Semigroup Representations of the Poincaré Group and Relativistic Gamow Vectors.

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Gamow vectors are generalized eigenvectors (kets) of self-adjoint Hamiltonians with complex eigenvalues \((E_R \mp i\Gamma/2)\) describing quasistable states. In the relativistic domain this leads to Poincaré semigroup representations which are characterized by spin \(j\) and by complex invariant mass square \(s = s_R = (M_R - \frac{i}{2}\Gamma_R)^2\). Relativistic Gamow kets have all the properties required to describe relativistic resonances and quasistable particles with resonance mass \(M_R\) and lifetime \(\hbar/\Gamma_R\).

I. INTRODUCTION

Following Wigner [1], an elementary relativistic quantum system, an elementary particle with mass \(m\) and spin \(s\) is in the mathematical theory described by the space of a unitary irreducible representation (UIR) of the Poincaré group \(P\). From these UIR, the relativistic quantum fields are constructed [2]. More complicated relativistic systems are described by direct sums of UIR (for “towers” of elementary particles) or by direct products of UIR (for combination of two or more elementary particles). A direct product of UIR may be decomposed into a continuous direct sum (integral) of irreducible representations [3,4]. The UIR are characterized by three invariants \((m^2, j, \text{sign}(p_0))\), where \(j\) represents the spin and the real number \(m\) represents the mass of elementary particle (we restrict ourselves here to \(\text{sign}(p_0) = +1\)).

The UIR of the Poincaré group \(P\) describe stable elementary particles (stationary systems). The vast majority of elementary particles are unstable and UIR provide only a more or less approximate description of them. The meaning of unstable elementary particles, in particular in the relativistic domain, has always been a subject of debate. This has recently flared-up in connection with the line shape analysis of the \(Z\)-boson, where one has difficulties to agree upon a definition of resonance mass \(m\) and width \(\Gamma\). Going back to Wigner’s definition of fundamental relativistic particles we want to present here a special class of (non-unitary) semi-group representations of \(P\) which describe quasistable relativistic particles.

Phenomenologically, one always takes the point of view that resonances are autonomous quantum physical entities, and decaying particles are no less fundamental than stable particles. Stable particles are not qualitatively different from quasistable particles, but only quantitatively by a zero (or very small) value of \(\Gamma\). Therefore both stable and quasistable states should be described on the same footing. This has been accomplished in the non-relativistic case, where a decaying state is described by a generalized eigenvector of the (self adjoint, semi-bounded) Hamiltonian with a complex eigenvalue \(z_R = E_R - i\Gamma/2\) [5] and exponential time evolution, called Gamow vectors. The stable state vectors with real eigenvalues \(E_S\) are the special case with \(\Gamma = 0\).

II. GAMOW VECTORS

In the standard Hilbert space formulation of quantum mechanics, such Gamow vectors can not exist and one has to employ a formulation based on the Rigged Hilbert Space (RHS) [6]. Dirac’s bras and kets are, mathematically, generalized eigenvectors with real eigenvalues, and Gamow vectors are generalizations of Dirac kets. They are described by kets \(\psi^G \equiv |z_R^-\rangle\sqrt{2\pi\Gamma}\) with complex eigenvalue \(z_R = E_R - i\Gamma/2\), where \(E_R\) and \(\Gamma\) are respectively interpreted as resonance energy and width. Like Dirac kets, the Gamow kets are functionals of a Rigged Hilbert Space:

\[\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times : \quad \psi^G \in \Phi_+^\times, \]

and the mathematical meaning of the eigenvalue equation \(H^\times|z_R^-\rangle = (E_R - i\Gamma/2)|z_R^-\rangle\) is:

\[\langle H\psi|z_R^-\rangle \equiv \langle \psi|H^\times|z_R^-\rangle = z_R\langle \psi|z_R^-\rangle \quad \text{for all} \quad \psi \in \Phi_+.\]
The conjugate operator $H^\times$ of the Hamiltonian $H$ is uniquely defined by the first equality in (2), as the extension of the Hilbert space adjoint operator $H^\dagger$ to the space of functionals $\Phi^\times_\mp$; on the space $\mathcal{H}$, the operators $H^\times$ and $H^\dagger$ are the same.

The non-relativistic Gamow vectors have the following properties:

1. They have an asymmetric (i.e., $t \geq 0$ only) time evolution and obey then an exponential law:
   \[
   \psi(t) = e^{-iH^\times t}|E_R - i\Gamma/2^-\rangle = e^{-iE_R t}e^{-i\Gamma t/2}|E_R - i\Gamma/2^-\rangle, \quad \text{only for } t \geq 0. \tag{3}
   \]

2. There is another Gamow vector $\tilde{\psi}(t) = |E_R + i\Gamma/2^+\rangle \in \Phi^\times_+,$ and another semigroup $e^{-iH^\times t}$ for $t \leq 0$ in another RHS $\Phi_- \subset \mathcal{H} \subset \Phi^\times_-$ (with the same $\mathcal{H}$) with the asymmetric evolution
   \[
   \tilde{\psi}(t) = e^{-iH^\times t}|E_R + i\Gamma/2^+\rangle = e^{-iE_R t}e^{i\Gamma t/2}|E_R + i\Gamma/2^+\rangle, \quad \text{only for } t \leq 0. \tag{4}
   \]

3. The Gamow vectors have a Breit-Wigner energy distribution
   \[
   \langle -E|\psi\rangle = i \sqrt{\frac{\Gamma}{2\pi}} \frac{1}{E - (E_R - i\Gamma/2)^+}, \quad -\infty < E < \infty, \tag{5}
   \]
   where $-\infty$ means that it extends to $-\infty$ on the second sheet of the S-matrix (whereas the standard Breit-Wigner extends to the threshold $E = 0$).

We want to present here a generalization of these non-relativistic Gamow vectors to the relativistic case.

In the non-relativistic case the inclusion of the degeneracy quantum numbers of energy, i.e., the extension of the Dirac-Lippmann-Schwinger kets
   \[
   |E^\pm\rangle = |E\rangle + \frac{1}{E - H \mp i0} V(E) = \Omega^\pm |E\rangle
   \]
   \[
   H |E^\pm\rangle = E |E^\pm\rangle; \quad (H - V)|E\rangle = E |E\rangle \tag{6}
   \]
   to the basis of the whole Galilei group is trivial.

For the two particle scattering states (direct product of two irreducible representations of the Galilei group [7]) one uses eigenvectors of angular momentum $(jj_3)$ for the relative motion and total momentum $p$ for the center of mass motion. Thus
   \[
   |E^\text{tot}_j\pm p j_3 (l, s) \rangle = |p\rangle \otimes |E j j_3 \pm\rangle \tag{7}
   \]
   where $E^\text{tot} = \frac{p^2}{2m} + E$ (the Hamiltonian in (6) is $H = H^\text{tot} - \frac{P^2}{2m}$).

The center-of-mass motion is usually separated by transforming to the center-of-mass frame and then ignoring the center-of-mass motion
   \[
   |p = 0\rangle \otimes |E j j_3 \pm\rangle \rightarrow |E, j j_3 \pm\rangle.
   \]

For the vector in (6) one then uses the generalized eigenvectors of $H$ and of angular momentum
   \[
   |E^\pm\rangle = |E j j_3 \pm\rangle \in \Phi^\times_\mp \supset \mathcal{H} \supset \Phi^\times_+ \tag{8}
   \]
   with
   \[
   H^\times |E j j_3 \pm\rangle = E |E j j_3 \pm\rangle, \quad 0 \leq E < \infty. \tag{9}
   \]

\textsuperscript{1}For (essentially) self adjoint $H$, $H^\dagger$ is equal to (the closure of) $H$; but we shall use the definition (2) also for unitary operators $\mathcal{U}$ where $\mathcal{U}^\times$ is the extension of $\mathcal{U}^\dagger$, but not of $\mathcal{U}$.

\hspace{1cm}2
The vectors (8) are the Dirac-Lippmann-Schwinger scattering states and $E$ runs along the cut on the positive real axis of the 1-st sheet of the $j$-th partial S-matrix. The proper eigenvectors of $H$ with $E = -|E_n|$ at the poles on the negative real axis of the 1-st sheet are the bound states $|E_n j j_3\rangle$. By the Galilei transformation one can transform these vectors (8) to arbitrary momentum $p$; $E$ and $p$ are not intermingled by Galilei transformations.

To obtain the non-relativistic Gamow kets one analytically continues the Dirac-Lippmann-Schwinger ket (8) into the second sheet of the $j$-th partial S-matrix to the position of the resonance pole $|z_R = E_R - i\Gamma/2, j, j_3\rangle$ and obtains the following representation [5]:

$$|z_R = E_R - i\Gamma/2, j, j_3\rangle = \frac{i}{2\pi} \int_{-\infty+i\epsilon}^{+\infty} dE|E, j, j_3\rangle \frac{1}{E - z_R}. \quad (10)$$

A Galilei transformation can boost this Gamow ket to any real momentum $p$:

$$|p, z_R, j j_3\rangle = U(p)|0\rangle \otimes |z_R j j_3\rangle.$$

Complex momenta cannot be obtained in this way since the Galilei transformations commute with the intrinsic energy operator $H$.

III. POINCARÉ GROUP REPRESENTATIONS WITH FOUR VELOCITY BASIS

In the relativistic case the Lorentz transformation – in particular Lorentz boosts – intermingle energy $E^{\text{tot}} = p^0$ and momenta $p^i$, $i = 1, 2, 3$. Thus if energy and/or mass were complex, this would also lead to complex momentum. To restrict the unwieldy set of Poincaré group representations with complex momenta we will consider a special class of “minimally complex” irreducible representations of $\mathcal{P}$ to describe relativistic resonances and decay ing elementary particles. Our construction will also lead to complex momenta $p^\mu$, but in our case the momenta will be “minimally complex” in such a way that the 4-velocities $p_\mu \equiv \frac{p^\mu}{m}$ remain real. This construction was motivated by a remark of D. Zwanziger [8] and is based on the fact that the 4-velocity eigenvectors $|p_{j3}(mj)\rangle$ furnish as valid a basis for the representation space of $\mathcal{P}$ as the usual Wigner basis of momentum eigenvectors $|p_{j3}(mj)\rangle$. This means every state $\phi \in \Phi \subset \mathcal{H}(m, j) \subset \Phi^\times$ of an UIR $(m^2, j)$, (where $\Phi$ denotes the space of well-behaved vectors and $\Phi^\times$ the space of kets for the Hilbert space $\mathcal{H}(m, j)$ of an UIR), can be written according to Dirac’s basis vector decomposition as

$$\phi = \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{p}, j_3\rangle \langle j_3, \hat{p}|\phi\rangle \quad (11)$$

where we have chosen the invariant measure

$$d\mu(\hat{p}) = \frac{d^3\hat{p}}{2\hat{p}^0} = \frac{1}{m^2} \frac{d^3p}{2E(p)}, \quad \hat{p}^0 = \sqrt{1 + \hat{p}^2}. \quad (12)$$

As a consequence of (12), the $\delta$-function normalization of these velocity-basis vectors is

$$\langle \xi, \hat{p} | \hat{p}', \xi' \rangle = 2\hat{p}^0 \delta^3(\hat{p} - \hat{p}') \delta_{\xi'\xi} = 2\hat{p}^0 m^2 \delta^3(p - p') \delta_{\xi'\xi}. \quad (13)$$

Here, $|\hat{p}, j_3\rangle \in \Phi^\times$ are the eigenkets of the 4-velocity operator $\hat{P}_\mu = P_\mu M^{-1}$ and $|\phi_{j3}(\hat{p})\rangle^2 = |\langle j_3, \hat{p}|\phi\rangle|^2$ represents the 4-velocity distribution of the vector $\phi$. The 4-velocity eigenvectors are often more useful for physical reasoning, because 4-velocities seem to fulfill to rather good approximation “velocity super-selection rules” which the momenta do not do [9]. Their use as basis vectors of the Poincaré group representation (11) does not constitute an approximation.

The relativistic Gamow vectors will be defined, not as momentum eigenvectors, but as 4-velocity eigenvectors in the direct product space of UIR spaces for the decay products of the resonance $R$. We want to obtain the relativistic Gamow vectors from the pole term of the relativistic S-matrix in complete analogy to the way the non-relativistic Gamow vectors were obtained [5]. In the absence of a vector space description of a resonance, we shall also in the relativistic theory define the unstable particle by the pole of the analytically continued partial S-matrix with angular momentum $j$ at the value $s = s_R = (M_R - i\Gamma_R/2)^2$ of the invariant mass square variable (Mandelstam variable)

$s = (p_1 + p_2 + \cdots)^2 = E_R^2 - p_R^2$, where $p_1, p_2, \ldots$ are the momenta of the decay products of $R$ [10,11]. This means that the mass $M_R$ and lifetime $\tau_R$ of the complex invariant mass $m_R = (M_R - i\Gamma_R/2) = \sqrt{s_R}$, in addition to spin $j$, are the intrinsic properties that define a quasistable relativistic particle $\ddagger$.

\ddagger Conventionally and equivalently one often writes
In order to make the analytic continuation of the partial S-matrix with angular momentum \( j \), we need the angular momentum basis vectors

\[
|\hat{\rho}_{j3}(w^j)\rangle = \int \frac{d^3\rho_1}{2E_1} \frac{d^3\rho_2}{2E_2} |\hat{\rho}_1\hat{\rho}_2[m_1m_2]\rangle|\hat{\rho}_1\hat{\rho}_2[m_1m_2]\rangle|\hat{\rho}_3(w^j)\rangle \tag{14}
\]

for any \((m_1 + m_2)^2 \leq w^2 < \infty\) \( j = 0, 1, \ldots \).

In the direct product space of the decay products of the resonance \( R \)

\[ \mathcal{H} \equiv \mathcal{H}(m_1, 0) \otimes \mathcal{H}(m_2, 0) = \int_{(m_1+m_2)^2}^{\infty} ds \sum_{j=0}^{\infty} \mathcal{H}(s, j), \tag{15} \]

where \( w^2 = s \), the Mandelstam variable defined above. For simplicity, we have assumed here that there are two decay products, \( R \rightarrow \pi_1 + \pi_2 \) with spin zero, described by the irreducible representation spaces \( \mathcal{H}^{\pi^2}(m_i, s_i = 0) \) of the Poincaré group \( P \).

The kets \(|\hat{\rho}_{j3}(w^j)\rangle\) are eigenvectors of the 4-velocity operators

\[
\hat{P}_\mu = (P_\mu^1 + P_\mu^2)M^{-1}, \quad M^2 = (P_\mu^1 + P_\mu^2)(P^{1\mu} + P^{2\mu}) \tag{16}
\]

with eigenvalues

\[
\hat{p}^\mu = \left( \frac{\hat{E}}{w} = \frac{\sqrt{1 + \hat{p}^2}}{w} \right) \quad \text{and} \quad w^2 = s. \tag{17}
\]

In (14) \(|\hat{\rho}_1\hat{\rho}_2[m_1m_2]\rangle = |\hat{\rho}_1m_1\rangle \otimes |\hat{\rho}_2m_2\rangle\) is the direct product basis of \( \mathcal{H} \) which are eigenvectors of \( \hat{P}_\mu \), the 4-velocity operators in the one particle spaces \( \mathcal{H}^{\pi_i}(m_i, s_i) \) with eigenvalues \( \hat{p}_\mu = \frac{P_\mu}{m}. \)

To obtain the Clebsch-Gordan coefficients \(|\hat{\rho}_1\hat{\rho}_2[m_1, m_2]|\hat{\rho}_{j3}(w^j)\rangle\) in (14), one follows the same procedure as given in the classic papers [3,4,12,13] for the Clebsch-Gordan coefficients for the Wigner (momentum) basis. This has been done in [14]. The result is

\[
\langle \hat{\rho}_1\hat{\rho}_2[m_1, m_2]|\hat{\rho}_{j3}(w^j)\rangle = 2\hat{E}(\hat{p})\delta^3(p - r)\delta(w - \epsilon)Y_{j3}(\hat{e})\mu_j(w^2, m_1^2, m_2^2) \tag{18}
\]

with \( \epsilon^2 = r^2 = (p_1 + p_2)^2, \quad r = p_1 + p_2, \)

where \( \mu_j(w^2, m_1^2, m_2^2) \) is a function that fixes the \( \delta \)-function “normalization” of \(|\hat{\rho}_{j3}(w^j)\rangle\). The unit vector \( \hat{e} \) in (18) is the three component of \( L^{-1}(r)q \), where \( q \) is the unit space like vector in the 2-plane defined by \( p_1, p_2 \) and orthogonal to \( r \) [13]. In the c.m. frame the direction of \( e \) is \( \hat{p}_1^{cm} = -\frac{m_2}{m}\hat{p}_2 \).

The normalization of the basis vectors (14) is chosen to be

\[
\langle \hat{\rho'}_{j3}'(w'j')|\hat{\rho}_{j3}(w^j)\rangle = 2\hat{E}(\hat{p})\delta(|\hat{\rho}' - \hat{\rho}|)\delta_{j,j'}\delta_{s,s'} \tag{19}
\]

where \( \hat{E}(\hat{p}) = \sqrt{1 + \hat{p}^2} = \frac{1}{w}\sqrt{w^2 + \hat{p}^2} \equiv \frac{1}{w}E(p, w). \)

This determines the weight function \( \mu_j(w^2, m_1^2, m_2^2) \) to be

\[
s_R \equiv M^2_R - iM_R\Gamma_R = M^2_R \left( 1 - \frac{1}{4} \left( \frac{\Gamma_R}{M_R} \right)^2 \right) - iM_R\Gamma_R \]

and calls \( M_R = M_R \sqrt{1 - \frac{1}{4} \left( \frac{\Gamma_R}{M_R} \right)^2} \) the resonance mass and \( \Gamma_R = \Gamma_R \left( 1 - \frac{1}{4} \left( \frac{\Gamma_R}{M_R} \right)^2 \right)^{-1/2} \) its width. We will see below that \( M_R \) is the mass and \( \hbar/\Gamma_R \), not \( \hbar/\Gamma_R \), is the lifetime.

\( ^{3} \) Though our discussions apply with obvious modifications to the general case of

\[ 1 + 2 + 3 + \cdots \rightarrow R_i \rightarrow 1' + 2' + 3' + \cdots, \]

these generalizations lead to enormously more complicated equations.
They are eigenvectors of the exact Hamiltonian $H$

The Dirac-Lippmann-Schwinger scattering states are obtained, in analogy to (6) (cf. also [2] sect. 3.1) by:

The basis vectors (14) are the eigenvectors of the free Hamiltonian $H = P_0^1 + P_0^2$

The relativistic Gamow kets are defined from the Dirac-Lippmann-Schwinger kets (24) or (23) by contour integrals

They are eigenvectors of the exact Hamiltonian $H = H_0 + V$

The basis vectors $|\hat{p}j_3(s_j)\rangle$ of the UIR $(s,j)$ are obtained from the basis vectors at rest $|Oj_3(wj)\rangle$ by the boost (rotation-free Lorentz transformation) $U(L(\hat{p}))$ whose parameters are the 4-velocities $\hat{p}^\mu$. The generators of the Lorentz transformations are the interaction-incorporating observables

i.e., the exact generators of the Poincaré group ([2] sec. 3.3). These vectors $|\hat{p}j_3(s_j)\rangle$ in (23), or $|Oj_3(wj)\rangle$ in (24) when boosted by $U(L(\hat{p}))$ or precisely $U^a(L(\hat{p}))$, span the unitary representation space of the Poincaré group (15) with the “exact generators” (26). We will be use these Dirac-Lippmann-Schwinger kets to define the relativistic Gamow kets by analytic continuation.

IV. RELATIVISTIC GAMOW KETS

The relativistic Gamow kets are defined from the Dirac-Lippmann-Schwinger kets (24) or (23) by contour integrals around the poles of the $j$-th partial $S$-matrix element. Starting with the $S$-matrix element

one deforms the contour of integration over $s$ from the physical values $(m_1 + m_2)^2 \leq s < \infty$ on the upper rim of the cut along the $s$-axis, into the second sheet past the pole at $s_R$. For the integration around the pole $s_R$ the integral (27) splits of a pole term which defines the Gamow vector $|\hat{p}j_3(s_R)\rangle^-$. This is done in exactly the same way as in the non-relativistic case [5] and leads to the relativistic analogue of (10):

For this analytic continuation to be possible the RHS formulation of quantum theory makes a new hypothesis:

The set of prepared in-states $\{\Phi_+\}$ and the set of detected out-states (decay products) $\{\Phi_\times\}$ form two different dense subspaces of the Hilbert space $\mathcal{H}$, cf. (15) and therewith two distinct RHS’s

For prepared in-states $\phi^+$ for detected out-states (observables) $\psi^-$. (29b)
where \( \Phi_- (\Phi_+) \) is of the Hardy class type in the lower (upper) half plane. This means \( \langle -p \mid j_3 s \mid \psi^- \rangle \left( \langle +p \mid j_3 s \mid \psi^+ \rangle \right) \) are well behaved Hardy class functions of the variable \( s \) in the upper (lower) half plane second sheet. This new hypothesis, which distinguishes meticulously between states (accelerator) and observables (detector) was justified in the non-relativistic case by some causality arguments [16]. All our new results can be derived from this new Hardy hypothesis which is different from the conventional assumptions of scattering theory \( \{ \phi^+ \} = \{ \psi^- \} = \mathcal{H} \).

The first equality in (28) is the definition that associates \( \psi^G \) to the pole term in the second sheet, and the second equality is a consequence of the Hardy class property [17]. As a consequence the wave function \( \langle -p \mid j_3 s \mid \psi^G \rangle \) of the Gamow ket is a Breit-Wigner function of \( s \) that extends over all physical values of \( s \) and the non-physical values of \( s \) on the second sheet to \(-\infty\).

The Lorentz transformations \( \Lambda \) are represented by unitary operators \( U^\dagger (\Lambda) \) in \( \mathcal{H} \). This means its conjugate operator \( \left( U^\dagger (\Lambda) \right)^* \) (usually denoted as \( \mathcal{U}(\Lambda) \)) acts in the space \( \Phi^\times_+ \) in the standard way:

\[
\left( U^\dagger (\Lambda) \psi \right)^* = \langle -p \mid j_3, s_{RJ}^- \rangle = \sum_{j_3} \langle \psi^- \mid A \mid j_3, s_{RJ}^- \rangle \hat{D}^j_{j_3} (R(\Lambda, \hat{p}))
\]

where \( R(\Lambda, \hat{p}) = L^{-1}(\Lambda \hat{p}) \mathcal{W}(\hat{p}) \) is the Wigner rotation. The \( U^\dagger (\Lambda) \) in (30) is the restriction of the unitary \( \mathcal{U}(\Lambda) \) to the dense subspace \( \Phi_+ \), which remains invariant under the action of \( \mathcal{U}(\Lambda) \) for all \( \Lambda \in SO(3,1) \). For the rotation free Lorentz boost one obtains in particular

\[
\left( U^\dagger (L(\hat{p})) \right)^* | \hat{p} = 0, j_3, s_{RJ}^- \rangle = | j_3, s_{RJ}^- \rangle,
\]

where the boost \( L^\mu_{\nu} \) is a function of the real parameters \( \hat{p}^\mu \) and not of the complex \( p^\mu \):

\[
L^\mu_{\nu} = \left( \begin{array}{ccc} m^2 & -p^\mu & -p_\nu \\ \frac{p^\mu}{m} & \frac{\delta^\mu}{m} & -\frac{p^\nu}{m} \\ \frac{p^\nu}{m} & \frac{\delta^\nu}{m} & \frac{\delta^\mu}{m} \end{array} \right), \quad L(\hat{p}) \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) = \hat{p}.
\]

Thus in these representations the velocities \( \hat{p}^\mu \) are real and the momenta \( p^\mu = m \hat{p}^\mu \) become complex only through the complex factor \( m = \sqrt{\mathcal{R}}. \) This is a property that leads to the semigroup representations.

The relativistic Gamow kets (28) are generalized eigenvectors of the invariant mass squared operator \( M^2 = P_\mu P^\mu \) with eigenvalue \( \mathcal{R} \) as can be seen in (28) \( M^2 \psi^- \in \Phi_+ \) in place of \( \psi^- \)

\[
\langle \psi^- \mid M^2 \mid j_3, s_{RJ}^- \rangle = i \int^{+\infty}_{-\infty} ds \langle \psi^- \mid j_3, s_{RJ}^- \rangle \frac{1}{s - s_R} = s_R \langle \psi^- \mid j_3, s_{RJ}^- \rangle \quad \text{for every} \quad \psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi^\times_+.
\]

To prove (33) one needs to use the properties of the Hardy class space [17]. Similarly one shows that the \( | j_3, s_{RJ}^- \rangle \) are generalized eigenvectors of the momentum operators of (26) [17]

\[
(P_\mu \psi^- | j_3, s_{RJ}^- \rangle = \langle \psi^- | P^\mu_\mu \mid j_3, s_{RJ}^- \rangle = i \int^{+\infty}_{-\infty} ds \sqrt{s_R} \hat{p}_\mu \langle \psi^- \mid j_3, s_{RJ}^- \rangle \frac{1}{s - s_R} = \sqrt{s_R} \hat{p}_\mu \langle \psi^- \mid j_3, s_{RJ}^- \rangle.
\]

Thus the generalized momentum eigenvalues are “minimally complex” \( p_\mu = \sqrt{s_R} \hat{p}_\mu \). The continuous linear combinations of the 4-velocity kets (23) with an arbitrary 4-velocity distribution function \( \phi_{j_3}(\hat{p}) \in S \) (Schwartz space),

\[
\psi^G(j_3, s_{RJ}, \hat{p}) = \int d^3 \hat{p} \phi_{j_3}(\hat{p}) | j_3, s_{RJ}^- \rangle,
\]

also represent relativistic Gamow states with the complex mass \( s_R = (RM - i\mathcal{R}/2)^2 \).

In contrast to the action of the Lorentz subgroup (30), the translation subgroup \( T^\dagger (x) = e^{iP^\mu x_\mu} \) does not leave the subspace \( \Phi_+ \) of \( \mathcal{H} \) invariant. However there is a semigroup of time-like translations \( (x^+, 1) \) into the forward light cone with \( \hat{p}^\mu x_\mu = (1 + \hat{p}^2)^{1/2} x^0 - \hat{p}.x \geq 0 \) whose (restrictions to \( \Phi_+ \) of) \( U^\dagger (x^+, 1) \) leave the subspace \( \Phi_+ \) invariant. The \( \{ (x^+, \Lambda) \} | \Lambda \in SO(3,1), x_\mu \text{with } (1 + \hat{p}^2)^{1/2} x^0 - \hat{p}.x \geq 0, \hat{p} \in \mathbb{R}^3 = P_+ \) form a semigroup and their representatives \( U^\dagger (x^+, 1) \) are continuous operators on \( \Phi_+, U^\dagger (x^+, \Lambda) \Phi_+ \rightarrow \Phi_+ \). For the other \( (x, \Lambda) \in P \) this is not fulfilled, cf. the analogy to the non-relativistic case [5].

For the particular case \( \hat{p} = 0, x^0 = t \geq 0 \) we obtain the time translation into the forward direction generated by the energy operator \( H = \hat{P}_0 \).
where $t$ is time in the rest system.

Thus relativistic Gamow states are representations of $\mathcal{P}_+$ with spin $j$ and complex mass $s_R = (M_R - i\Gamma_R/2)^2 \equiv m_0^2 - im_0\Gamma_0$, for which the Lorentz subgroup is unitarily represented. They are obtained from the resonance pole of the relativistic partial S-matrix $S_j(s)$, and thus lead to a representation of the $j$-th partial scattering amplitude

$$a_j(s) = a_j^{BW}(s) + B(s),$$

where $a_j^{BW}(s)$ is a relativistic Breit-Wigner amplitude given by

$$a_j^{BW}(s) = \frac{\Gamma_{jR}}{s - (M_R - i\Gamma_R)^2}, \quad -\infty < s < +\infty,$$

and $B(s)$ is a background term not associated to the resonance pole at $s_R$. The background is slowly varying in the neighborhood of the resonance peak $(M_R^2 - \Gamma_R^2/4)$ of $|a_j^{BW}(s)|^2$, unless there is another resonance in the same partial wave at a nearby $s_{R_2}$ in which case the true representation at $s_{R_2}$ has to be treated in the same way and leads to $B(s) \to a_j^{BW_2}(s) + B'(s)$.

V. SUMMARY

The Gamow vector obeys an exact exponential decay law with a lifetime $\tau_R$ given precisely by $\tau_R = \hbar/\Gamma_R$, according to (36), and not by $\hbar/\Gamma_0$ or any other $\Gamma$. The separation (37a) of an exact Breit-Wigner (37b) and the isolation of an exactly exponential decaying Gamow state $\psi^G$ associated to each Breit-Wigner of each $S$-matrix pole is achieved by the hypothesis (29) of the Hardy class spaces. Only for the Gamow ket (28) can one prove (36) which leads to the exact exponential decay law for the decay rate [18] and therewith to the precise relation $\tau_R = \hbar/\Gamma_R$. Without the postulate (29) this cannot be derived, though it has always been assumed on the basis of some “approximate” derivations [19]. The Gamow vector also helps to decide the debate about the right definition of the $Z$-boson mass and width [20]. According to (36) it is probably $M_R$ and certainly $\Gamma_R$ (if one wants $\tau_R = \hbar/\Gamma_R$ to hold) which should be called the mass and width, not the peak position $M_\rho$ of the Breit-Wigner (37b) and not $M_Z = \sqrt{M_R^2 \left(1 + \frac{1}{2} \left(\frac{\Gamma_R}{M_R}\right)^2\right)}$.

The above are all features which one may welcome or easily accept for states that are to describe relativistic resonances. In addition, Gamow vectors have a semigroup time evolution $t \geq 0$ (36), expressing irreversibility on the microphysical level. This may be puzzling and disturbing to many, but a fundamental time asymmetry of quantum physics has been noticed independently and in more general contexts [21,22]. The Gamow kets can represent the “causal links” between two events [22] and for microphysical “states” representing causal links a semigroup time evolution is quite natural.

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The projectively equivalent irreducible representations \((m, E, s) \sim (m, 0, s) \equiv [m, s]\) of the Galilei group are characterized by non-relativistic mass \(m\) and spin \(s\). In the direct product of \([m_1, s_1] \otimes [m_2, s_2]\), the two spins are combined (using SO(3) Clebsch-Gordan coefficients) to give the total spin \(s\). Then, using orbital angular momentum \(l\) for the relative orbital motion, one couples \(s\) with \(l\) to form the total angular momentum \(j\) with the 3-component \(j_3\). This gives the basis vectors \(|E^\text{tot}; \mathbf{p}jj_3(s, l)\rangle\) defined by their scalar products in the center of mass frame \(\mathbf{p} = 0\) (Clebsch-Gordan coefficients of the Galilei group) with the basis vector \(|\mathbf{p}_1, s_1, (s_1)_{3} \otimes |\mathbf{p}_2, s_2, (s_2)_{3}\rangle = |\mathbf{p}_1, (s_1)_{3}, \mathbf{p}_2, (s_2)_{3}(s_1, s_2)\rangle\)

\[
(E^\text{tot} - \frac{p_1^2}{2m_1} - \frac{p_2^2}{2m_2} ) \times \sum_{s_3,l_3} (s_1 s_3 s_2)_{3} (s_3 s_2) \langle l|\Delta s_3 j(j_3) Y_{l_3}(\mathbf{e}) \rangle
\]

where \(\mathbf{e}\) is the unit vector in the direction of \(\mathbf{p}\) and \(N\) is a suitable normalization factor.