A "POMERANCHUK" THEOREM FOR ELASTIC DIFFRACTION PEAKS

H. Cornille

Institut de Physique Nucléaire *)
Division de Physique Théorique

and

A. Martin

CERN - Geneva

A B S T R A C T

Under weak smoothness assumptions on the modulus of elastic scattering amplitudes in the physical region, we prove, using positivity properties of the absorptive parts, that the ratio of the widths of the diffraction peaks for reactions $A+B \rightarrow A+B$ and $\bar{A}+B \rightarrow \bar{A}+B$ approaches unity at infinite energy. It follows that if persistent shrinking is observed for one of the two reactions, the same must be true for the other one at sufficiently high energies.

*) Laboratoire associé au C.N.R.S.

Recently, some progress has been made in the rigorous study of the asymptotic behaviour of

$$\frac{d\sigma^P(s,t)}{dt} / \frac{d\sigma^A(s,t)}{dt}$$

at infinite energy ¹⁾. P and A refer, respectively, to the line reversed reactions

$$\begin{cases} A + B \rightarrow C + D \\ \bar{C} + B \rightarrow \bar{A} + D \end{cases} \quad (1)$$

It has been shown that for fixed momentum transfer $t < 0$ the ratio $d\sigma^A/d\sigma^P$ can only have a limit unity if the phases of the two amplitudes (1) grow less fast than the logarithm of the energy. Generalizations to momentum transfers decreasing with energy have also been considered in Ref. 1).

In the special case of elastic scattering :

$$\begin{cases} A + B \rightarrow A + B \\ \bar{A} + B \rightarrow \bar{A} + B \end{cases} \quad (2)$$

positivity properties of the absorptive part of the amplitude allow to obtain some information about the phase. A preliminary report on our work is contained in the lectures given by one of us at the summer schools of Erice and Les Houches in 1971 ²⁾.

The simplest case ³⁾ is, of course, that of forward scattering, where $\text{Im } F_{AB \rightarrow AB}$ and $\text{Im } F_{\bar{A}B \rightarrow \bar{A}B}$ are both positive. Then the phases of both amplitudes lie therefore between 0 and π . Hence, if

$$\lim_{s \rightarrow \infty} \frac{d\sigma^P(s, t=0)}{dt} / \frac{d\sigma^A(s, t=0)}{dt}$$

exists in a generalized sense, this limit is unity, and more generally, unity belongs to the set of limiting values of $d\sigma^P/d\sigma^A$ if the amplitudes are continuous.

The next relatively simple situation that we want to consider here is that of the diffraction peak. Here we shall use a definition of the diffraction peak due to Leader ⁴⁾. We call $\Delta(s)$ the width of the diffraction peak if $-t = \Delta(s) > 0$ is the smallest value, such that

$$\left| \frac{F(s, t = -\Delta(s))}{F(s, t = 0)} \right| = \frac{1}{2} \quad (3)$$

A simple condition about σ_{TOT} can be found in order that $\Delta(s)$ exists. In fact ⁴⁾ from

$$s^{-2} \left(\frac{d\sigma}{dt} \right)_{t=0} \frac{\Delta(s)}{4} < \sigma_{elastic} < \sigma_{Total} \quad (4)$$

it is easy to get

$$\Delta(s) < \frac{\text{const.}}{\sigma_{Total}} \quad (5)$$

Therefore $\Delta(s)$ exists if σ_{TOT} decreases less fast than $1/s$.

If we want to compare this definition with the more familiar definition through an exponential fit to $d\sigma/dt$

$$\frac{d\sigma}{dt} = A \exp(t/T(s)) \quad (6)$$

we see that

$$\Delta(s) = 2 \log_2 T(s) \quad (7)$$

if the cross-section has an exponential behaviour. Definition (3) is, however, more general. One could invent other definitions by considering the value of t such that

$$\left| \frac{F(s, t)}{F(s, 0)} \right| = \alpha \quad (8)$$

for some constant α value such that $0 < \alpha < 1$. For each α a width could be defined. All the considerations that follow are valid for any α . However, we treat first $\alpha = \frac{1}{2}$, corresponding to definition (8). It is an easy exercise for the reader to treat the case of $\alpha > \frac{1}{2}$.

From (3) we have for $t > -\Delta(s)$

$$|F(s, t)| > \frac{1}{2} |F(s, t=0)| > \frac{1}{2} \operatorname{Im} F(s, t=0) \quad (9)$$

On the other hand $\operatorname{Im} F$ or, more exactly, the absorptive part of the amplitude F , can be expanded in Legendre polynomials. Positivity tells us that the expansion coefficients are positive

$$\operatorname{Im} F(s, t) = \sum (2\ell+1) \operatorname{Im} f_{\ell}(s) P_{\ell}\left(1 + \frac{t}{2k^2}\right)$$

where k is the centre-of-mass momentum.

From the inequality ⁵⁾

$$\left. \begin{aligned} -\frac{1}{2} &\leq P_{\ell}(\cos \theta) \leq 1 \\ \text{for } 0 &\leq \cos \theta \leq 1 \end{aligned} \right\} \quad (10)$$

one gets

$$\left. \begin{aligned} \operatorname{Im} F(s, t) &> -\frac{1}{2} \operatorname{Im} F(s, 0) \\ \text{for } t &> -2k^2 \end{aligned} \right\} \quad (11)$$

If we confront (9) and (11), we see that inside the diffraction peak (3) it is impossible to have at the same time $\operatorname{Re} F = 0$ $\operatorname{Im} F < 0$, and so the phase of F cannot take the value $-\pi/2$. Since F is continuous and non-vanishing inside the diffraction peak, we see that the phase variation for $-\Delta(s) \leq t \leq 0$ must necessarily be less than 2π . On the other hand, the phase of the forward amplitude lies between 0 and π . Hence for

$$-\Delta(s) \leq t \leq 0, \quad s > s_0 \quad (12)$$

$$-\frac{\pi}{2} \leq \operatorname{Arg} F \leq \frac{3\pi}{2} \quad (13.a)$$

Notice that inside the two-dimensional domain (12), where we assume the continuity of F , the definition of the phase is unique.

Other definitions of the diffraction peak, as in Eq. (8) (for instance $0 < \alpha < \frac{1}{2}$) lead to similar results. In fact, as will be shown elsewhere ⁶⁾, the phases of the physical amplitudes for negative transfers T satisfy the following bound :

$$|\text{Arg. } F(s, T)| \leq \frac{3\pi}{2} + \frac{\left(T(\frac{1}{4})\right)^4}{2\pi^3} \max_{T < t < 0} \left| \frac{\text{Im } F(s, 0)}{F(s, t)} \right|^2 \quad (13.b)$$

It follows that if in Eq. (8) $\alpha = \text{const}$ then $|\text{Arg } F(s, t = -\Delta)| < \text{const.}$

All these considerations apply to both reactions $A+B \rightarrow A+B$ and $\bar{A}+B \rightarrow \bar{A}+B$. We can therefore define $\Delta^P(s)$ and $\Delta^A(s)$ and the question is to see if

$$\Delta^P(s) / \Delta^A(s) \rightarrow 1 \quad (14)$$

when $s \rightarrow \infty$.

We can assume that for s big enough, one of the two widths is larger than the other. If it is not the case, it means - since Δ^P and Δ^A are continuous - that $\Delta^P - \Delta^A$ vanish on an infinite sequence and that therefore unity belongs to the set of limiting values of $\Delta^P(s)/\Delta^A(s)$, which is what we want to prove. So, without loss of generality, we can assume for s large enough

$$0 < \Delta^P(s) < \Delta^A(s) \quad (15)$$

- A) First let us put aside the case $\Delta^P(s) \rightarrow \infty$. Then it is not clear that the analyticity domain obtained from field theory is large enough to lead to some results. Anyway from inequality (5) we see that if the total cross-section does not go to zero at infinite energy, the width of the diffraction peak stays finite.

B) The next case is that where

$$\Delta^P(s) \rightarrow \Delta \neq 0 \quad (16)$$

for $s \rightarrow \infty$.

From (15) we see that both the phase of F^P and the phase of F^A satisfy the inequalities (13). Then we can use fixed t dispersion relations for $t = -\Delta$ and hence, from Ref. 1) the limit of

$$\frac{d\sigma^P(s, -\Delta)}{dt} / \frac{d\sigma^A(s, -\Delta)}{dt} \quad (17)$$

has to be unity. Since, on the other hand, the limit of

$$\frac{d\sigma^P(s, 0)}{dt} / \frac{d\sigma^A(s, 0)}{dt} \quad (18)$$

must be unity, we see that

$$\lim \frac{d\sigma^P(s, -\Delta)}{d\sigma^P(s, 0)} = \lim \frac{d\sigma^A(s, -\Delta)}{d\sigma^A(s, 0)} = \frac{1}{4} \quad (19)$$

The width of both diffraction peaks is therefore asymptotically equal. If (17) and (18) have no limit separately, it is still possible to find a sequence on which

$$\left| \frac{F(s, -\Delta) F(\Sigma M^2 - s, 0)}{F(s, 0) F(\Sigma M^2 + \Delta - s, -\Delta)} \right| \rightarrow 1 \quad (20)$$

Finally, let us notice that in (17)-(19), the ratios are considered for fixed $t = -\Delta$ transfer values. In fact a study of the quantities (17)-(19) can also be done if we replace Δ by $\Delta^P = \Delta - t^P(s)$ where $t^P(s) \rightarrow 0$ when $s \rightarrow \infty$. Therefore it is necessary to compare $\lim |F^{P,A}| / |F^{P,A}|$ for non-fixed transfers approaching a value different from zero in the high energy limit. Such a study is given in detail in C for the physically interesting case where the limiting transfer is zero.

C) We disregard the case where Δ^P oscillates between two finite limits. We shall now consider the case where $\Delta^P(s)$ is a monotonous decreasing function tending to zero as $s \rightarrow \infty$. Then it is convenient to use the variable

$$E = s - \sum \frac{M^2}{2} \quad (21)$$

such that exchanging s and u in the high energy limit (t then goes to zero inside the diffraction peak) is approximately equivalent to changing E into $-E$. One introduces, as was done in Ref. 1),

$$t(E) = -\frac{\sqrt{-E}}{\pi} \int_0^\infty \frac{\Delta^P(E') dE'}{\sqrt{E'}(E' - E)} \quad (22)$$

where, for simplicity, we kept the same name for Δ .

Then the amplitude

$$\left. \begin{aligned} f(E, t(E)) &= F^P(s, t(E)) \\ f(-E - t(-E), t(-E)) &= F^A(s, t(-E)) \end{aligned} \right\} \quad (23)$$

is analytic in the E cut plane. The unavoidable complication is that $t(E)$ is complex for $E > 0$ and $t(E)$ does not coincide with $-\Delta(|E|)$ for $E < 0$. In Ref. 1) we have shown that

$$\left| t(E) + \Delta(|E|) \right| < \frac{C \Delta(|E|)}{\log |E|} \quad (24)$$

$$E \rightarrow \pm \infty$$

if above a certain energy one has $\Delta(E)$ decreasing and

$$\Delta(E) (\log E)^n \quad (25)$$

increasing, for some n .

In fact, (25) is a very natural assumption if one believes, as suggested by experiment

$$\frac{d\sigma}{dt}(s, t=0) \gg \text{const.} \frac{d\sigma}{dt}(s, t \neq 0) \quad (26)$$

If (26) is true, from the Bessis ⁷⁾ study, it is easy to get

$$\Delta(E) > \text{const.} (\log E)^{-2} \quad (27)$$

for almost every E .

We must show that

$$\left| \frac{f^P(E, t(E))}{f^P(E, -\Delta(E))} - 1 \right| \rightarrow 0 \text{ for } E \rightarrow \pm\infty \quad (28)$$

To do this, it is enough to show that

$$\left| \frac{f^{A,P}(E, -\Delta(E)(1+\gamma/\log E))}{f^{A,P}(E, -\Delta(E))} - 1 \right| \rightarrow 0 \text{ for } E \rightarrow +\infty \quad (29)$$

where γ is a bounded complex number. In the following, we always assume that

$$\lim_{s \rightarrow \infty} \left| \frac{F^{A,P}(s+\rho(s), \tau(s))}{F^{A,P}(s, \tau(s))} \right| = 1, \text{ and } \tau \text{ real} \quad (30)$$

$\rho \rightarrow 0$
 τ bounded ;

in fact it is clear that unity is the only acceptable limit for this quantity. To prove (29) we use again the assumption (26) which says that the ratio of the modulus of the physical amplitude to its forward part is bounded by a constant. Then, following Bessis ⁷⁾, one can prove that

$$\left| \frac{f(E, t)}{f(E, 0)} \right| < \text{const.} \quad (31)$$

for t inside the ellipse with foci $t=0$, $t=-4k^2$ and right extremity $t = \text{const}/(\log E)^2$. Then one finds that the circle

$$|t + \Delta(E)| < C \frac{\sqrt{\Delta(E)}}{\log E} \quad (32)$$

is inside the ellipse. So inside (32), using (31) and the definitions of the widths (3)-(8)

$$\left| \frac{f(E, t)}{f(E, -\Delta(E))} \right| < \left| \frac{f(E, t) f(E, 0)}{f(E, 0) f(E, -\Delta(E))} \right| < C$$

and using the Schwarz lemma

$$\left| \frac{f(E, t)}{f(E, -\Delta(E))} - 1 \right| < \frac{|t + \Delta(E)| \log s}{\sqrt{\Delta(E)}} \times \text{const.} \quad (33)$$

one sees immediately from (33) and (24) that (29) is true if $\Delta(E)$ goes to zero at infinite energy, which is the case.

Then one finds :

- i) that the phase of $f(E, t(E))$ is arbitrarily close to that of $f(E, -\Delta(|E|))$ for $E \rightarrow \pm\infty$ and $\lim |f(E, t(E))/f(E, -\Delta(|E|))|$ is unity if it exists;
- ii) that the phase of $f(E, t(E))$ which is equal to the phase of $F^P(s, t(E))/F^P(s, -\Delta(E))$ plus the phase of $F^P(s, -\Delta(E))$ is bounded. Similarly the phase of $f(-E, t(-E))$ is bounded and if we apply the results of Ref. 1), one has $\lim |f(E, t(E))/f(-E, t(-E))| = 1$ if it exists;
- iii) finally we remark that $F^P(s, -\Delta(E))/F^A(s, -\Delta(E))$ can be written as the product of three terms : $f(E, -\Delta(E))/f(E, t(E))$, $f(E, t(E))/f(-E, t(-E))$ and $f(-E, t(-E))/f(-E, \Delta(E), -\Delta(E))$, each of them having a modulus going to one asymptotically.

We deduce

$$\left| \frac{F^P(s, -\Delta(E))}{F^A(s, -\Delta(E))} \right| \rightarrow 1$$

and following the same argument as in case B), we see that the ratio of the widths of the diffraction peaks tend to unity.

In conclusion, we prove in all reasonable physical cases, without any a priori assumptions about the phases of the amplitudes, that the ratio of the widths of the diffraction peaks defined by (3)-(8) tend to unity. In particular, if $A+B \rightarrow A+B$ exhibits persistent shrinking, $\bar{A}+B \rightarrow \bar{A}+B$ should eventually shrink at sufficiently high energy. It would be interesting to make these considerations quantitative to apply them to pp and $p\bar{p}$ scattering. This is not completely hopeless because the key inequality on the phases, inequality (13) contains no arbitrary constant.

In fact one could be more ambitious and try to prove that the shape of the peaks has to be the same for particles and antiparticles. This can be done by varying α in the definition of the width. This allows to explore the peak at different heights. In this way we have found that in the interval $-\Delta_\epsilon(s) \leq t \leq 0$ where $\Delta_\epsilon(s)$ is the minimum value, such that

$$\frac{d\sigma^P(s, -\Delta_\epsilon(s))}{d\sigma^P(s, 0)} = \epsilon$$

ϵ fixed arbitrarily small but fixed,

$$\frac{d\sigma^P(s, t)}{d\sigma^A(s, t)} \rightarrow 1.$$

R E F E R E N C E S

- 1) H. Cornille and A. Martin, CERN preprint TH.1503 (1972).
- 2) A. Martin, Lectures given at the 1971 Erice Summer School, CERN preprint TH.1386 (1971), and Lectures given at Les Houches Summer School (1971).
- 3) A. Martin, Nuovo Cimento 39, 704 (1965).
- 4) E. Leader, Phys.Letters 5, 75 (1963).
- 5) T. Kinoshita and A. Martin, unpublished;
A. Martin and F. Cheung, "Analyticity Properties and Bounds of the
Scattering Amplitudes", Gordon and Breach, Science Publ., New York,
London and Paris, p.107 (1970).
- 6) H. Cornille and A. Martin, CERN preprint, to be published (Constraints
on the phases of scattering amplitudes due to positivity properties).
- 7) J.D. Bessis, Nuovo Cimento 45A, 974 (1966).