The World of Baby Skyrmions:
Numerical Studies of (2+1)D Topological Skyrme-like Solitons
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Abstract

This is one of the 'New Talents' seminars at 'Erice International School of Subnuclear Physics 1999' and looks at numerical studies of (2+1)D topological Skyrme-like solitons; the baby skyrmions. We explain the concept of integrable and topological solitons. Then we introduce the non-linear $\sigma$ model in 2D in order to discuss the 3D nuclear Skyrme model. We explain that the baby Skyrme models can be viewed as a (2+1)D version of the Skyrme model. We describe the numerical methods needed to study the baby skyrmions. Finally, we present some results on skyrmion scattering and our work on skyrmions with higher topological charge.

1 Introduction to solitons

The concept of solitons goes back to J. Scott Russell’s discovery of 'great waves of translations' on the Edinburgh-Glasgow canal in 1834. He observed that a boat which suddenly stops creates a single, localised water wave; now named ‘a solitary wave’. Solitons are those solitary waves which keep their identity after interactions. Later work by Boussinesq (1871), by Lord Rayleigh (1876) and by Korteweg and de Vries (1895) shows that this solitary wave can be described by the KdV equation:

$$u_t + u_x + uu_x + u_{xxx} = 0.$$  \hspace{1cm} (1)

The stability of the KdV solitary wave is due to the fine dynamical balance between the non-linear and the dispersive term in the differential equation. In 1968, Miura, Gardner and Kruskal showed that the KdV equation conserves an infinite number of quantities of motion. The KdV system is an integrable system. Techniques like Lax pair, Inverse Scattering and Bäcklund transformations are available and they enable us to find explicit solutions. This makes integrable solitonic systems good toy models to probe physical ideas; albeit not very realistic ones. The main weak points are: finding integrable systems in other than (1+1)D, imposing Lorentz invariance, adding a small perturbative term without breaking integrability and ensuring the presence of annihilation processes for solitons. (see [?] for an introduction to solitons)
The simplest example of a topological soliton is the Sine-Gordon model whose field is described by
\[ L = \partial_\mu \phi \partial^\mu \phi + (1 - \cos \phi). \] (2)
The lagrangian is invariant under \( \phi \rightarrow \phi + 2\pi n, n \in \mathbb{Z} \). \( \phi \) is an angle in field space, the circle \( S^1 \). Let us re-emphasise our definition of a soliton: it is a localised finite-energy field configuration which keeps localised after interactions. The field has to go to the vacuum sufficiently fast for the soliton to be localised and of finite energy. Therefore, we can identify the spatial infinity in each direction with one single point and compactify the one-dimensional space \( \mathbb{R}^1 \) to \( S^1 \). The field theory of the Sine-Gordon model can be described by the map
\[ \mathcal{M}(t) : S^1 \rightarrow S^1 \] (3)
at a given time \( t \). This non-trivial mapping gives us the possibility to partition the space of all possible field configurations into equivalence classes having the same topological charge or winding number. We can visualise this concept with a belt. We can trivially close it or we can twist one side by 180 degrees and close it or we can anti-twist it by 180 degrees i.e. twist it by -180 degrees and close it. We introduce a twist into the belt which cannot be un-done unless you open the belt. Topological solitons are very much twisted field configurations fixed by boundary conditions. In a formal language, we get a non-trivial homotopy group \( \Pi_1(S^1) = \mathbb{Z} \) from the map of the Sine-Gordon field which allows us to add or subtract field configurations. For example, if we twist the belt twice and anti-twist it twice, we get back to an untwisted belt: very much like an annihilation process in particle physics. The ‘twist’ i.e. the topological charge is fixed by boundary conditions and conserved. Further, we can easily impose Lorentz invariance by only using covariant lagrangian terms and work in any dimension by construction. The weak points of topological solitons are: generally they are non-integrable systems, few techniques are available (only Ansätze, topological bounds, etc.) and a computational approach is required. The Sine-Gordon model is a very good toy model, because it is topological and integrable at the same time: it is close to real physics and exactly solvable. Further, we can compare our numerical methods with exact results. (see [7, section 2.5] for more details)

2 Non-linear \( \sigma \) model in 2D

The non-linear \( \sigma \) model is described by
\[ L = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} \] (4)
where \( \vec{\phi} \) is a three-dimensional field vector on the sphere \( S^2 \) i.e. \( \vec{\phi} \cdot \vec{\phi} = 1 \). We could also use a complex field \( W \) which represents the stereographic projection of the sphere. Again, we have a map
\[ \mathcal{M}(t) : S^2 \rightarrow S^2 \] (5)
which leads to the second homotopy group \( \Pi_2(S^2) = \mathbb{Z} \). The existence of the topological charge i.e. the twisted field configuration representing a soliton is ensured by topology. However, we also need to make sure that the soliton is dynamically stable i.e. that it has a stable scale. Derrick’s theorem provides a necessary but not sufficient condition for stability:
\[ \left. \frac{dE[\tilde{\phi}(\lambda x)]}{d\lambda} \right|_{\lambda=1} = 0 \]
\[ \left. \frac{d^2E[\tilde{\phi}(\lambda x)]}{d\lambda^2} \right|_{\lambda=1} \geq 0. \] (6)
The lagrangian is scale invariant and does not satisfy the second condition. A change of scale does not change the energy. Numerical simulations are unstable, because numerical errors change the scale of the soliton without loss of energy. Another important feature is the existence of a topological bound, the Bogomolnyi bound

$$E \geq 4\pi n.$$  \hfill (7)

If we were to extend the model to (3+1)D, the $\sigma$ model term would expand the soliton: another term needs to be added to ensure stability. (See [?, section 3.2 and 3.3] for further details)

3 The nuclear Skyrme model in 3D

In the early 60’s, Skyrme came up with a theory to describe the hadronic spectrum i.e. a theory of mesons and baryons [?]. Its lagrangian

$$L = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \theta_S \left[ (\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi})^2 - (\partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi})(\partial^\mu \vec{\phi} \cdot \partial^\nu \vec{\phi}) \right]$$  \hfill (8)

is a straightforward extension of the non-linear $\sigma$ model plus the addition of a fourth-order term called the Skyrme term. We need to include this extra term to ensure stability according to Derrick’s theorem. The mapping becomes

$$\mathcal{M}(t) : S^3 \longrightarrow S^3.$$  \hfill (9)

In effect, the most commonly used notation is in terms of a coordinate set on the $SU(2)$ group manifold. Skyrme had the ingenious idea to construct an effective field theory of mesons where the baryons are the topological solitons of the theory. Baryon conservation is equivalent to the conservation of the topological charge. His ideas were put aside by the success of QCD in the description of hadrons in terms of quarks. A crucial step forward in the understanding of QCD is the emergence of the idea of confinement. In QED, the running coupling constant decreases with distance of interaction and a perturbation expansion is very effective. In QCD, the inverse is true: only at small distance ergo high energy does the coupling become small and the quarks are free particles. At low energies, people believe that the quarks form colourless colour-singlet states: the hadrons. This must be true as there is no experimental evidence to the contrary. So far, no-one has come up with a decent way of showing confinement in QCD. The lack of a small parameter to do a perturbation expansion is evident. In the mid 70’s, t’Hooft [?] generalised QCD to be invariant under a $SU(N_C)$ gauge group. He realised that the quantum treatment of QCD simplified considerably and it is possible to derive some qualitative statements in the large $N_C$ and low energy limit. This comes from the fact that non-planar diagrams and internal quark loops are suppressed by a factor of $N_C^{-2}$ respectively $N_C^{-1}$. For large $N_C$, QCD is a weakly interacting theory of mesons with the meson-meson interaction of order $N_C^{-1}$. For $N_C \rightarrow \infty$, QCD is therefore a theory of mesons which are free and non-interacting. Witten [?] realised that weakly coupled theories may exhibit non-perturbative states like solitons or monopoles. He showed that baryons behave as if they are solitons in a large-$N_C$ meson theory. The derivation of effective actions from the QCD lagrangian is an unsolved problem. The Skyrme model is the simplest of a candidate theory for the low-energy effective lagrangian of QCD. Of course, one may add some higher order correction terms. See [?, chapter 9] for a simple review on the relation between QCD and the Skyrme model.

There are difficulties surrounding the quantization of the (3+1)-dimensional Skyrme model. First of all, the quantisation is ambiguous, because the theory is not renormalisable. One can do a renormalisation to a given order, but it will depend on the scheme. Therefore, it is essential to compare the scheme-dependent results with experimental data. In 1981, Adkins, Nappi and Witten...
[?] partially overcame these problems by using a spinning-top approximation; the 1-skyrmion i.e. skyrmion of charge one is radially symmetric and rotating without deformation. Effectively, they only include the rotational zero mode of the 1-skyrmion. This procedure allows to quantise the 1-skyrmion as a proton and gives reasonable agreement with experiments. The numerical work by Braaten et al. and Battye and Sutcliffe [?] on the structure of classical multi-skyrmions, i.e. skyrmions of charge two and more, supports further the idea that an appropriate quantization around these minimal-energy solutions for a given topological sector could lead to an effective description of atomic nuclei. However, the calculation of quantum properties of the multi-skyrmions is very difficult, because these minimal-energy solutions are not radially symmetric, for example. This is rather frustrating, for the claim that the Skyrme model descending from a large N-QCD approximation models mesons, baryons and higher nuclei is a very compelling one. There have been various attempts to extract quantum properties, notably by Walet [?] and Leese et al. [?]. A recent zero mode quantization of multi-skyrmions from charge four to nine has been undertaken by Irvine [?]. Reliable quantitative results were not to be expected, but even the correct quantum number for the ground states of some nuclei, skyrmions with odd charge 5, 7 and 9, could not be obtained. All these attempts rely on theoretical approximations like moduli-space or rational map Ansatz and mostly include only zero modes.

4 The baby Skyrme models

The nuclear Skyrme model has two big problems: quantization is ambiguous and very hard and numerical simulations are computationally very intense even on supercomputers. It makes sense to look for a (2+1) version. The baby Skyrme model is a modified version of the $S^2$ sigma model and the lagrangian is

$$L = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \theta_S \left[ (\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi})^2 - (\partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi})(\partial^\mu \vec{\phi} \cdot \partial^\nu \vec{\phi}) \right] - \theta_V V(\vec{\phi})$$

(10)

The addition of a potential and a Skyrme term to the lagrangian ensures stable solitonic solutions. The Skyrme term has its origin from the nuclear Skyrme model and the baby Skyrme model can therefore be viewed as its (2+1)D analogue. It plays the role of a toy model to explore Skyrme-like properties. However, in recent years people have also tried to use the baby Skyrme model to model aspects of the Quantum Hall Effect. Further, in (2+1)D, a potential term is necessary in the baby Skyrme models to ensure stability of skyrmions; this term is optional in the (3+1)D nuclear Skyrme model. We find the minimal-energy solution for topological charge one by using the Hedgehog Ansatz;

$$\vec{\phi} = \begin{pmatrix} \sin[f(r)] \sin(n\theta - \chi) \\ \sin[f(r)] \cos(n\theta - \chi) \\ \cos[f(r)] \end{pmatrix}.$$  \hspace{1cm} (11)

Note that $n$ is a non-zero integer (it is the topological charge), $\theta$ the polar angle, $\chi$ a phase shift and $f(r)$ the profile function satisfying certain boundary conditions. We end up with a one-dimensional energy functional

$$E = (4\pi)^{1/2} \int_0^\infty r dr \left( f'^2 + n^2 \sin^2 f (1 + 2\theta_S f'^2) + \theta_V \tilde{V}(f) \right).$$ \hspace{1cm} (12)

The corresponding Euler-Lagrange equation with respect to the invariant field $f(r)$ leads to a second-order ODE,

$$\left( r + 2\theta_S n^2 \sin^2 f \right) f'' + \left( 1 - 2\theta_S n^2 \sin^2 f + \frac{2\theta_S n^2 \sin f f'}{r} \right) f'$$
\[- \frac{n^2 \sin f \cos f}{r} - r \frac{\theta_V}{2} \frac{d\tilde{V}(f)}{df} = 0, \]  
(13)

which we re-write in terms of the second derivative of the profile function:

\[ f'' = F(f, f', r). \]  
(14)

The profile function \( f(r) \) is a static solution of the baby Skyrme model. These static solutions are certainly critical points, but not necessarily global minima. However, it is reasonable to assume that the hedgehog solution for topological charge one is the minimal-energy solution.

Taking into account the \( S^2 \) constraint, the equation of motion is

\[ \partial_{\mu} \frac{\partial_{\mu}}{\partial_{\nu}} \phi_a - (\vec{\partial} \cdot \partial_{\nu} \vec{\partial}) \phi_a - 2\theta_S [ (\partial_{\nu} \vec{\partial} \cdot \partial_{\nu} \vec{\partial}) \partial_{\mu} \partial_{\mu} \phi_a + (\partial_{\mu} \vec{\partial} \cdot \partial_{\mu} \vec{\partial}) \partial_{\nu} \phi_a - (\partial_{\mu} \vec{\partial} \cdot \partial_{\mu} \vec{\partial}) \partial_{\nu} \phi_a + (\partial_{\nu} \vec{\partial} \cdot \partial_{\nu} \vec{\partial}) \partial_{\mu} \phi_a ] + 2\theta_V \frac{dV}{d\phi_3} (\delta_{a3} - \phi_a \phi_3) = 0 \]  
(15)

which we re-write in terms of the acceleration of the field \( \phi_a \):

\[ \partial_{tt} \phi_a = K_{ab}^{-1} F_b \left( \vec{\partial}_t \phi_a, \vec{\partial}_t \phi_a \right) \]  
(16)

with

\[ K_{ab} = (1 + 2\theta_S \partial_{\mu} \vec{\partial} \cdot \partial_{\nu} \vec{\partial}) \delta_{ab} - 2\theta_S \partial_{\mu} \phi_a \partial_{\nu} \phi_b. \]  
(17)

We find that the inverse matrix of \( K \) exists in an explicit, but rather messy form. The equation of motion is a second order PDE.

One drawback of the model is that the potential term is free for us to choose. The most common choices are: \( V = (1 + \phi_3)^4 \) (the holomorphic model chosen so that the 1-skyrmion solution is \( W = \lambda z \) (discussed in [?]), \( V = (1 + \phi_3) \) (a 1-vacuum potential (studied in [?])) , \( V = (1 - \phi_3)(1 + \phi_3) \) (a 2-vacua potential (studied in [?])). Except for the first potential, no minimal-energy solutions are known explicitly. The baby Skyrme model is a non-integrable system and explicit solutions to its resulting differential equations are nearly impossible to find. Numerical methods are the only way forward.

## 5 Numerical Techniques

We are interested in two kinds of studies: the minimal-energy solution in a given topological sector and the time-evolution of configurations. We have used numerical techniques based on solving the corresponding Euler-Lagrange equation. A randomisation technique e.g. Simulated Annealing might be more effective in finding the global minimal energy solution: an initial field is randomly perturbed and changes are selected via the Metropolis algorithm which accepts changes to lower energy and allows changes to higher energies according to a certain probability. (see paper by Hale, Schwindt and Weidig [?])

We need (16) for the time-evolution and relaxation of an initial configuration and use (14) to find static hedgehog solutions. These DEs are re-written as sets of two first order DEs. We discretise DEs by restricting our function to values at lattice points and by reducing the derivatives to finite differences.\(^1\) (as explained in [?], see also [?]). We take the time step to be half the lattice spacing: \( \delta t = \frac{1}{2} \delta x \). We use fixed boundary conditions i.e. we set the derivatives to zero at the boundary. We check our numerical results via quantities conserved in the continuum limit and by changing lattice spacing and number of points. Moreover, we compare them with theoretical predictions.

\(^1\) we use the 9-point laplacian
Minimal Energy solutions of the hedgehog Ansatz  The static hedgehog solutions of (14) are found by the shooting method using the 4th order Runge-Kutta integration and the boundary conditions imposed by the finite energy condition. Alternatively, one can use a relaxation technique like the Gauss-Seidel over-relaxation ([?]) applied to an initial configuration with the same boundary conditions.

Time-evolution and Non radially symmetric Minimal Energy Solutions The time-evolution of an initial configuration is determined by the equation of motion (16). We are using the 4th order Runge-Kutta method to evolve the initial set-up and correction techniques to keep the errors small. We relax i.e. take out kinetic energy by using a damping (or friction) term. We construct n-skyrmions by relaxing an initial set-up of n 1-skyrmions with relative phase shift of $\pi_n$. Using the dipole picture developed by Piette et al. [?], two baby 1-skyrmions attract each other for a non-zero value of the relative phase; phase shift of $\frac{\pi}{2}$ for maximal attraction. A circular set-up is crucial as they maximally attract each other and 1-skyrmions do not form several states that repel each other. We have run simulations for non-circular set-ups, but either the 1-skyrmions take longer to fuse together or they fuse into many bound-states and repel each other. We run our simulations on grids with $200^2$ or $300^2$ lattice points and the lattice was $\delta x = 0.1$ or 0.05. However, for large topological charge, we need larger grids and the relaxation takes a long time. The corresponding hedgehog solution as an initial set-up usually works well and is faster, but biased due to its large symmetry group.

Initial set-up: The initial field configuration is a linear superposition of static solutions with or without initial velocity; typically we use a circular set-up of n 1-skyrmions with a $\frac{\pi}{n}$ phase-shift between each other (for maximal attraction). The superposition is justified, because the profile function decays exponentially. The superposition is done in the complex field formalism i.e. where $W$ is the stereographic projection of the $\vec{\phi}$ field of $S^2$ (see [?]). We use the profile function of a static solution (typically of topological charge one) to obtain

$$W = \tan \left( \frac{f(r)}{2} \right) e^{-i n \theta}. \quad \text{(18)}$$

This equation holds in the rest frame of a static skyrmion solution centred around its origin and $\frac{dW}{dr} = 0$. We may introduce moving solutions by switching to a different frame of reference. This can be done by performing a Lorentz boost on the rest frame of a given $W$, because $W$ is a Lorentz scalar. It is important that the different skyrmions are not too close to each other. Finally, the different $W$ fields are added together and the complex field is re-written in terms of the field $\vec{\phi}$ and its derivative.

Correction techniques on $S^2$: The integration method introduces small errors which eventually add up. In terms of the $S^2$ constraint, this corresponds to the field leaving the two-sphere. Hence, we need to project the field back onto the sphere. The simplest and sufficient projections are $\phi_a \rightarrow \frac{\phi_a}{\sqrt{\phi \cdot \phi}}$ and $\partial_t \phi_a \rightarrow \partial_t \phi_a - \frac{\partial_t \vec{\phi} \cdot \vec{\phi}}{\vec{\phi} \cdot \vec{\phi}} \phi_a$. Of course, the space derivatives may also be corrected. See [?] for further discussions.

Relaxation technique: A damping term in the equation of motion will gradually take the kinetic energy out of the system. The equation (16) changes to

$$\partial_{tt} \phi_a = K_{ab}^{-1} F_b \left( \vec{\phi}, \partial_t \vec{\phi}, \partial_t \vec{\phi} \right) - \gamma \partial_t \phi_a \quad \text{(19)}$$
where $\gamma$ is the damping coefficient. We set $\gamma$ to 0.1, but most values will do as long as they are not too large. Another approach would be to absorb the outwards traveling kinetic energy waves in the boundary region.

6 Numerical Studies

We present three numerical studies of the baby Skyrme models.

- **Check for Stability**
  It is important to check whether a solution of the Hedgehog Ansatz is a global minimum or not. We have taken the exact solution of the first potential and perturbed it by a scale transformation. The initial configuration has twice the size of the static solution. The skyrmion ‘shakes’ off its extra energy and radiates out kinetic energy waves. The stable skyrmion remains; the stability of the 1-skyrmion is fine and it probably is the minimum energy solution.

- **Scattering of baby skyrmions**
  First, the holomorphic baby skyrmions repel each other and only scatter if you collide them above a critical speed. Skyrmions of the 1-vacuum and 2-vacua potential have attractive channel and scatter in the same way. Figure 1 shows how the two 1-skyrmions attract each other, form a bound-state, scatter away at 90 degrees, get slowed down by their mutual attraction, attract each other again and so on. This oscillating but stable bound-state is an excited state of the 2-skyrmion solution. Taking out the kinetic energy, the bound-state relaxes to the 2-skyrmion; a ring.

- **Multi-skyrmion solutions**
  The holomorphic model does not have any stable n-skyrmions where $n$ is greater than one. The common 1-vacuum potential possesses rather beautiful, non-radially symmetric multi-skyrmions. They look like crystal with 2-skyrmions as building blocks. (for example, see a 7-skyrmion in figure 2.) The 2-vacua potential gives rise to a remarkably different structure for multi-skyrmions. In fact, we show that the energy density of all n-skyrmions turns out to be radially symmetric configurations, namely rings of larger and larger radii. Figure 3 shows the relaxation to a ‘vibrating string’ configuration (further relaxation leads to a perfect ring). Clearly, the choice of the potential term has a major impact on the formation of multi-skyrmions and their shape. However, it is not clear what controls the structure. (see our paper \cite{?})
7 Conclusion

We have studied classical aspects of the non-integrable baby Skyrme models. No exact solutions are known in general and we could only achieve our study by using numerical methods. We have shown that the choice of potential term dictates the structure of the multi-skyrmions. However, it is not clear to us what properties of the potential controls the structure. Our numerical approach is symptomatic for the increased use of computers in extracting valuable information from theories untrackable by mathematical tools. ‘Numerical experiments’ are becoming a well established area between ‘real’ experiments and pure theory. Further, the author is in no doubt that solitons will play an increasingly important role in fundamental physics. Monopoles, instantons and D-branes are other examples of solitonic objects. Physicists have two ways of thinking about physical systems; in terms of waves or particles. Solitons combine both fundamental concepts and are waves with particle behaviour. This new concept is exciting and very appealing to theorists in all domains – a new tool to use with the help of computers.

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