by \( \xi = a_0N^{1/2} \).

The two-point correlation function is also calculated in the grand canonical ensemble. A conventional grand canonical partition function is given as
\[ Z_{\kappa_0} = \sum_{N=1}^{\infty} \kappa_0^N Z_N = \frac{V\kappa_c\kappa_0}{\kappa_c - \kappa_0} \]  

where \( \kappa_0 \) is fugacity and \( \kappa_c = (\hat{f}(0))^{-1} \). For later convenience, we first define an unnormalized two-point function by
\[ \hat{G}_{\kappa_0}^{(2)}(p) = \sum_{N=1}^{\infty} \kappa_0^N \sum_{i,j} \hat{f}(p)^{i-j} \hat{f}(0)^{N-|i-j|-1} \]

\[ = \frac{\kappa_0}{(1-\kappa_0)^2 (1-\kappa_0\hat{f}(p))} \frac{2}{(1-\kappa_0\hat{f}(0))^2} - \frac{\kappa_0}{\kappa_0\kappa_c^2} \frac{2}{(\kappa_c - \kappa_0)^2 (\kappa_c - \kappa_0\hat{h}(p))} - \frac{\kappa_0\kappa_c}{(\kappa_c - \kappa_0)^2}. \]

Hence the normalized two-point function becomes
\[ \hat{g}^{(2)}_{\kappa_0}(p) = \frac{\hat{G}_{\kappa_0}^{(2)}(p)}{Z_{\kappa_0}} \]

\[ = \frac{1}{V(1-\kappa_0\hat{f}(0)) (1-\kappa_0\hat{f}(p))} \frac{2}{V\kappa_c} - \frac{1}{V\kappa_c - \kappa_0} \]

\[ = \frac{2\kappa_c}{\delta \kappa \delta H(p) + \frac{\delta \kappa}{\kappa_c} H(p)} \left(1 + \frac{\delta \kappa}{\kappa_c} H(p)\right) \]

(2.13)

where \( \delta \kappa = \kappa_c - \kappa_0 \). Since \( \langle N \rangle = \kappa_c / \delta \kappa \), the correlation function behaves as that of massive scalar particles
\[ \hat{g}^{(2)}_{\kappa_0}(p) \sim \frac{2\langle N \rangle}{V H(p) + 1/\langle N \rangle} \]  

(2.14)
and agrees with the result in the canonical ensemble calculation. This result gives the correlation length $\xi = a_0 N^{1/2}$, which indicates the Hausdorff dimension of random walk $d_H = 2$.

Here we give two different definitions of Hausdorff dimensions. The first one is defined in terms of the relation between the system size $N$ and the extent of the system. The infrared behavior of the above two-point function shows that the correlation damps rapidly over the length scale $\xi = a_0 N^{1/2}$. Since the extent of the system $L \sim \xi$ is proportional to $N^{1/2}$, the Hausdorff dimension is given by $d_H^{(1)} = 2$ where $d_H^{(1)}$ is defined as

$$L = a_0 N^{1/d_H^{(1)}}. \quad (2.15)$$

The second definition is to use the behavior of the correlation function at much shorter length scale than the system size $L = \xi$. In $d$-dimensional coordinate space, the density correlation behaves as

$$g^{(2)}(x) \sim \frac{\exp(-m|x|)}{|x|^{d-2}} \quad (2.16)$$

where $m = 1/\xi$. If $|x| \ll 1/m$, the mass term can be neglected and $g^{(2)}(x) \sim \frac{1}{|x|^{d-2}}$. The total number of points within a ball of radius $R$ ($R \ll L$) around a certain point is evaluated as

$$N(R) = \int_0^R d^d x g^{(2)}(x) \sim \left(\frac{R}{a_0}\right)^2 \quad (2.17)$$

and gives the Hausdorff dimension $d_H^{(2)} = 2$. Here the definition of $d_H^{(2)}$ is

$$N(R) = \left(\frac{R}{a_0}\right)^{d_H^{(2)}}. \quad (2.18)$$

The second definition of the Hausdorff dimension is determined only through the behavior of correlation functions in the scaling region $a_0 \ll x \ll \xi$ and has nothing to do with the behaviors near the infrared cutoff. Hence, it is a more appropriate definition than the first one from the thermodynamic viewpoint. Of course, it is quite natural that we should obtain the same dimension $d_H$ from the both definitions since the extent of the system is approximately evaluated at the length $L$ where $N(L) = N$.

Finally in this section, we comment on generalized types of grand canonical ensembles. Since the motivation of introducing grand canonical ensembles is to reproduce the same thermodynamic quantities as those in the canonical one, we may assign different weights in summing over different $N$. We define generalized grand canonical ensembles by

$$Z_{\kappa_0,l} = \sum_{N=1}^\infty N^l \kappa_0^N Z_N, \quad \hat{G}_{\kappa_0,l}^{(2)}(p) = \sum_{N=1}^\infty N^l \kappa_0^N \hat{G}_N^{(2)}(p), \quad \text{where } \kappa_0 \text{ is fugacity.} \quad (2.19)$$

The criterion for a ‘good’ grand canonical ensemble is such that we can take the correct thermodynamic limit, or, in other words, we can correctly take the universal part in the sum over $N$. That is, the correlation functions in the grand canonical ensemble
at the critical value of fugacity should reproduce those in the canonical ensemble for large $N$:

$$\lim_{N \to \infty} g_{N}^{(2)} = \lim_{\kappa_0 \to \kappa_{0,c}} g_{\kappa_0,l}^{(2)}. \quad (2.20)$$

This criterion does hold if the grand canonical ensemble is dominated by large $N$ systems. If the above criterion is satisfied for some value of $l$, it does hold for larger values of $l$. In the case of random walks, it already holds for the conventional grand canonical ensemble of $l = 0$ and we do not need to introduce the generalized ensembles of $l > 0$.

From eqs. (2.11), (2.12), the generalized partition functions and the correlation functions are given by

$$Z_{\kappa_0,l} = \left( \frac{\partial}{\partial \kappa_0} \right)^l Z_{\kappa_0},$$

$$\hat{G}_{\kappa_0,l}^{(2)}(p) = \left( \frac{\partial}{\partial \kappa_0} \right)^l \hat{G}_{\kappa_0}^{(2)}. \quad (2.21)$$

We are interested in the behaviors near the critical point $\delta \kappa \sim 0 \,(N \to \infty)$ and in the scaling region where $1/(\kappa_c - \kappa_0) > 1/(\kappa_c - h(p)\kappa_0) \gg 1/\kappa_c$. In this region, they become

$$Z_{\kappa_0,l} \sim \frac{V\kappa_c^{l+2}!}{(\kappa_c - \kappa_0)^{l+1}},$$

$$\hat{G}_{\kappa_0,l}^{(2)}(p) \sim \frac{2\kappa_c^{l+4}(l+1)!}{(\kappa_c - h(p)\kappa_0)(\kappa_c - \kappa_0)^{l+2}} \left( 1 + \frac{l}{l+1} \frac{\kappa_c - \kappa_0}{\kappa_c - h(p)\kappa_0} + \cdots \right)$$

$$\sim \frac{2\kappa_c^{l+4}(l+1)!}{(\kappa_c - \kappa_0)^{l+2} \kappa_0 H(p)} \left( 1 - \frac{\delta \kappa}{(l+1)\kappa_0 H(p)} + \cdots \right). \quad (2.22)$$

Since $\langle N \rangle = Z_{\kappa_0,l+1}/Z_{\kappa_0,l} = \kappa_c (l+1)/\delta \kappa$ for the generalized grand canonical ensembles of $l$, the normalized correlation functions become

$$\hat{g}_{\kappa_0,l}^{(2)}(p) \sim \frac{2\langle N \rangle}{V H(p)} \frac{1}{\langle N \rangle H(p)} \left( 1 - \frac{1}{\langle N \rangle H(p)} + \cdots \right) \quad (2.23)$$

which agrees with the previous result (2.14) of $l = 0$.

## 3 Branched polymer dynamics

### 3.1 Canonical Ensemble

Branched polymers are a statistical system of $N$ points connected by $N - 1$ bonds whose lengths are of order $a_0$. The canonical partition function is defined as

$$Z_N = \frac{1}{N!} \sum_{G:\text{tree graph}} \int \prod_{i=1}^{N} d^d y^i \prod_{(ij):\text{bond of } G} f(y^i - y^j), \quad (3.1)$$

where $f(x)$ is a function assigned to each bond in each graph, and it damps sufficiently fast at long distances compared to the typical length scale $a_0$. The presence of the factor $1/N!$ is due to the fact that the $N$ points are regarded identical.
We can calculate a partition function for each $N$, by counting the number of all possible tree graphs:

\[
Z_1 = V, \\
Z_2 = \frac{1}{2!} \hat{f}(0)V, \\
Z_3 = \frac{1}{3!} 3\hat{f}(0)^2V, \\
Z_4 = \frac{1}{4!} 16\hat{f}(0)^3V, \\
\vdots \\
Z_N = \frac{1}{N!} N^{N-2} \hat{f}(0)^{N-1}V, 
\]

(3.2)

where $\hat{f}(p)$ is a Fourier transform of $f(x)$;

\[
\hat{f}(p) = \int d^d x \, e^{ipx} f(x),
\]

(3.3)

and $V$ is the total volume of the system. A derivation of the general form (3.2) is given in the appendix A.

We define an (unnormalized) $m$-point correlation function of density operators as

\[
G_N^{(m)}(x^1, \cdots, x^m) = Z_N < \rho(x^1) \cdots \rho(x^m) >_N \\
= \frac{1}{N!} \sum_{G:\text{tree graph}} \prod_{i=1}^{N} d^d y^i \prod_{(ij): \text{bond of } G} f(y^i - y^j) \rho(x^1) \cdots \rho(x^m), 
\]

(3.4)

where the density operator is defined by

\[
\rho(x) = \sum_{i=1}^{N} \delta^{(d)}(x - y^i). 
\]

(3.5)

Due to the translational invariance, one point function is proportional to the partition function;

\[
G_N^{(1)} = \frac{N}{V} Z_N. 
\]

(3.6)

The one-point function is nothing but the partition function with one marked point.

### 3.2 Grand canonical ensemble

We then define partition functions and $m$-point correlation functions in the generalized grand canonical ensembles as in section 2:

\[
Z_{\kappa_0,l} = \sum_{N=1}^{\infty} N^l \kappa_0^N Z_N, \\
G_{\kappa_0,l}^{(m)}(x^1, \cdots, x^m) = \sum_{N=1}^{\infty} N^l \kappa_0^N G_N^{(m)}(x^1, \cdots, x^m). 
\]

(3.7)
\( \kappa_0 \) is the fugacity.

The criterion for a ‘good’ grand canonical ensemble is such that we can take the correct thermodynamic limit in the following sense. The correlation functions in the grand canonical ensemble at the critical value of fugacity should reproduce those in the canonical ensemble for large \( N \):

\[
\lim_{N \to \infty} g_N^{(m)} = \lim_{\kappa_0 \to \kappa_{0,c}} g^{(m)}_{\kappa_0,l},
\]

where we have defined normalized correlation functions as

\[
g_N^{(m)} = \frac{G_N^{(m)}}{Z_N}, \quad g_{\kappa_0,l}^{(m)} = \frac{G_{\kappa_0,l}^{(m)}}{Z_{\kappa_0,l}}.
\]

In the case of random walks, we have confirmed that it does hold for any nonnegative value of \( l \) but we need to check it in the case of branched polymers. To satisfy this criterion, the grand canonical ensembles with \( N \)-dependent weights should be dominated by large \( N \) systems. This is not assured only by taking the fugacity near the critical value. This is because, if \( G_N^{(m)} \) behaves as \( (\kappa_{0,c})^{-N} N^\alpha \) for large \( N \) and \( l + \alpha < -1 \), the summation over \( N \) is dominated by small \( N \) system, not by the large \( N \sim \kappa_c/\Delta \kappa \) even near the critical point. On the other hand, if we take a sufficiently large \( l \), we can obtain the correct large \( N \) correlation functions in the grand canonical ensembles, which are, of course, independent of \( l \).

We illustrate the above mentioned criterion by taking the partition function as an example. Since the canonical ensemble partition function (3.2) behaves at large \( N \) as

\[
Z_N \sim \frac{N^{-5/2}}{\sqrt{2\pi e^{-N} f(0)}} N^{-1} V,
\]

the grand canonical ensemble is approximated by

\[
Z_{\kappa_0,l} \sim \frac{V}{\sqrt{2\pi f(0)}} \sum_{N=1}^{\infty} N^{l-5/2} (\kappa/\kappa_c)^N \approx \frac{V}{\sqrt{2\pi f(0)}} \int_0^\infty dN \ N^{l-5/2} e^{-N\Delta \kappa/\kappa_c},
\]

where

\[
\kappa = f(0)\kappa_0, \quad \kappa_c = e^{-1}, \quad \Delta \kappa = \kappa_c - \kappa.
\]

If we take \( l \) sufficiently large, the integrand in eq. (3.12) has a peak at \( N \sim \kappa_c/\Delta \kappa \) and we can make \( N \) large by letting \( \kappa \) approach \( \kappa_c \). On the other hand, if \( l \) is not sufficiently large, a non-universal small \( N \) behavior dominates the summation and we cannot obtain the correct answer of the large \( N \) limit by a grand canonical ensemble.
3.3 Schwinger-Dyson eq.

In this subsection, we recapitulate the arguments that the correlation functions for branched polymers in the conventional grand canonical ensemble are given by massless $\phi^3$ theory. Let’s consider the following correlation functions $G^{(m)}_{\kappa_0} = G^{(m)}_{\kappa_0,l=0}$, which are suitable for Schwinger-Dyson analysis:

$$G^{(m)}_{\kappa_0}(x^1, \ldots, x^m) = \sum_{N=1}^{\infty} \kappa_0^N G^{(m)}_{N}(x^1, \ldots, x^m)$$

(3.14)

We write a Fourier transform of $G^{(m)}_{\kappa_0}(x^1, \ldots, x^m)$ as $\hat{G}^{(m)}_{\kappa_0}(p^1, \ldots, p^{m-1})$:

$$(2\pi)^d \delta^{(d)}(p^1 + \cdots + p^m) \hat{G}^{(m)}_{\kappa_0}(p^1, \ldots, p^{m-1})$$

$$= \int d^d x^1 \cdots d^d x^m e^{ip^1 x^1} \cdots e^{ip^{m-1} x^{m-1}} G^{(m)}_{\kappa_0}(x^1, \ldots, x^m).$$

(3.15)

Schwinger Dyson equation for 1-point function $\hat{G}^{(1)}_{\kappa_0}$ becomes

$$b = \kappa e^b,$$

(3.16)

where

$$b \equiv \hat{f}(0)\hat{G}^{(1)}_{\kappa_0},$$

(3.17)

as can be seen from figure 1. Figure 2 illustrates eq. (3.16).

![Figure 1: Schwinger-Dyson equation for one point function. A grey blob and a black point mean $\hat{G}^{(1)}_{\kappa_0}$ and $\kappa_0$, respectively.](image)

At the critical point,

$$b_c = 1,$$

$$\kappa_c = e^{-1},$$

(3.18)

$\partial b/\partial k$ diverges. Near this critical point, $N$ becomes large;

$$\Delta b \sim \sqrt{2e} \sqrt{\Delta \kappa} \sim 1/\sqrt{N},$$

(3.19)
Figure 2: Schwinger-Dyson equation, $\kappa = b e^{-b}$. At the critical point, $b_c = 1, \kappa_c = e^{-1}$.

where

$$\Delta b = b_c - b,$$

$$\Delta \kappa = \kappa_c - \kappa. \tag{3.20}$$

The one-point function (which is equal to the partition function of $l = 1$) now behaves as follows.

$$Z_{\kappa_0, l=1} = \sum_N N \kappa_0^N Z_N = V \sum_N \kappa_0^N G_N^{(1)} = V \hat{G}_0^{(1)}$$

$$= \frac{bV}{f(0)} \sim \frac{V}{f(0)} (1 - \sqrt{2}e\sqrt{\Delta \kappa}). \tag{3.21}$$

Next, we consider the 2-point function $\hat{G}_0^{(2)}(p)$. When we pick up any two points on a tree graph, we can fix the path connecting these two points. Thus, as can be seen from figure 3, 2-point function is calculated to be

$$\hat{G}_0^{(2)}(p) = \sum_{s=0}^{\infty} \hat{f}(p)^s (\hat{G}_0^{(1)})^{s+1}$$

$$= \frac{b}{\hat{f}(0)(1 - bh(p))}, \tag{3.22}$$

where

$$h(p) \equiv \hat{f}(p)/\hat{f}(0) \equiv 1 - H(p)$$

$$= 1 - ca_0^2 p^2 + \cdots. \tag{3.23}$$

Here $c$ is a positive constant of order one. Recall that $f(x)$ damps rapidly out of the region $0 < x < a_0$.

Near the critical point, $b \sim b_c = 1$, the 2-point correlation function behaves as

$$\hat{G}_0^{(2)}(p) = \frac{b}{\hat{f}(0)} \frac{1}{H(p) + \Delta bh(p)}$$

$$\sim \frac{1}{H(p) + N^{-1/2}}. \tag{3.24}$$
Here we used eq. (3.19). Thus, the correlation length is \( \xi = a_0 N^{1/4} \), which shows the first definition of the Hausdorff dimension defined in (2.15) to be \( d^{(1)}_H = 4 \). Let us consider the following region:

\[
a_0 < x < \xi = a_0 N^{1/4} \tag{3.25}
\]

or in momentum space

\[
\frac{1}{\xi} = N^{-1/4}/a_0 < p < 1/a_0. \tag{3.26}
\]

\( a_0 \) gives an ultraviolet cut-off whereas \( \xi \) gives an infrared cut-off length over which correlation functions damp rapidly. In the above scaling region of (3.26), the correlation function behaves as an ordinary massless field \( \hat{G}^{(2)}(p) \sim 1/p^2 \) and gives the second definition of the Hausdorff dimension of (2.18), \( d^{(2)}_H = 2 \). This is different from the Hausdorff dimension \( d^{(1)}_H \) determined from the relation of the system size \( N \) and the extent of the system. In the next section, we show that the above behavior of the correlation function in the scaling region is not correct and hence gives the incorrect Hausdorff dimension \( d^{(2)}_H \).

Finally, we consider \( m > 2 \) point correlation functions. As in the case of the two-point function, when \( m \) points are fixed on each tree graph, we can uniquely fix the path connecting these points. Therefore, an \( m \)-point function \( \hat{G}^{(m)} \) is represented as a summation over all tree diagrams with \( m \) fixed points in which \( \hat{G}^{(2)} \) appear as propagators. For example,

\[
\hat{G}^{(3)}(p, q) = (\hat{G}^{(1)})^{-2} \hat{G}^{(2)}(p) \hat{G}^{(2)}(q) \hat{G}^{(2)}(p+q). \tag{3.27}
\]

In general, we obtain the following result for \( m \)-point correlation functions;

\[
\hat{G}^{(m)} \sim \text{correlation functions of massless } \phi^3 \text{ theory at tree level.} \tag{3.28}
\]

In the next subsection, we consider generalized grand canonical ensembles with larger \( l \) and we point out that the results (3.28) do not correspond to the correct thermodynamic limit.

### 3.4 Correlation functions in thermodynamic limit

Let us consider the generalized \( m \)-point correlation functions with \( l \geq 1 \). From the definition (3.8), they can be obtained by applying \( l \)-th derivative to the \( l = 0 \) case:

\[
\hat{G}^{(m)}(p^1, \ldots, p^{m-1}) = (\kappa_0 \frac{\partial}{\partial \kappa_0})^l G^{(m)}_{\kappa_0, l=0}(p^1, \ldots, p^{m-1})
= \left( \frac{b}{1 - b \frac{\partial}{\partial b}} \right)^l G^{(m)}_{\kappa_0, l=0}(p^1, \ldots, p^{m-1}) \tag{3.29}
\]

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Partition functions are obtained from (3.21) as
\[
Z_{\kappa, l} = \left(\frac{b}{1 - b} \frac{\partial}{\partial b}\right)^{l-1} \frac{bV}{f(0)}.
\] (3.30)
Near the critical point \((b_c = 1)\), they become
\[
Z_{\kappa, l \geq 2} \sim \frac{V}{f(0)} \frac{(2l - 5)!!}{(1 - b)^{2l - 3}}.
\] (3.31)
2-point function with \(l = 1\) is given by
\[
\hat{G}^{(2)}_{\kappa, l = 1}(p) = \left(\frac{b}{1 - b} \frac{\partial}{\partial b}\right) \left(\frac{b}{f(0)(1 - bh(p))}\right)
\sim \frac{1}{f(0)(1 - b)(1 - bh(p))^2}
\sim \frac{1}{p^4}
\] (3.32)
It is because in the scaling region (3.26) and near the critical point, the following inequality holds:
\[
\frac{1}{b} \sim 1 \ll \frac{1}{1 - bh(p)} \sim \frac{1}{c(a_0 p)^2 + N^{-1/2}} \ll \frac{1}{1 - b} \sim \sqrt{N}.
\] (3.33)
This behavior is different from that of \(\hat{G}^{(2)}_{\kappa, l = 0}\) \(\sim 1/p^2\). Similarly for \(l \geq 2\), the behavior of 2-point function becomes
\[
\hat{G}^{(2)}_{\kappa, l \geq 2}(p) \sim \frac{1}{f(0)} \frac{(2l - 3)!!}{(1 - b)^{2l - 1}(1 - bh(p))^2}[1 + 2 \frac{1 - b}{1 - bh(p)} + 6 \frac{l - 2}{2l - 3} \left(\frac{1 - b}{1 - bh(p)}\right)^2 + \cdots].
\] (3.34)
Due to the inequality (3.33), the derivative \(\frac{\partial}{\partial b}\) acts dominantly on \(\frac{1}{1 - b}\), not on \(\frac{1}{1 - bh(p)}\). The normalized 2-point functions now become
\[
\hat{g}^{(2)}_{\kappa, l \geq 2}(p) \sim \frac{1}{V} \frac{2l - 3}{(1 - b)^2(1 - h(p))^2}(1 - 3 \left(\frac{1 - b}{1 - h(p)}\right)^2 \cdots)
\sim \frac{\langle N \rangle}{V} \frac{1}{H(p)^2} \frac{1}{(1 - 3 \left(\frac{1}{\langle N \rangle H(p)^2}\right))} + O((\frac{1}{\langle N \rangle H(p)^2})).
\] (3.35)
Here we have used \(\langle N \rangle = Z_{\kappa, l + 1}/Z_{\kappa, l} = (2l - 3)/(1 - b)^2\) for the generalized grand canonical ensembles of \(l\). Their \(p\)-dependences are all the same except \(l = 0\) case. The \(l = 0\) case, which can be obtained directly from the Schwinger-Dyson equation, does not reproduce the correct thermodynamic result. Instead, we should consider a ‘good’ grand canonical correlation function with \(l \geq 1\), otherwise a non-universal small \(N\) behavior affects the summation and we cannot obtain the universal result. In the appendix B, we estimate the large \(N\) asymptotic behavior of the two-point function by the saddle point method. Such an explicit analytical result is completely consistent with the analysis here.

The behavior of \(\hat{g}^{(2)}_{\kappa, l \geq 1}(p) \sim 1/p^4\) gives (the second definition of) the Hausdorff dimension (2.18) \(d_H^{(2)} = 4\), which is now consistent with (the first definition of) the Hausdorff dimension.
discussed in the previous subsection. An argument expected from the figure 3 is that the
effect of branching could be absorbed by renormalizing the mass. If so, the propagator would
behave as that of random walks with a renormalized mass and we might obtain the identical
result with the $l = 0$ case. We discuss in the next section why this argument is not correct.

Similarly, 3-point functions become

$$\hat{g}_{k_0,l=0}(p,q) \sim g(p)g(q)g(p+q),$$
$$\hat{g}_{k_0,l \geq 1}(p,q) \sim g(p)g(q)g(p+q) + g(p)g(q)g(p+q) + g(p)g(q)g(p+q), \quad (3.36)$$

where

$$g(p) = \frac{1}{1 - bh(p)} \sim \frac{1}{p^2},$$
$$g'(p) = \frac{1}{(1 - bh(p))^2} \sim \frac{1}{p^4}. \quad (3.37)$$

The behaviors do not change above $l = 1$. Only one propagator in a graph is replaced by
$g'(p)$. This is because the derivative $\partial / \partial b$ dominantly act on the factor $1/(1 - b)$ rather than
on $1/(1 - bh(p))$, as in the case of 2-point functions. Here again, we might be tempted to
argue that the only effect of branching is mass renormalization and to give rise to a three
point vertex. If so, we might conclude that the 3-point function behaves as the result of
$l = 0$ case. As we discuss in detail in the next section, this argument is not correct in the
thermodynamic limit and we should only retain $l \geq 1$ cases. Hence correlation functions
for branched polymers are expressed in terms of $\phi^3$ theory at tree level with a single mass
insertion.

For $m$-point functions ($m > 3$), we can obtain the same result.

$$\hat{g}_{m,l=0}(p_{m+1}, \cdots, p_m = 0) \sim \text{correlation functions for } \phi^3 \text{ theory at tree level}$$
$$\hat{g}_{m,l \geq 1}(p_{m+1}, \cdots, p_m = 0) \sim \text{correlation functions for } \phi^3 \text{ theory at tree level with a mass insertion.} \quad (3.38)$$

The universal correlation functions with $l \geq 1$ represent the correct correlation functions in
the thermodynamic limit.

As a consistency check, the following relation between an $(m+1)$-point function and an
$m$-point function must hold:

$$\hat{g}_{m+1,l \geq 1}(p^1, \cdots, p^m = 0) = N \hat{g}_{m,l \geq 1}(p^1, \cdots, p^{m-1}). \quad (3.39)$$

It actually holds because in the L.H.S. of eq. (3.39), the special class of diagrams dominate
in which the $m$-th end point is attached to a propagator $g'(p)$. It is due to the inequality
$g'(p = 0)g(p) \gg g'(p)g(p = 0)$. Therefore, it is equal to the R.H.S. of eq. (3.39).

4 Physical interpretation by a single mother universe

In this section, we give a physical interpretation why the propagator behaves as $1/p^4$ instead
of $1/p^2$. As we can see from the figure 3, the effect of branching seems to be absorbed by
renormalizing the mass and we might conclude that the two-point function behaves as that of random walks. If this is the case, the propagator should be given by an ordinary massive scalar particle with a renormalized mass \((a_0 N^{1/4})^{-1}\), instead of \((a_0 N^{1/2})^{-1}\). Similarly, higher point correlation functions should be given by tree graphs of \(\phi^3\) field theory. As we saw in the previous section, these are not the correct behaviors of correlations. In this section, we give a physical interpretation why the propagator behaves as \(1/p^4\) instead of a conventional behavior \(1/p^2\) and why the higher point correlation functions behave as \(\phi^3\) theory with a single mass insertion.

Let us consider the two-point function as an example. Two-point function \(\hat{G}_{\kappa=0}^{(2)}(p)\) was defined by a sum of two-point functions in the canonical ensemble (eq.(3.14)):

\[
\hat{G}_{\kappa_0}^{(2)}(p) = \sum_{N=1}^{\infty} \kappa_0^N \hat{G}_N^{(2)}(p).
\]  

(4.1)

On the other hand, from the equation (3.22), it is written as a sum of all contributions over \(s\) where \(s\) is the length between the two points in concern. Each term \(\hat{G}_{\kappa_0}^{(1)}\) represents a gray blob in figure 3. When we fix the total number of the system \(N\), the \(N\) points are distributed among \((s+1)\) blobs. We will show here that the most dominant contributions to the correlation functions are those that most of \(N\) points are concentrated on only a single blob. We call this blob the mother universe. In the branched polymer, there is only one mother universe and the other universes (blobs) contain much fewer points than the mother universe.

To show this, we first note that the 1-point function (blob) is expanded as

\[
\hat{G}_{\kappa_0}^{(1)} = \sum_{n=1}^{\infty} \kappa_0^n u_n
\]

(4.2)

where \(u_n\) is given at large \(n\) as

\[
u_n = G_n^{(1)} \sim \frac{n^{-3/2} \hat{f}(0) n^{-1} e^n}{\sqrt{2\pi}}.
\]

(4.3)

By using this expansion and eq.(3.22), we obtain the two-point function for fixed \(N\) as

\[
\hat{G}_N^{(2)}(p) = \sum_{s=0}^{\infty} \hat{f}(p)^s \left( \sum_{n_0=1}^{\infty} \cdots \sum_{n_s=1}^{\infty} u_{n_0} \cdots u_{n_s} \delta_{N,n_0+n_1+\cdots+n_s} \right).
\]

(4.4)

Each contribution in the bracket comes from a graph in which the first blob contains \(n_0\) points, the second \(n_1\) points, and so on. Using (4.3), the term in the bracket becomes

\[
\left( \frac{1}{\sqrt{2\pi}} \right)^{s+1} \hat{f}(0)^{N-s-1} e^N \sum_{n_0=1}^{\infty} \cdots \sum_{n_s=1}^{\infty} n_0^{-3/2} n_1^{-3/2} \cdots n_s^{-3/2} \delta_{N,n_0+n_1+\cdots+n_s}.
\]

(4.5)

In the case of \(s = 1\), the summation

\[
\sum_{n=1}^{N-1} n^{-3/2} (N-n)^{-3/2}.
\]

(4.6)
is dominated by terms of $n \sim 0$ and $n \sim N$. For the general case with the exponent $\alpha$,  
\[ \sum_{n=1}^{N-1} n^\alpha (N-n)^\alpha, \]  
(4.7)  
the sum is dominated at the boundaries for $\alpha < -1$ and asymmetry arises between two blobs. On the contrary, for $\alpha > -1$, $N$ points are distributed equally and neither blob is special. This argument can be generalized to $s > 1$. To conclude, most of $N$ points belong to a single blob along the propagator. We then have to divide the propagator with length $s$ at the mother universe. Since the other blobs contain only a finite number of points, the effect of branching other than dividing the propagator into two pieces is simply to renormalize the mass of propagator. Hence, the propagator behaves as a product of two ordinary ones with a renormalized mass;  
\[ \hat{G}_N^{(2)}(p) \sim \left( \frac{1}{p^2 + m^2} \right)^2. \]  
(4.8)  

We can also apply a similar argument for the higher-point functions. If the effect of branching is only to renormalize the mass term, the higher-point correlation functions will be represented by diagrams of $\phi^3$ theory with propagators $1/(p^2 + m^2)$, which is not the correct thermodynamic behavior of the correlation functions as we saw in the previous section. Similar to the case of 2-point functions above, we can argue that there is only one mother universe in which most of the $N$ points reside. Since only a single blob contains an infinitely many points, we have to divide one of the propagators at the mother universe and this propagator behaves as $1/(p^2 + m^2)^2$. This mother universe corresponds to the mass insertion. The other blobs contain a finite number of points and the effect of branching can be absorbed into mass renormalization. This is the physical reason why the higher point correlation functions are represented by tree diagrams of $\phi^3$ field theory with a single mass insertion.

The above statement that there is only one mother universe in the branched polymer is explained differently as follows. Let us consider again the original branched polymer systems with $N$ points and $N-1$ bonds. By counting the number of ways in which we can divide a branched polymer into two parts by cutting a bond, we obtain a relation  
\[ (N-1)Z_N/V = \hat{f}(0) \sum_{N'=1}^{N-1} [N'Z_{N'}/V] [(N-N')Z_{(N-N')}/V]. \]  
(4.9)  
The factor $(N-1)$ in the L.H.S. is interpreted as the number of bonds we can cut to divide the whole into two parts. The factors $N'$ and $(N-N')$ in the R.H.S. are interpreted as the number of points to which the bond connecting the two parts is attached. Since $Z_N$ behaves as in eq.(3.11) at large $N$,  
\[ NZ_N/V \gg \hat{f}(0) \sum_{N'=N\epsilon}^{N(1-\epsilon)} [N'Z_{N'}/V] [(N-N')Z_{(N-N')}/V], \]  
(4.10)  
which means the summation in the R.H.S. of eq.(4.9) is dominated by terms of $N' \sim 0$ or $N' \sim N$. This formula can be interpreted as follows. If we divide any graph into two parts by cutting some bond, we find only finite points in one of them and most of them belong
to the other. The mother universe belongs to the larger part. Consider the 2-point function as an example. By dividing the graph of figure 3 into two parts at some bond, we find that most points are dominantly distributed on only one of them.

We can apply the same procedure to the dominant part with infinite points. After repeating it several times to divide the total graph into several pieces, we find that only a single part consists of infinitely many points and the others consist of finite points. We can find out which blob in figure 3 is the mother universe when, after the several repetitions, the dominant part with infinite points is detached from the path in figure 3. We can also apply the same argument to the higher-point functions. Here we point out the difference of the argument given here and that for random surfaces[9]. Although a similar inequality appears for random surfaces, the factor $N$ is absent on the L.H.S. of the inequality (4.10) in that case.

5 Conclusion and discussion

In this paper we have shown that the correlation functions for branched polymers are given by those for $\phi^3$ theory at tree level with a single mass insertion if we correctly take the thermodynamic limit. It is not given by those for $\phi^3$ theory at tree level themselves. We have interpreted the single mass insertion as the presence of a single mother universe in branched polymers.

We have reviewed random walks in order to clarify the relation between the canonical and grand canonical ensemble. We have introduced generalized grand canonical ensembles which assign different weights for different system sizes. We have emphasized that a ‘good’ grand canonical ensemble is such that the ensemble average should be dominated by the systems of large size. In branched polymers, this criterion is not satisfied in the conventional grand canonical ensemble. Nevertheless we can consider ‘good’ grand canonical ensembles in branched polymers. Our conclusion follows as the universal prediction of ‘good’ grand canonical ensembles. It represents the correct scaling behavior of the correlation functions in canonical ensembles of large system size $N$.

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Appendix A

In this appendix, we derive the canonical ensemble partition function (3.2) from the Schwinger-Dyson equation (3.16). Let us solve $b$ as a form of expansion in $\kappa$. Each coefficient is calculated to be

$$\frac{1}{2\pi i} \int_{\kappa_0} d\kappa \frac{b}{\kappa^{N+1}} = \frac{1}{2\pi i} \int_{b_0} db (1 - b) e^{-b} \frac{b}{(b e^{-b})^{N+1}} = \frac{N^{N-1}}{N!}.$$
Hence,
\[ b = \sum_{N=1}^{\infty} \frac{N^{N-1}}{N!} \kappa^N. \] (A.3)

From eqs. (3.6), (3.14) and (3.17), \( b \) is expanded as
\[ b = \hat{f}(0) \sum_{N=1}^{\infty} \frac{N}{N} Z_N \kappa_0^N. \] (A.4)

Comparing these two expansions, we get the result of eq. (3.2).

**Appendix B**

In this appendix, we derive the two-point correlation function in the canonical ensemble for fixed but large \( N \). Similar to the calculation in appendix A, we can obtain the two-point function for a fixed \( N \) from eq.(3.22);
\[
\left( \frac{1}{f(0)} \right)^N \hat{G}_N^{(2)}(p) = \frac{1}{2\pi i} \oint_{b=0} \frac{d\kappa}{\kappa^{N+1}} \hat{G}_N^{(2)}(\kappa) \]
\[ = \frac{1}{2\pi i} \oint_{b=0} \frac{db}{b} \frac{1}{\hat{f}(0)} \left( 1 - b \right) b^{-N} e^{bN}. \] (B.1)

This integration can be estimated for large \( N \) by the saddle point approximation. Since \( b^{-N} e^{bN} = e^{N(b - \log b)} \), the saddle point is at \( b = 1 \). The steepest descent direction is along the imaginary direction. We change the variable from \( b \) to \( t \) around the saddle point as
\[ b - \log b = 1 - \frac{t^2}{2}. \] (B.2)

The new parameter \( t \) is written in terms of \( b \) by
\[ t = i(1 - b)(1 + \frac{1 - b}{3} + \frac{7}{36}(1 - b)^2 + \cdots). \] (B.3)

Solving this, we obtain
\[ \frac{1}{b} - 1 = -it(1 - \frac{2}{3}it - \frac{13}{36}t^2 + \cdots). \] (B.4)

Therefore in the saddle point approximation, the correlation function becomes
\[
\left( \frac{1}{f(0)} \right)^{N-1} \hat{G}_N^{(2)}(p) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{t}{1/b - 1 + H(p)} e^{N(1 - \frac{t^2}{2})}
\]
\[ = \frac{e^N}{2\pi} \int_{-\infty}^{\infty} dt \frac{t^2(1 - 13t^2/36 + \cdots)}{(H(p) - 2t^2/3 + \cdots)^2 + t^2(1 - 13t^2/36 + \cdots)^2} e^{-Nt^2/2}
\]
\[ = \frac{N^{-3/2} e^N}{2\pi} \int_{-\infty}^{\infty} dt \frac{t^2(1 - 13t^2/36N + \cdots)}{(H(p) - 2t^2/3N + \cdots)^2 + t^2/3N(1 - 13t^2/36N + \cdots)^2} e^{-t^2/2}. \] (B.5)
where $H(p) = 1 - h(p) \sim c(a_0 p)^2$. Hence the normalized correlation function becomes

$$\hat{g}_N^{(2)}(p) = \frac{\hat{G}_N^{(2)}(p)}{Z_N}$$

$$= \frac{N}{V} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \frac{t^2(1 - 13t^2/36N + \cdots)}{H(p) - 2t^2/3N + \cdots} + \frac{t^2/Ne^{-t^2/2}}{3N + \cdots} e^{-t^2/2}$$  \hspace{1cm} (B.6)

$$= \frac{N}{V} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \frac{t^2}{H(p)^2 - 4t^2H(p)/3N + t^2/N} e^{-t^2/2} (1 + O(1/N)).$$  \hspace{1cm} (B.7)

The correlation function for branched polymers behaves very differently from the case of simpler random walks. In the scaling region $1/\sqrt{N} < H(p) \ll 1$, the 2-point function can be evaluated as

$$\hat{g}_N^{(2)}(p) \sim \frac{N}{V} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} \frac{t^2}{H(p)^2 + t^2/N} e^{-t^2/2}$$  \hspace{1cm} (B.8)

This result for canonical ensemble is completely consistent with the result of the grand canonical ensemble (3.35).

References


