Opposite momenta lead to opposite directions

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Abstract

When a particle decays into two fragments, the wavefunctions of the latter are spherical shells with expanding radii. In spite of this spherical symmetry, the two particles can be detected only in opposite directions.

When particles at rest decay into two fragments, the latter have opposite momenta,

\[ p_1 + p_2 = 0. \] (1)

Yet, the statistical distribution of each fragment is isotropic. The same holds for pairs of particles resulting from a collision described in the center-of-momentum frame. In classical physics, it is obvious that particles with opposite momenta will be found in opposite directions. The problem is whether this property still holds in quantum mechanics. Indeed, the operator \( p_1 + p_2 \) does not commute with \( q_1 + q_2 \). Rather, there are uncertainty relations,

\[ \Delta(p_1 + p_2) \Delta(q_1 + q_2) \geq \hbar, \] (2)

for each one of the Cartesian components of these vectors. This equation says that if a system is prepared in such a way that \( p_1 + p_2 \) is sharp, then the midpoint between the two particles has a broad distribution. However, Eq. (2) does not restrict the angular alignment of the two particles. The purpose of the present article is to show that the operator equation (1) leads to an observable alignment of the detection points of the two particles.

First, consider a simpler problem, namely the motion of a single free particle. Why does a typical wavepacket move along a straight line? Let us write the initial wavefunction as a Fourier integral,

\[ \psi(r, 0) = \int f(p) e^{ip \cdot r / \hbar} dp. \] (3)

After a time \( t \), this wavefunction becomes

\[ \psi(r, t) = \int f(p) e^{i(p \cdot r - E t) / \hbar} dp, \] (4)

where \( E = p^2 / 2m \) for nonrelativistic particles (only the nonrelativistic case is considered here\(^1\)). Let us write

\[ f(p) = |f(p)| e^{iS(p) / \hbar}, \] (5)

and let us assume that \( |f(p)| \) is peaked around \( p \simeq k \), so that the main contribution to the integral in Eq. (4) comes from values of \( p \) in the vicinity of \( k \). However, this is not the only condition on the parameters appearing in that integral. Its value is usually very small because of the rapid oscillations of the exponent. The integral will be appreciably different from zero only if the phase of the exponent is stationary, namely

\[ \frac{\partial S}{\partial p} + r - \frac{\partial E}{\partial p} t = 0, \] (6)

where all the above expressions have to be evaluated for \( p = k \). Recall that \( \partial E / \partial p = \mathbf{v} \) is the (classical) velocity of a particle with momentum \( p \), and define \( r_0 := \partial S / \partial p \) (evaluated at \( p = k \)). We then have

\[ r = r_0 + \mathbf{v} t. \] (7)
We see that for a wavepacket given by Eq. (4), $\psi(r, t)$ is large only in the vicinity of the above value of $r$. In other words, the wavepacket moves in a way similar to that of a classical particle, provided that $|f(p)|$ is indeed peaked near some value of $p$, and that $S(p)$ is well behaved there.

If there are two particles, we write likewise

$$
\psi(r_1, r_2, t) = \int f(p_1, p_2) e^{i(p_1 \cdot r_1 + p_2 \cdot r_2 - Et)/\hbar} \, dp_1 \, dp_2.
$$

(8)

Then, if $|f(p_1, p_2)|$ is peaked around $p_1 \simeq k_1$ and $p_2 \simeq k_2$, the above $\psi$ describes two wavepackets, with approximate positions

$$
r_1 = r_{10} + v_1 t,
$$

(9)

and

$$
r_2 = r_{20} + v_2 t,
$$

(10)

where $r_{j0} = -\partial S/\partial p_j$ and $v_j = \partial E/\partial p_j$ (evaluated at $p_1 = k_1$ and $p_2 = k_2$).

However, this description of two wavepackets moving along straight lines does not correspond to the physical situation discussed at the beginning of this article, namely two particles having opposite momenta and an isotropic distribution. In that case, each wavepacket is an expanding shell. However, the two spherical shells are correlated, and it will be shown below that the two particles in each pair are observed along diametrically opposite directions.

Let $F(p_0, E_0)$ be the momentum and energy distribution of the particle that decays into two fragments (or of the two incoming particles in a collision). From energy and momentum conservation, we have

$$
f(p_1, p_2) = \int F(p_0, E_0) \delta(p_0 - p_1 - p_2) \, dE_0 \, dp_0,
$$

(11)

$$
= F(p_1 + p_2, E_1 + E_2).
$$

(12)

We assume that $F$ is peaked around $p_0 = 0$, so that Eq. (1) holds for the expectation values of $p_1$ and $p_2$. Moreover, let us assume that the phase of $F$ satisfies $\partial S/\partial p_j = 0$, so that the decaying system is localized near the origin of the coordinates. It is always possible to choose the coordinate system so that these conditions hold. We then have, as before,

$$
\psi(r_1, r_2, t) = \int F(p_1 + p_2, E_1 + E_2) e^{i(p_1 \cdot r_1 + p_2 \cdot r_2 - Et)/\hbar} \, dp_1 \, dp_2,
$$

(13)

where $E = E_1(p_1) + E_2(p_2)$. As usual, the phase of the exponent has to be stationary to yield a non-negligible value of $\psi$. However, in the present case, $F$ is not peaked at definite values of $p_1$ and $p_2$, but has spherical symmetry. We therefore introduce spherical coordinates such as

$$
p_{1x} = p_1 \sin \theta_1 \cos \phi_1,
$$

(14)

and

$$
r_{1x} = r_1 \sin \theta'_1 \cos \phi'_1,
$$

(15)

and likewise for the other components. We thus have

$$
p_1 \cdot r_1 = p_{1x} r_1 \cos \xi_1,
$$

(16)

where $\xi_1$ is the angle between $p_1$ and $r_1$, explicitly given by

$$
\cos \xi_1 = \cos \theta_1 \cos \theta'_1 + \sin \theta_1 \sin \theta'_1 \cos(\phi_1 - \phi'_1),
$$

(17)

and likewise for the second particle.

For given values of $r_1$ and $r_2$, the phase in Eq. (13) has to be stationary with respect to variations of the six integration variables $p_{1x}$, $\theta_1$ and $\phi_1$. Since the various angles appear in that phase only in the expressions for $\cos \xi_1$ and $\cos \xi_2$, we can as well request $\cos \xi_j$ to be stationary. This implies that $\sin \xi_j = 0$, or simply $\xi_j = 0$. (We also have $\sin \xi_j = 0$ when $\xi_j = \pi$, namely when $p_j$ and $r_j$ have opposite directions. However, this case corresponds to $t \to -\infty$ and is irrelevant to the present problem.)
The exponent thus becomes $i(p_1 r_1 + p_2 r_2 - E_1 t - E_2 t)/\hbar$ and stationarity with respect to $p_1$ and $p_2$ leads to

$$r_1 = v_1 t \quad \text{and} \quad r_2 = v_2 t,$$

where $v_j = dE_j/dp_j$ is the classical velocity of a particle of energy $E_j$.

Recall now that $F(p_1 + p_2, E_1 + E_2)$ is peaked at $p_1 + p_2 = 0$ and at $E_1 + E_2 = E_0$. In spherical coordinates, these peaks occur at

$$\theta_1 + \theta_2 = \pi \quad \text{and} \quad |\phi_1 - \phi_2| = \pi,$$

and

$$p_1^2 = p_2^2 = 2\mu E_0,$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the pair of outgoing particles.

We have seen above that the demand of a stationary phase in the integrand in Eq. (13) gives $\xi_j = 0$. It follows that the directions of $r_1$ and $r_2$ have to satisfy relations similar to Eq. (19):

$$\theta'_1 + \theta'_2 = \pi \quad \text{and} \quad |\phi'_1 - \phi'_2| = \pi.$$  

We thus see that opposite momenta lead to opposite directions, as intuitively expected. A spectacular experimental verification of this property was recently given by Pittman et al.\(^2\)

Finally, we have to evaluate how large may be deviations from perfect alignment. There are two causes for these deviations. One is that each wavepacket has a width which increases with time (until now we were only concerned by the motion of its centroid). The expressions $pr \cos \xi/\hbar$ in the exponent in Eq. (13) are maximal for $\xi = 0$ (that is, when $p$ and $r$ are parallel) and rapid oscillations start at $pr\xi^2/\hbar \sim 1$, or $\xi \sim \sqrt{\hbar/pr} = \sqrt{\lambda/r}$, where $\lambda$ is the de Broglie wavelength of the particles. As expected, the transversal deviation $\sqrt{\lambda r}$ is identical to the standard quantum limit\(^3\) $\sqrt{\hbar/m}$. The other cause of deviations from perfect alignment is that $p_0 = p_1 + p_2$ is not exactly zero, but has a width $\Delta p_0$. The resulting angular spread is of the order of $(\Delta p_0)/p_j = (\Delta p_0)/\sqrt{2\mu E_0}$. This spread does not decrease as $r \to \infty$, so that it is the main cause of deviations for $r > \hbar p/((\Delta p_0)^2)$.

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\(^1\)For the relativistic case, see A. Peres, Ann. Phys. (NY) 37, 179 (1966).

\(^2\)T. B. Pittman, Y. H. Shih, D. V. Strekalov, and A. V. Sergienko, Phys. Rev. A 52, 3429 (1995). This experiment does not involve massive particles with opposite momenta (as I consider in the present article), but correlated photons resulting from parametric down-conversion. However, the reason for their observed spatial correlation is the same as explained here.