Large Gauge Ward Identity

Ashok Das

Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627

Gerald Dunne∗

Department of Physics, Technion - Israel Institute of Technology, Haifa 32000, Israel

J. Frenkel

Instituto de Física, Universidade de São Paulo, São Paulo, SP 05315-970, Brazil.

Abstract

We study the question of the Ward identity for “large” gauge invariance in 0+1 dimensional theories. We derive the relevant Ward identities for a single flavor fermion and a single flavor complex scalar field interacting with an Abelian gauge field. These identities are nonlinear. The Ward identity for any other complicated theory can be derived from these basic sets of identities. However, the structure of the Ward identity changes since these are nonlinear identities. In particular, we work out the “large” gauge Ward identity for a supersymmetric theory involving a single flavor of fermion as well as a complex scalar field. Contrary to the effective action for the individual theories, the solution of the Ward identity in the supersymmetric theory involves an infinity of Fourier component modes. We comment on which features of this analysis are likely/unlikely to generalize to the 2 + 1 dimensional theory.

∗permanent address: Department of Physics, University of Connecticut, Storrs, CT 06269
1 Introduction:

The question of “large” gauge invariance at finite temperature has been a very interesting one for several years now [1]. It is well known that the Chern-Simons action in an odd dimensional non-Abelian gauge theory is not invariant under “large” gauge transformations [2]. Rather, it shifts by a constant proportional to the topological winding number associated with the “large” gauge transformation. To be specific, consider the three dimensional Chern-Simons action

\[ S_{CS} = M \int d^3x \text{Tr} \epsilon^{\mu\nu\lambda} A_\mu (\partial_\nu A_\lambda + \frac{2g}{3} A_\nu A_\lambda) \]

where \( A_\mu \) is a matrix valued gauge field in some representation of the gauge algebra. Then, under a gauge transformation \( A_\mu \rightarrow U^{-1} A_\mu U + \frac{1}{g} U^{-1} \partial_\mu U \) the Chern-Simons action changes as

\[ S_{CS} \rightarrow S_{CS} - \frac{4\pi M}{g^2} \times 2\pi W \]

where \( W \) is the winding number of the gauge transformation defined to be

\[ W = \frac{1}{24\pi^2} \int d^3x \text{Tr} \epsilon^{\mu\nu\lambda} \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\lambda U U^{-1} \]

The winding number is a topological number (integer) and unless the gauge transformation belongs to the trivial topology class, it is clear that the Chern-Simons action will not be gauge invariant. However, if the coefficient of the Chern-Simons term is quantized in units of \( \frac{g^2}{4\pi} \), then the path integral, which involves \( \exp(iS_{CS}) \), will be invariant and we can define a consistent quantum theory [2].

Chern-Simons actions can be induced radiatively, with a perturbatively calculable coefficient. For example, for massive fermions interacting with a gauge field at zero temperature, radiative corrections due to the fermions induce a Chern-Simons term with a coefficient \( \frac{1}{2} \) (in units of \( \frac{g^2}{4\pi} \) for every flavor) [3]. Taking into account the intrinsic global anomaly [4], the effective action is in fact invariant. Alternatively, we can simply consider an even number of fermion flavors.

These radiative corrections at finite temperature are even more interesting. At one loop, in the static limit, they induce a Chern-Simons term with a temperature dependent coefficient [5] such that

\[ M \rightarrow M - \frac{g^2}{8\pi} \frac{m}{|m|} \tanh \frac{\beta|m|}{2} \]

where \( \beta \) is the inverse temperature (in units of the Boltzmann constant). This is now a continuous function of temperature and, consequently, it can no longer be quantized in units of \( \frac{g^2}{4\pi} \) even for an even number of fermion flavors. It would appear, therefore, that “large” gauge invariance would be lost at finite temperature which is quite mysterious since gauge invariance has no direct relation with temperature.
An interesting possible resolution to this puzzle comes from a study of the Chern-Simons theory in 0 + 1 dimensions [6] which has all the features of the 2 + 1 dimensional theory and yet is much simpler so that the theory can be exactly solved. It was observed there that, at finite temperature, the radiative corrections give rise to an infinite number of non-extensive terms besides the induced Chern-Simons term and that the effective action due to radiative corrections can be exactly summed in a closed form. This has to be contrasted with the case at zero temperature, where the only nontrivial radiative correction was the Chern-Simons term. Furthermore, it was observed that the summed effective action at finite temperature is invariant under “large” gauge transformations once the tree level coefficient is quantized and we have an even number of fermion flavors. This is, indeed, quite interesting since it points out that even when the Chern-Simons term itself may violate “large” gauge invariance, there may be other terms in the effective action which can compensate to make the total effective action gauge invariant.

This mechanism extends [7, 8, 9, 10, 11, 12] to 2+1 dimensional Abelian theories for a restricted class of static backgrounds $A_{\mu} = (A_0(t), \vec{A}(\vec{x}))$. However, this is not the full answer, since for these backgrounds (and for their trivial non-abelian generalizations) the “large” gauge transformations in fact have zero winding number - the shift in the Chern-Simons actions comes from a total derivative term, not from the winding number piece. Furthermore, such backgrounds only address the static limit, while the non-static limit is known to be very different [13, 14]. Of course, calculations are more difficult in the 2 + 1 dimensional theory and one does not expect to be able to sum all the terms in the effective action of this theory. Therefore, to study the problem of “large” gauge invariance in this theory, we must develop a systematic procedure. The natural idea, of course, would be to write a Ward identity for “large” gauge invariance, which relates different amplitudes and, therefore, can be perturbatively checked even if the complete effective action is difficult to evaluate. It is with this in mind that we have chosen to study the question of the Ward identity for “large” gauge transformations in the 0 + 1 dimensions. Clearly, the derivation of the Ward identity for gauge transformations which are topologically nontrivial is hard, but, at least, in 0 + 1 dimensions, we have the exact effective actions in closed forms and, therefore, such theories provide a natural starting point. In this paper, we carry out such a study in detail. In section 2, we recapitulate briefly all the relevant facts known from the studies in the 0 + 1 dimensional theories. In section 3, we try to derive the relevant Ward identity for a single flavor fermion theory from the effective action, directly by brute force. As we will show, this is quite hard since the Ward identities are extremely nonlinear. An alternate method is to look at the Ward identities in terms of the exponential of the effective action which we do in section 4. These identities are linear and easier to handle. (Of course, the nonlinearity creeps in when we transform back to the effective action.) The nonlinearity of these identities brings in many interesting features which we are not used to. Thus, for example, unlike the Ward identities for small gauge invariance, here the identities do not obey superposition. Consequently, if a theory has two distinct sectors, the sum of the effective action coming from the two sectors does not have the same structure of the identity as satisfied by the individual contributions. We point out all such features and present a brief conclusion in section 5 pointing out which features are likely/unlikely to extend to the 2 + 1 dimensional theory.
2 Recapitulation of Results:

We first recapitulate all the relevant results known from studies in 0 + 1 dimensional theories. Recall that the theory of massive fermions with $N_f$ flavors interacting with an Abelian gauge field including a Chern-Simons term is described by the action (We assume $m > 0$ for simplicity.)

$$S_{\text{fermion}} = \int dt \overline{\psi} (i \partial_t - m - A) \psi - \kappa \int dt A$$

(5)

where we have suppressed the flavor index for the fermions and we note that the last term is the Chern-Simons term in 0 + 1 dimension. It is worth emphasizing that even though the gauge field here is Abelian, this theory has all the properties of a 2 + 1 dimensional non-Abelian theory. Under a gauge transformation, $\psi \to e^{-i\lambda} \psi$, $A \to A + \partial_t \lambda$, the fermion action is invariant, but the complete action changes:

$$S_{\text{fermion}} \to S_{\text{fermion}} - \kappa 2\pi N$$

(6)

where $N$ is the appropriate winding number and it is clear that the tree level coefficient $\kappa$ of the Chern-Simons term must be quantized for the theory to be consistent. The mass term for the fermion breaks charge conjugation invariance and, consequently, the radiative corrections due to the fermions generate a Chern-Simons term at finite temperature modifying the coefficient as

$$\kappa \to \kappa - \frac{N_f}{2} \tanh \frac{\beta m}{2}$$

(7)

This is analogous to the behavior in the 2 + 1 dimensional theory. In particular, we note that, at zero temperature ($\beta \to \infty$) with an even number of flavors, this is compatible with the quantization of the coefficient of Chern-Simons term, but it poses a problem at finite temperature suggesting that gauge invariance may be violated if the temperature is nonzero.

In this case, of course, the effective action can be exactly evaluated and has the form [6, 15]

$$\Gamma = \Gamma_f^{(N_f)} - \kappa a = -iN_f \log \frac{\cosh (\beta m + ia)}{\cosh \frac{\beta m}{2}} - \kappa a$$

$$= -iN_f \log \left( \cos \frac{a}{2} + i \tanh \frac{\beta m}{2} \sin \frac{a}{2} \right) - \kappa a$$

(8)

where we have defined (We have normalized the effective action so that it vanishes for $A = 0$.)

$$a = \int dt A(t)$$

(9)

There are several things to note from the form of this effective action. First, this is a non-extensive action (involves powers of an integrated quantity). Non-extensive actions do not arise at zero temperature from requirements of locality, but locality is not necessarily respected at finite temperature. In fact, it is easily seen from small gauge invariance that, in this theory, if higher order terms do not vanish, the effective action must be non-extensive. For example, let us note [15] that
if we have a quadratic term in the effective action, of the form \( S_q = \frac{1}{2} \int dt_1 dt_2 A(t_1) F(t_1 - t_2) A(t_2) \), then invariance under small gauge transformations would require
\[
\delta S_q = \int dt_1 dt_2 \partial_1 \lambda F(t_1 - t_2) A(t_2) = - \int dt_1 dt_2 \lambda \partial_1 F(t_1 - t_2) A(t_2) = 0 \tag{10}
\]
whose general solution is \( F = \text{constant} \). If the constant is nonzero, the quadratic action becomes non-extensive (a quadratic function of \( a \)). Thus, one of the important features of this theory is that \( \Gamma_f^{(N_f)} \) and, therefore, \( \Gamma \) is a function of \( a \). This simple feature is unlikely to generalize to the 2 + 1 dimensional theory. In fact, it is already known [16] that this does not hold even in 1 + 1 dimensions because the Ward identities for small gauge invariance are not restrictive enough.

Another feature to note is that, under a “large” gauge transformation, for which
\[ a \rightarrow a + 2\pi N, \tag{11} \]
the effective action coming from the radiative corrections due to the fermions transforms as
\[
\Gamma_f^{(N_f)} (a) \rightarrow \Gamma_f^{(N_f)} (a + 2\pi N) = \Gamma_f^{(N_f)} (a) + \pi N_f N \tag{12}
\]
so that the theory continues to be well defined for an even number of fermion flavors. That is, even though the coefficient of the Chern-Simons term is no longer quantized at finite temperature, the noninvariance of this term is completely compensated for by all the higher order terms in the effective action. In fact, the most important thing to observe in this connection is that the inhomogeneous transformation of the fermion effective action is independent of temperature, as we should expect since gauge transformations are not related to temperature.

In 0 + 1 dimensions, we also know the results for the effective action of a massive, complex scalar field interacting with the Abelian gauge field [17]. Consider the theory with action
\[
S_{\text{scalar}} = \int dt \left( (\partial_t - iA) \phi^* (\partial_t + iA) \phi - m^2 \phi^* \phi \right) - \kappa \int dt A \tag{13}
\]
where, we have again suppressed the number of flavors for simplicity. The mass term in this theory does not break parity and, consequently, there is no Chern-Simons term generated. In fact, at zero temperature, the radiative corrections due to the scalar fields identically vanishes which follows from a combination of invariance under small gauge transformation and the absence of parity violation. Nonetheless, at finite temperature, the effective action coming from the scalar fields is nontrivial and has the form
\[
\Gamma_s^{(N_f)} = iN_f \log \frac{\sinh \left( \frac{\beta m + ia}{2} \right)}{\sinh^2 \frac{\beta m}{2}} - iN_f \log \left( \cos^2 a + \coth^2 \frac{\beta m}{2} \sin^2 a \right)
\]
\[
= iN_f \log \left( \frac{\cosh \beta m - \cos a}{2 \sinh^2 \left( \frac{\beta m}{2} \right)} \right) \tag{14}
\]
We see that even though there is no Chern-Simons term, we would have run into the problem of “large” gauge invariance had we done a perturbative calculation and looked at the quadratic terms.
The effective action, once again, is non-extensive and is a function of \(a\), namely, \(\Gamma = \Gamma(a)\). Furthermore, under a large gauge transformation,

\[
\Gamma^{(N_f)}_s(a) \rightarrow \Gamma^{(N_f)}_s(a + 2\pi N) = \Gamma^{(N_f)}_s(a)
\]

Namely, the action is invariant independent of the temperature.

Finally, consider a simple supersymmetric model in \(0 + 1\) dimensions, with action [17]

\[
S_{\text{super}} = \int dt \left( (\partial_t - iA)\phi^* (\partial_t + iA)\phi - m^2 \phi^* \phi + \overline{\psi}(i\partial_t - m - A)\psi \right)
+ \int dt \left( \frac{1}{2}(A + \dot{\theta})^2 + \frac{i}{2}(\lambda + \xi)(\dot{\lambda} + \dot{\xi}) \right) - \kappa \int dt A
\]

Here, in addition to the usual scalar and fermionic fields with identical number of flavors, we also have a stuckelberg multiplet of fields. The effective action for this theory is quite simple

\[
\Gamma = \Gamma^{(N_f)}_{\text{susy}}(a) + \int dt \left( \frac{1}{2}(A + \dot{\theta})^2 + \frac{i}{2}(\lambda + \xi)(\dot{\lambda} + \dot{\xi}) \right) - \kappa \int dt A
\]

where we recognize that

\[
\Gamma^{(N_f)}_{\text{susy}}(a) = \Gamma^{(N_f)}_f(a) + \Gamma^{(N_f)}_s(a) = -iN_f \log \frac{2\sinh^2 \frac{\beta m}{2}(\cos \frac{a}{2} + i \tanh \frac{\beta m}{2} \sin \frac{a}{2})}{(\cosh \beta m - \cos a)}
\]

and the transformation properties follow from our earlier discussion. With these basics, we are now ready to get into the question of the Ward identity for “large” gauge invariance.

### 3 Ward Identity (Hard Way):

To begin with, let us consider the model for a single (flavor) massive fermion interacting with an Abelian gauge field. The Lagrangian is trivially obtained from eq. (5). Let us denote by \(\Gamma^{(1)}_f\) the effective action which results from integrating out the fermions. From the general arguments of the last section, we know that this effective action will be a function of \(a\) such that under a large gauge transformation

\[
\Gamma^{(1)}_f(a) \rightarrow \Gamma^{(1)}_f(a + 2\pi N) = \Gamma^{(1)}_f(a) + \pi N
\]

The Taylor expansion of this relation gives

\[
\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{a^m(2\pi N)^n}{m!n!} \frac{\partial^{n+m} \Gamma^{(1)}_f}{\partial a^{n+m}} \bigg|_{a=0} = \pi N
\]
This is an infinite number of constraints which can also be rewritten in the form
\[ \sum_{n=1}^{\infty} \frac{(2\pi N)^n}{n!} \frac{\partial^n \Gamma_f^{(1)}}{\partial a^n} \bigg|_{a=0} = \pi N \]

\[ \frac{a^m}{m!} \sum_{n=1}^{\infty} \frac{(2\pi N)^n}{n!} \frac{\partial^{n+m} \Gamma_f^{(1)}}{\partial a^{n+m}} \bigg|_{a=0} = 0 \quad \text{for } m > 0 \tag{21} \]

Solving this set of equations is tedious, but with a little bit of work, we can determine that the simplest relation which will satisfy the infinity of relations has the form (This is, however, not the most general relation as we will discuss later in the case of the supersymmetric theory.)

\[ \frac{\partial^2 \Gamma_f^{(1)}}{\partial a^2} = i \left( \frac{1}{4} - \left( \frac{\partial \Gamma_f^{(1)}}{\partial \Gamma_f^{(1)}} \right)^2 \right) \tag{22} \]

This relation (as well as the ones following from it) can be checked explicitly for the first few low order amplitudes of the theory [15, 18] and, consequently, eq. (22) can be thought of as the Ward identity for “large” gauge invariance (or better yet, the Master equation from which all the relevant relations can be obtained by taking further derivatives with respect to \( a \)). This relates higher point functions to lower ones as we would expect a Ward identity to do and must hold at zero as well as nonzero temperature. However, unlike conventional Ward identities associated with small gauge invariance, we note that this relation is nonlinear. In some sense, this is to be expected for “large” gauge transformations and we comment on the consequences of this feature later.

We note from the explicit form of the effective action determined earlier (see eq. (8) for \( N_f = 1 \)) that eq. (22) indeed holds for the single flavor fermion theory. Let us also note here that all the higher point functions can be determined from the Ward identity in eq. (22) in terms of the one point function which has to be calculated from the theory and is known to be [15, 18]

\[ \left. \frac{\partial \Gamma_f^{(1)}}{\partial a} \right|_{a=0} = \frac{1}{2} \tanh \frac{\beta m}{2} \tag{23} \]

It follows from this (using the identity in eq. (22)) that, at zero temperature, the two point and all other higher point functions vanish. In fact, from the Ward identity (22), we can determine the form of the effective action to be (recall that \( \Gamma_f(a = 0) = 0 \))

\[ \Gamma_f^{(1)}(a) = -i \log \left( \cos \frac{a}{2} + 2i \left. \frac{\partial \Gamma_f^{(1)}}{\partial a} \right|_{a=0} \sin \frac{a}{2} \right) \tag{24} \]

Namely, the effective action can be completely determined from the knowledge of the one point function which, of course, coincides with the exact calculations. It is clear, however, that this way of deriving the Ward identity is extremely hard and, in particular, if the theory is complicated (remember that so far we have only considered a single flavor of massive fermion), then, it may be much more difficult to determine the Ward identity.
4 Simple Derivation of the Ward Identity:

To find a simpler way of deriving the “large” gauge Ward identity, let us define

\[ \Gamma(a) = \mp i \log W(a) \] (25)

where the upper sign is for a fermion theory while the lower one corresponds to a scalar theory. Namely, we are interested in looking at the exponential of the effective action (i.e. up to a factor of \(i\), \(W\) is the basic determinant that would arise from integrating out a particular field). Once again, we will restrict ourselves to a single flavor massive fermion or a single flavor massive complex scalar since any other theory can be obtained from these basic components. The advantage of studying \(W(a)\) as opposed to the effective action lies in the fact that, in order for \(\Gamma(a)\) to have the right transformation properties under a large gauge transformation (see eqs. (12),(14)), \(W(a)\) simply has to be quasi-periodic. Consequently, from the study of harmonic oscillator (as well as Floquet theory), we see that \(W(a)\) has to satisfy a simple equation of the form

\[ \frac{\partial^2 W(a)}{\partial a^2} + \nu^2 W(a) = g \] (26)

where \(\nu\) and \(g\) are parameters to be determined from the theory. In particular, let us note that the constant \(g\) can depend on parameters of the theory such as temperature whereas we expect the parameter \(\nu\), also known as the characteristic exponent, to be independent of temperature and equal to an odd half integer for a fermionic mode or an integer for a scalar mode. However, all these properties should automatically result from the structure of the theory. Let us also note here that the relation (26) is simply the equation for a forced oscillator whose solution has the general form

\[ W(a) = \frac{g}{\nu^2} + A \cos(\nu a + \delta) = \frac{g}{\nu^2} + \alpha_1 \cos \nu a + \alpha_2 \sin \nu a \] (27)

The constants \(\alpha_1\) and \(\alpha_2\) appearing in the solution can again be determined from the theory. Namely, from the relation between \(W(a)\) and \(\Gamma(a)\), we recognize that we can identify

\[ \nu^2 \alpha_1 = - \frac{\partial^2 W}{\partial a^2} \bigg|_{a=0} = \left( \frac{\partial \Gamma}{\partial a} \right)^2 \mp i \frac{\partial^2 \Gamma}{\partial a^2} \bigg|_{a=0} \]

\[ \nu \alpha_2 = \frac{\partial W}{\partial a} \bigg|_{a=0} = \pm i \frac{\partial \Gamma}{\partial a} \bigg|_{a=0} \] (28)

From the general properties of the scalar and fermion theories we have discussed, we intuitively expect \(g_f = 0\) and \(\alpha_{2,s} = 0\). However, these should really follow from the structure of the theory and they do, as we will show shortly.

The identity (26) is a linear relation as opposed to the Ward identity (22) in terms of the effective action, and holds both for a fermionic as well as a scalar mode. In fact, rewriting this in terms of the effective action (using eq. (25)), we have

\[ \frac{\partial^2 \Gamma(a)}{\partial a^2} = \pm i \left( \nu^2 - \left( \frac{\partial \Gamma(a)}{\partial a} \right)^2 \right) \mp i g e^{\mp i \Gamma(a)} \] (29)
This is reminiscent of the identity in eq. (22), but is not identical. So, let us investigate this a little bit more in detail, first for a fermionic mode. In this case, we know that the fermion mass term breaks parity and, consequently, the radiative corrections would generate a Chern-Simons term, namely, in this theory, we expect the one-point function to be nonzero. Consequently, by taking derivative of eq. (29) (as well as remembering that $\Gamma(\alpha = 0) = 0$), we determine

\[
\left(\nu_f^{(1)}\right)^2 = \left[ \frac{\partial \Gamma_f^{(1)}}{\partial \alpha} \right]_{\alpha=0} = \left[ \frac{\partial^2 \Gamma_f^{(1)}}{\partial \alpha^2} - 3i \frac{\partial \Gamma_f^{(1)}}{\partial \alpha} \right]_{\alpha=0}^{-1} \left( \frac{\partial^3 \Gamma_f^{(1)}}{\partial \alpha^3} \right)_{\alpha=0}
\]

\[
g_f^{(1)} = -2i \frac{\partial^2 \Gamma_f^{(1)}}{\partial \alpha^2} + \left( \frac{\partial \Gamma_f^{(1)}}{\partial \alpha} \right)^{-1} \left( \frac{\partial^3 \Gamma_f^{(1)}}{\partial \alpha^3} \right)_{\alpha=0}
\]  

This is quite interesting, for it says that the two parameters in eq. (26) or (29) can be determined from a perturbative calculation. Let us note here some of the perturbative results in this theory [15, 18], namely,

\[
\left. \frac{\partial \Gamma_f^{(1)}}{\partial \alpha} \right|_{\alpha=0} = \frac{1}{2} \tanh \frac{\beta m}{2}
\]

\[
\left. \frac{\partial^2 \Gamma_f^{(1)}}{\partial \alpha^2} \right|_{\alpha=0} = \frac{i}{4} \sech^2 \frac{\beta m}{2}
\]

\[
\left. \frac{\partial^3 \Gamma_f^{(1)}}{\partial \alpha^3} \right|_{\alpha=0} = \frac{1}{4} \tanh \frac{\beta m}{2} \sech^2 \frac{\beta m}{2}
\]

Using these, we immediately determine from eq. (30) that

\[
\left(\nu_f^{(1)}\right)^2 = \frac{1}{4}, \quad g_f^{(1)} = 0
\]

so that the equation (29) coincides with (22) for a single fermion flavor. Furthermore, we determine now from eq. (28)

\[
\alpha_{1,f}^{(1)} = 1, \quad \alpha_{2,f}^{(1)} = \pm i \tanh \frac{\beta m}{2}
\]

The two signs in of $\alpha_{2,f}^{(1)}$ simply corresponds to the two possible signs of $\nu_f^{(1)}$. With this then, we can solve for $W(a)$ in the single flavor fermion theory and we have (independent of the sign of $\nu_f^{(1)}$)

\[
W_f^{(1)}(a) = \cos \frac{a}{2} + i \tanh \frac{\beta m}{2} \sin \frac{a}{2}
\]

which can be compared with eq. (8).

For a scalar theory, however, we know that the mass term does not break parity. Consequently, we do not expect a Chern-Simons term to be generated simply from symmetry arguments. In fact,
in the scalar theory with parity as a symmetry, there cannot be any odd terms (in $a$) in the effective action. Consequently, taking derivatives of eq. (29) and keeping this in mind, we obtain

$$\nu_s^{(1)} = 3i \left[ \frac{\partial^2 \Gamma_s^{(1)}}{\partial a^2} \right]_{a=0} = \left. \frac{1}{\sinh^2(\beta m/2)} \right|_{a=0}$$

$$g_s^{(1)} = -\frac{i}{2} \left[ \frac{\partial^4 \Gamma_s^{(1)}}{\partial a^4} \right]_{a=0} = \left. \frac{i}{2} \left( 1 + \frac{3}{2 \sinh^2(\beta m/2)} \right) \right|_{a=0}$$

Once again, we see that the two parameters in the Ward identity can be determined from the first two nontrivial amplitudes of the theory. For the scalar theory, the necessary nontrivial amplitudes can be easily computed [17]. (Calculationally, the scalar theory coincides with two fermionic theories with masses of opposite sign if we neglect the negative sign associated with fermion loops. The amplitudes can also be determined from the explicit form of the effective action in eq. (14).)

$$\frac{\partial^2 \Gamma_s^{(1)}}{\partial a^2} \bigg|_{a=0} = \frac{i}{2 \sinh^2(\beta m/2)}$$

$$\frac{\partial^4 \Gamma_s^{(1)}}{\partial a^4} \bigg|_{a=0} = -\frac{i}{2 \sinh^2(\beta m/2)} \left( 1 + \frac{3}{2 \sinh^2(\beta m/2)} \right)$$

It follows from this that

$$\nu_s^{(1)} = 1, \quad g_s^{(1)} = \frac{\cosh \beta m}{2 \sinh^2(\beta m/2)}$$

This is indeed consistent with our expectations. Furthermore, we now determine from eq. (28)

$$\alpha_{1,s}^{(1)} = -\frac{1}{2 \sinh^2(\beta m/2)}, \quad \alpha_{2,s}^{(1)} = 0$$

so that we can write

$$W_s^{(1)}(a) = \frac{(\cosh \beta m - \cos a)}{2 \sinh^2(\beta m/2)}$$

This can be compared with eq. (14).

Thus, we see that the Ward identities for the single flavor fermion and scalar theories are given, in terms of the effective actions, respectively by

$$\frac{\partial^2 \Gamma_f^{(1)}}{\partial a^2} = i \left( \frac{1}{4} - \left( \frac{\partial \Gamma_f^{(1)}}{\partial a} \right)^2 \right)$$

$$\frac{\partial^2 \Gamma_s^{(1)}}{\partial a^2} = -i \left( \frac{1}{4} - \left( \frac{\partial \Gamma_s^{(1)}}{\partial a} \right)^2 \right) + \frac{i \cosh \beta m}{2 \sinh^2(\beta m/2)} e^{\Gamma_s^{(1)}}$$

As we have already pointed out, these are nonlinear identities and, therefore, superposition does not hold. In fact, even if we are considering only fermions (or scalars) of $N_f$ flavors, the identity
modifies in a nontrivial manner (which can be derived by simply noting that $W^{(1)} = e^{\pm (i/N_f) \Gamma^{(N_f)}}$), namely,

$$
\frac{\partial^2 \Gamma^{(N_f)}_f}{\partial a^2} = i N_f \left( 1 - \frac{1}{N_f^2} \left( \frac{\partial \Gamma^{(N_f)}_f}{\partial a} \right)^2 \right)
$$

$$
\frac{\partial^2 \Gamma^{(N_f)}_s}{\partial a^2} = -i N_f \left( 1 - \frac{1}{N_f^2} \left( \frac{\partial \Gamma^{(N_f)}_s}{\partial a} \right)^2 \right) + \frac{i N_f \cosh \beta m}{2 \sinh^2(\beta m/2)} e^{\Gamma^{(N_f)}_s} \tag{41}
$$

Incidentally, let us note that although the Ward identity is linear in $W^{(a)}$, for $N_f$ flavors $W^{(N_f)}$ is a product of $W^{(1)}$’s, not a sum, and so superposition is lost.

It is clear, therefore, that while the Ward identity is simple for the basic fermion and scalar modes, for arbitrary combinations of these modes, the identity is bound to be much more complicated. However, we can still derive the Ward identity from the basic identities for a single flavor fermion and scalar theory. Thus, as a simple example, let us consider the supersymmetric theory discussed in section 2 (see eq. (16-18)) for a single flavor. In this case, we have simply a sum of a fermionic and a complex bosonic degree of freedom, and defining

$$
\Gamma^{(1)}_{\text{susy}}(a) = -i \log W^{(1)}_{\text{susy}}(a) \tag{42}
$$

where

$$
W^{(1)}_{\text{susy}}(a) = \frac{W^{(1)}_f(a)}{W^{(1)}_s(a)} \tag{43}
$$

we see that we can no longer write a single identity for $W^{(1)}_{\text{susy}}(a)$. Rather, we will have a coupled set of identities, one of which, say the one for the fermions will be decoupled.

On the other hand, since the fermion equation is uncoupled and can be solved (see eq. (34)), we can use the solution to write a single identity for $W^{(1)}_{\text{susy}}(a)$ and, therefore, $\Gamma^{(1)}_{\text{susy}}(a)$:

$$
i \frac{\partial^2 \Gamma^{(1)}_{\text{susy}}}{\partial a^2} + \left( \frac{\partial \Gamma^{(1)}_{\text{susy}}}{\partial a} \right)^2 - \frac{3}{4} = \tanh(\frac{\beta m + ia}{2}) \frac{\partial \Gamma^{(1)}_{\text{susy}}}{\partial a}
$$

$$
= -\frac{\cosh \beta m \cosh \frac{\beta m}{2}}{2 \sinh^2 \frac{\beta m}{2}} e^{i \Gamma^{(1)}_{\text{susy}}} \tag{44}
$$

This is very different from eq. (22) and yet satisfies the infinite set of constraints in eq. (21). (Let us note here that the invariance properties of $\Gamma^{(1)}_{\text{susy}}(a)$ is the same as that for a single flavor fermion theory.) Thus, as we had mentioned earlier, the identity for a basic single flavor theory is simpler. The identities following from eq. (44) can be perturbatively checked. In fact, the identity can even be solved in the following way. Consider eq. (44). Using the method of Fourier decomposition, it
is seen after some algebra that the solution to eq. (44) has the form

$$W^{(1)}_{\text{susy}}(a) = (1 - e^{-\beta m}) \sum_{k=0}^{\infty} e^{-k\beta m} \left[ \cos(k + \frac{1}{2})a + i \tanh^2 \frac{\beta m}{2} \sin(k + \frac{1}{2})a \right]$$

(45)

It is interesting that even for this simple model, which contains just a single flavor of fermion and a complex scalar, $W(a)$ becomes a sum over infinitely many distinct Fourier modes as opposed to the case of either a single fermion or a single complex scalar where $W(a)$ involves only a single Fourier component. Let us note here that, although $W^{(1)}_{\text{susy}}(a)$ in eq. (45) appears different from that in eq. (18), the series in (45) can, in fact, be summed [19] and coincides with the result in (18) for a single flavor.

5 Conclusion:

In this paper, we have systematically studied the question of “large” gauge invariance, in the $0 + 1$ dimensional Chern-Simons theory. The effective actions, in $0 + 1$ dimensions, are functions of $a = \int dt A(t)$ which is a consequence of Ward identities for small gauge invariance and makes the derivation of the “large” gauge identities relatively simple. This is a feature that we do not expect to generalize to $2 + 1$ dimensions, except for special static backgrounds. Explicitly, we have derived the “large” gauge Ward identities for a single flavor fermion theory as well as a single flavor complex scalar theory interacting with an Abelian gauge field. These identities are simple, but nonlinear. The Ward identity for any other theory can be derived from them and is, in general, more complicated because of the nonlinear nature of the identities. In particular, we have shown that the solutions of the Ward identity for a single flavor fermion theory or a single flavor complex scalar theory involves a single characteristic index and is simple while for a more complex theory (even for a sum of just a single fermion and a single scalar), it involves a sum over an infinity of Fourier modes. This is a feature which we believe would generalize to $2 + 1$ dimensions.

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