Abstract

We consider models of scalar fields coupled to gravity which are higher-dimensional generalizations of four dimensional supergravity. We use these models to describe domain wall junctions in an anti-de Sitter background. We derive Bogomolnyi equations for the scalar fields from which the walls are constructed and for the metric. From these equations a BPS-like formula for the junction energy can be derived. We demonstrate that such junctions localize gravity in the presence of more than one uncompactified extra dimension.
1 Introduction

One of the very few robust predictions of string theory is the existence of extra dimensions of spacetime. The apparent conflict between this prediction and experiment is traditionally resolved by postulating a direct product structure in which the additional spatial dimensions describe a compact space of Planckian scale. While this view leads to a potentially rich phenomenology, the difficulty in choosing amongst compactification scenarios and understanding their cosmological evolution encourages the search for alternative pictures.

It has recently become widely recognized that extra dimensions can be much larger than the Planck scale if the non-gravitational fields of the Standard Model are confined to a (3+1)-dimensional brane [1] (see [2] for precursors). Taking this idea to its extreme, Randall and Sundrum [3] have shown that gravity can be effectively localized to a flat four-dimensional submanifold with a single infinite extra dimension if the five-dimensional bulk geometry is locally anti-de Sitter (AdS); see also [4, 5]. It is more difficult, however, to make this idea work for more than one extra dimension. Arkani-Hamed et al. [6] have suggested that this obstacle can be overcome by considering $n$ individual branes, each of worldvolume dimension $2+n$, in a $(4+n)$-dimensional spacetime. If these branes intersect at a four-dimensional subspace, with the solution in the bulk consisting of $2^n$ patches of $(4+n)$-dimensional anti-de Sitter space, gravity can be localized to the four-dimensional intersection. This scenario has been further elaborated on in [7, 8], where the importance of determining the tension associated with the junction itself was emphasized.

In a brane-world picture, the branes themselves may be either fundamental objects or solitons (domain walls) constructed from fundamental fields. Supersymmetric domain walls have been extensively studied (see e.g. [9]); in supersymmetric theories with multiple discrete vacua, there are typically BPS domain wall solutions which preserve half of the underlying supercharges. In (3+1) dimensions, domain wall junctions have been studied [10, 11, 12, 13, 14], and it has been shown that there can be a nonzero tension associated with the junction. In [13, 14], it was argued that wall junctions can be BPS states, preserving a single Hermitian supercharge (of the four in an $N = 1$ theory).
We are therefore interested in studying (3+1)-dimensional junctions of domain walls in higher dimensional supersymmetric theories, coupled to gravity. Some general properties of supergravity domain walls are well established [15]. The simplest context in which (3+1)-dimensional junctions with \( N = 1 \) SUSY in 4D can arise is in six-dimensional theories such as gauged six-dimensional supergravity with sixteen supercharges, coupled to matter multiplets. Unfortunately this set of theories has not yet been constructed. However, it has recently been pointed out that a general class of theories of scalar fields coupled to gravity, with potentials \( V(\phi^i) \) derived from a “superpotential”, give rise to classical Bogomolnyi bounds for the tensions of domain walls, independently of whether or not they can be derived from a specific supergravity Lagrangian [16, 17]. We therefore consider models in this class and derive a similar bound for the tension of domain wall junctions, bypassing the construction of the supergravity Lagrangians in which they may or may not be embedded. While BPS domain walls in supergravity are generally singular, in this paper we calculate the contribution to the energy of the nonsingular region, leaving aside the question of resolving the singularities.

2 Setup

In this section we describe how to adapt the construction of Skenderis and Townsend [16] and DeWolfe, Freedman, Gubser and Karch [17], describing domain walls in supergravity-like theories, to a context appropriate for wall junctions.

We consider a set of complex scalar fields \( \{ \phi^i \} \) coupled to gravity in \( D \) spacetime dimensions, with an action (in units where \( \kappa_D^2 \equiv 8\pi G_D = 1 \))

\[
S = \int d^D x \left( \frac{1}{2} \sqrt{|g|} R + L_{\text{matter}} \right),
\]

where the matter Lagrange density is

\[
L_{\text{matter}} = \sqrt{|g|} \left[ -\frac{1}{2} K_{ij} g^{ab} (\partial_a \phi^i) (\partial_b \bar{\phi}^j) - V(\phi, \bar{\phi}) \right].
\]

We have assumed that the number of real scalars is even, allowing us to group them into complex fields, and that the metric on field space has the Kähler form \( K_{ij} = \partial^2 K / \partial \phi^i \partial \bar{\phi}^j \), with \( K(\phi, \bar{\phi}) \) the Kähler potential. We will assume
further that the potential \( V(\phi, \bar{\phi}) \) can be written in terms of a holomorphic function \( W(\phi) \) as

\[
V = \left( \frac{D-2}{4} \right) \exp(K) \left[ (D-2)K^{ij} W_{ij} \bar{W}_{ij} - 2(D-1)W \bar{W} \right],
\]

where

\[
W_{ij} \equiv \frac{\partial W}{\partial \phi^i} + \frac{\partial K}{\partial \phi^i} W.
\]

For obvious reasons we will refer to \( W \) as the “superpotential", even though our higher-dimensional examples will not be based on a well-defined supergravity theory. For the case of trivial Kähler metric, the form (3) was shown to be necessary for vacuum stability about an AdS background [20].

Since our ultimate objective is to construct domain wall junctions rather than the individual walls, we consider an ansatz for the fields which allows for dependence on two spatial dimensions, \( x^{D-2} \) and \( x^{D-1} \), which we combine into complex coordinates:

\[
z \equiv x^{D-2} + i x^{D-1},
\]

\[
\bar{z} \equiv x^{D-2} - i x^{D-1}.
\]

Our metric ansatz in these coordinates is

\[
ds^2 = e^{2A(z, \bar{z})} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{2} e^{2B(z, \bar{z})} (dz d\bar{z} + d\bar{z} dz),
\]

where \( \mu = 0, 1, 2, \ldots, D - 3 \). We assume in addition that the scalar fields depend only on the extra dimensions, \( \partial_\mu \phi^i = 0 \). This ansatz is the most general compatible with \( (D-2) \)-dimensional Poincare invariance.

Skenderis and Townsend [16] consider a theory with a single scalar field. They study a domain-wall ansatz in which the field only depends on one spatial coordinate \( r \), and derive “BPS equations” of the form \( B_I = 0 \) (with \( B_I \) first-order in derivatives of the fields) by showing that the energy functional can be written in the form

\[
E = \int_{-\infty}^{\infty} dr \left[ (B_1)^2 - (B_2)^2 \right] + \text{surface terms}.
\]

Such a functional will have a saddle point (although not necessarily a minimum, due to the minus sign) at configurations with \( B_I = 0 \), which one can verify also solve the full equations of motion of the theory. DeWolfe et al. [17], meanwhile, use an equivalent set of equations similar to those derived
from a five-dimensional supergravity theory in [18]. In supergravity, one can derive BPS equations by setting to zero the supersymmetry transformations generated by certain spinors $\zeta_\alpha$, representing the fact that BPS states leave some supersymmetry unbroken. DeWolfe et al. point out that, in the context of scalar fields coupled to gravity, solutions to such equations continue to solve the full equations of motion whether or not the superpotential derives from supergravity.

The additional complication introduced by our extra degrees of freedom and extra spatial dimension renders the procedure of searching for BPS equations by writing the energy functional in a form analogous to (7) prohibitively unwieldy. Instead, we would like to employ supergravity transformations to obtain the BPS equations. However, an appropriate six-dimensional supergravity theory from which we could derive BPS-like equations has not been constructed. What we can do is to first derive the equations appropriate to ordinary $D = 4$ supergravity by setting supersymmetry variations to zero, then generalize those equations to higher dimensions. Our strategy for deriving BPS equations is thus as follows:

- Posit the ansatz for the metric and scalar fields appropriate to wall junctions.
- Derive first-order BPS equations in $D = 4$ supergravity by setting to zero the supersymmetry transformations of the matter fermions $\psi_{i}^{\alpha}$ and the gravitino $\psi_{a\alpha}$, for some spinor parameter $\zeta_{\alpha}$.
- Generalize these equations to higher dimensions by introducing constant coefficients, and determining their values by the requirement that solutions to the BPS equations also solve the equations of motion of a theory of scalars coupled to gravity (which may or may not be derivable from supergravity).

Once the BPS equations are found, they may be used to calculate the tension of a junction satisfying them (even in the absence of specific solutions). Following [19], it is very plausible that any such solutions are singular in the cores of the walls. However, we expect that the appearance of such a singularity may under favorable conditions be resolved in a more complete theory (see for example [24],[25],[26]), and that our results about the existence of BPS configurations and the junction tension should still hold.
3 BPS Equations for Junctions

We begin with a derivation of the BPS equations in $D = 4$ supergravity for our ansatz (6).

For calculational convenience we represent the gravitino $\psi_{\alpha a}$ and the matter fermions $\psi^i_\alpha$ as Weyl spinors, and the supersymmetry transformation parameter $\zeta_\alpha$ as a Majorana spinor. The supersymmetry transformations for the matter fermions are then

$$\delta\psi^i_\alpha = (P_+ \Gamma^a)_{\alpha\beta}\zeta_\beta \partial_a \phi^i - \exp \left( \frac{K}{2} \right) K^{ij} W_{ij} P_+ \zeta_\beta ,$$

and for the gravitino

$$\delta\psi_{\alpha a} = P_+ \zeta_\beta \left( \bar{\nabla}_a \zeta_\beta + \frac{1}{2} (P_+ \Gamma_a)_{\alpha\beta} \zeta_\beta \exp \left( \frac{K}{2} \right) W \right) ,$$

where

$$\bar{\nabla}_a \equiv \nabla_a + \frac{i}{2} \Gamma \tilde{A}_a .$$

$\nabla_a$ is the spacetime covariant derivative acting on spinors, and we have introduced the projection matrices $P_\pm = (1/2)(1 \pm \Gamma)$. Here $\Gamma = -i \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3$ is the four-dimensional chirality matrix, and $\tilde{A}_a$ is an auxiliary axial-vector gauge field defined by

$$\tilde{A}_a = \text{Im} \left( K_{,\alpha} \partial_\alpha \phi^i \right) .$$

In presenting the supersymmetry transformations (8) and (9) we have not chosen a Kähler gauge, thereby preserving covariance under a Kähler transformation in which we assign the matter fermions, the gravitino, and the local SUSY parameter a nontrivial transformation law:

$$K \rightarrow K + f(\phi) + \bar{f}(\bar{\phi})$$

$$W \rightarrow \exp (-f(\phi)) W$$

$$\psi^i_\alpha \rightarrow \exp \left( \frac{i}{2} \text{Im} [f(\phi)] \right) \psi^i_\alpha$$

$$\psi_{\alpha a} \rightarrow \exp \left( -\frac{i}{2} \text{Im} [f(\phi)] \right) \psi_{\alpha a}$$

$$\zeta_\alpha \rightarrow \exp \left( -\frac{i}{2} \text{Im} [f(\phi)] \Gamma \right) \zeta_\beta .$$
We have chosen the auxiliary axial gauge field (11) in such a way that under this transformation the covariant derivative of the SUSY parameter transforms in a linear way:

\[
\begin{align*}
\tilde{A}_a & \rightarrow \tilde{A}_a + \partial_a \{ \text{Im}[f(\phi)] \} \\
\tilde{\nabla}_a & \rightarrow \exp \left(-\frac{i}{2} \text{Im}[f(\phi)] \Gamma \right) \cdot \tilde{\nabla}_a \cdot \exp \left(\frac{i}{2} \text{Im}[f(\phi)] \Gamma \right) \\
\tilde{\nabla}_a \zeta & \rightarrow \exp \left(-\frac{i}{2} \text{Im}[f(\phi)] \Gamma \right) \cdot (\tilde{\nabla}_a \zeta)
\end{align*}
\]  

(13)

Maintaining this invariance explicitly is useful in deriving the higher-dimensional analogues of the \( D = 4 \) BPS equations.

We now look for a set of equations corresponding to a \( 1/4 \)-BPS configuration, i.e., one which is invariant under a single spinor \( \zeta \). These equations will describe the junction states that we seek.

The first BPS equation comes from setting the supersymmetric variation of the matter fermions to zero, \( \delta \psi^i_\alpha = 0 \). This condition is

\[
(P_+ \Gamma^a)_{\alpha \beta} \zeta_\beta \partial_a \phi^i - \exp \left(\frac{K}{2} \right) K^{ij} \tilde{W}_{ij} P_{+ \alpha \beta} \zeta_\beta = 0,
\]

(14)

where \( \Gamma^a \equiv e^a_A \Gamma^A \) and \( \Gamma_a \equiv e^A_a \Gamma_A \), with \( A, B, \ldots \) being tangent space indices, and we have chosen vielbeins \( e^a_\mu = \exp(-A) \delta^a_\mu \), \( e^2_2 = e^2_3 = -ie^3_3 = ie^3_3 = \exp(-B) \). For simplicity we will choose the spinor \( \zeta \) so that the particular linear combination of the resulting conditions that we obtain is an equation for \( \partial \phi \) (and \( \bar{\partial} \bar{\phi} \)). This implies that

\[
\zeta = \begin{pmatrix} g \\ 0 \\ g^* \\ 0 \end{pmatrix},
\]

(15)

where \( g \) is a function of \( z \) and \( \bar{z} \), and we are working in a spinor basis where the \( \Gamma \)-matrices take the form

\[
\begin{align*}
\Gamma^0 & = i \sigma^2 \otimes \sigma^1 \\
\Gamma^1 & = \sigma^1 \otimes \sigma^1 \\
\Gamma^2 & = \sigma^3 \otimes \sigma^1 \\
\Gamma^3 & = 1 \otimes \sigma^2
\end{align*}
\]

(16)
It is also convenient to introduce
\[ \Gamma \equiv \Gamma^2 + i\Gamma^3 = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad (17) \]
implying
\[ P_+ \Gamma = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (18) \]

With these conventions, and the assumption that the solution respects 2-dimensional Poincare invariance, we find that the first term in (14) can be written as
\[
e^a_A (P_+ \Gamma^A \zeta) \partial_a \phi^i = e^a_A (P_+ \Gamma^A \zeta) \partial_a \phi^i \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} g \\ 0 \\ g^* \end{pmatrix} = 2 \exp (-B) \partial \phi^i \cdot \begin{pmatrix} g \\ 0 \\ g^* \end{pmatrix}. \quad (19) \]
The other part of \( \delta \psi^i_a \) is equal to
\[
-\exp \left( \frac{K}{2} \right) K_{ij} \bar{W}_{ij} P_+ \zeta = -\exp \left( \frac{K}{2} \right) K_{ij} \bar{W}_{ij} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g \\ 0 \\ g^* \end{pmatrix} = -\exp \left( \frac{K}{2} \right) K_{ij} \bar{W}_{ij} \begin{pmatrix} g \\ 0 \\ g^* \end{pmatrix}. \quad (20) \]
Therefore, adding (19) to (20) and equating to zero, the relevant BPS equation becomes
\[
\partial \phi^i = \frac{1}{2} \left( \frac{g}{g^*} \right) \exp \left( B + \frac{K}{2} \right) K_{ij} \bar{W}_{ij}. \quad (21) \]

We now move on to the BPS equations coming from the SUSY variation of the gravitino \( \psi_{\mu a} \). Requiring \( \delta \psi_{\mu a} = 0 \) implies
\[
(P_+ \bar{\nabla}_\mu \zeta)_a + \frac{1}{2} (P_+ \Gamma_\mu \zeta)_a \exp \left( \frac{K}{2} \right) W = 0. \quad (22) \]
With $\tilde{A}_\mu = \partial_\mu \zeta = 0$ (since $\mu$ runs from 0 to 1), this becomes

$$\frac{1}{4} \omega_{\mu AB}(P_+ \Gamma^{AB})_{\alpha\beta} \zeta_\beta + \frac{1}{2}(P_+ \Gamma_\mu \zeta)_\alpha \exp \left( \frac{K}{2} \right) W \equiv T_1 + T_2 = 0 ,$$

(23)

where $\Gamma^{AB} \equiv [\Gamma^A, \Gamma^B]/2$. The first term, $T_1$, can be shown to be

$$T_1 = \frac{1}{2} \eta_{\mu p} \exp (A - B) \tilde{\partial} A (\Gamma^p \tilde{\Gamma} P_+ \zeta)_\alpha$$

(24)

(with $p = 0, 1$), for a spinor chosen as above. The second term, $T_2$, is equal to

$$T_2 = \frac{1}{2} (P_+ \Gamma_\mu \zeta)_\alpha \exp \left( \frac{K}{2} \right) W$$

$$= \frac{1}{2} \exp \left( \frac{K}{2} \right) W \eta_{\mu p} (\Gamma^p P_+ \zeta)_\alpha$$

$$= \frac{1}{2} \exp \left( \frac{K}{2} \right) W \exp (A) \eta_{\mu p} (\Gamma^p P_+ \zeta)_\alpha .$$

(25)

Therefore, adding (24) to (25) and setting the sum to zero, we obtain that the SUSY variation of $\psi_{\mu \alpha}$ vanishes if and only if

$$\exp (-B) \tilde{\partial} A \tilde{\Gamma} P_+ \zeta + \exp \left( \frac{K}{2} \right) W P_+ \zeta = 0 .$$

(26)

In components, this equation reads

$$\exp (-B) (\tilde{\partial} A) \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 \\
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \cdot \begin{pmatrix} g \\
0 \\
0 \\
0 \\
\end{pmatrix} + \exp \left( \frac{K}{2} \right) W \cdot \begin{pmatrix} 0 \\
0 \\
0 \\
g^* \\
\end{pmatrix} = 0 ,$$

(27)

which yields the second BPS equation:

$$\tilde{\partial} A = -\frac{1}{2} \left( \frac{g^*}{g} \right) \exp \left( B + \frac{K}{2} \right) W .$$

(28)

In a similar manner, using the combined variations of $\psi_{z \alpha}$ and $\delta \bar{\psi}_{z \alpha}$, we obtain the condition

$$g = \exp \left( \frac{A - iJ}{2} \right) ,$$

(29)

where $J$ is some real function, and a third BPS equation. Writing

$$N \equiv B + \frac{K}{2} + iJ ,$$

(30)
the final 4-dimensional BPS equations take the form
\[ \partial \phi^i = \frac{1}{2} \exp (N^*) K^{i\bar{j}} \bar{W} \bar{j}, \quad (31) \]
\[ \partial A = -\frac{1}{2} \exp (N^*) \bar{W}, \quad (32) \]
\[ \partial N = \frac{1}{2} \exp (N^*) (K^{i\bar{j}} K_{i\bar{j}} \bar{W} - \bar{W}). \quad (33) \]

To generalize these BPS equations to higher dimensions, we make the ansatz that they retain the same form up to potential dimension-dependent constants, while implying the full equations of motion in a higher-dimensional theory of scalars coupled to gravity.

The independent components of the Ricci tensor for our metric are
\[ R_{\mu\nu} = -4 \eta_{\mu\nu} \exp (2(A - B)) \left[ \partial \bar{\partial} A + (D - 2)(\partial A)(\bar{\partial} A) \right], \]
\[ R_{zz} = (D - 2) \left[ 2(\partial A)(\partial B) - (\partial A)^2 - \partial^2 A \right], \quad (34) \]
\[ R_{zz} = -2(\partial \bar{\partial} A + (\partial A)(\bar{\partial} A)) - 2\partial \bar{\partial} B. \]

(The \( \bar{z}z \) component of \( R_{ab} \), like that of the energy-momentum tensor below, is obtained from the \( zz \) component by replacing \( \partial \leftrightarrow \bar{\partial} \).) The energy-momentum tensor is
\[ T_{\mu\nu} = -2 \eta_{\mu\nu} \exp (2(A - B)) K_{i\bar{j}} \left[ (\partial \phi^i)(\bar{\partial} \bar{\phi}^j) + (\bar{\partial} \phi^i)(\partial \bar{\phi}^j) \right] - \eta_{\mu\nu} \exp (2A) V(\phi, \bar{\phi}), \]
\[ T_{zz} = 2 K_{i\bar{j}}(\partial \phi^i)(\bar{\partial} \bar{\phi}^j), \quad (35) \]
\[ T_{zz} = -\frac{1}{2} \exp (2B) V(\phi, \bar{\phi}). \]

There are thus three independent Einstein equations; the \( \mu\nu \) equation gives
\[ 2(D - 3)\partial \bar{\partial} A + 2\partial \bar{\partial} B + (D - 2)(D - 3)(\partial A)(\bar{\partial} A) = K_{i\bar{j}} \left[ (\partial \phi^i)(\bar{\partial} \bar{\phi}^j) + (\bar{\partial} \phi^i)(\partial \bar{\phi}^j) \right] - \frac{1}{2} \exp (2B) V(\phi, \bar{\phi}), \quad (36) \]
the \( zz \) equation gives
\[ (D - 2) \left[ 2(\partial A)(\partial B) - (\partial A)^2 - \partial^2 A \right] = 2 K_{i\bar{j}}(\partial \phi^i)(\bar{\partial} \bar{\phi}^j), \quad (37) \]
and the \( z\bar{z} \) equation gives
\[ (D - 2) \left[ \partial \bar{\partial} A + (D - 2)(\partial A)(\bar{\partial} A) \right] = -\frac{1}{2} \exp (2B) V(\phi, \bar{\phi}). \quad (38) \]
In addition we have the scalar field equation of motion,

\[(D - 2) \left[ 2(\partial A)(\bar{\partial} \phi^i) + 2(\bar{\partial} A)(\partial \phi^i) \right] + 4\partial \bar{\partial} \phi^i + 4K^{i\bar{m}} K_{j\bar{k}n}(\partial \phi^j)(\bar{\partial} \phi^k) = \exp (2B) K^{i\bar{j}} V_{\bar{j}} . \tag{39} \]

Substitution of the four-dimensional BPS equations, augmented by appropriate constant coefficients, back into these \(D\)-dimensional equations of motion yields the following \(D\)-dimensional BPS equations:

\[
\partial \phi^i = \frac{(D - 2)}{4} \exp (\kappa_D N^*) K^{i\bar{j}} \bar{W}_{\bar{j}} , \tag{40} 
\]

\[
\partial A = -\frac{\kappa_D}{2} \exp (\kappa_D N^*) \bar{W} , \tag{41} 
\]

\[
\partial N = \frac{\kappa_D}{2} \exp (\kappa_D N^*) \left[ \frac{(D - 2)}{2} K^{i\bar{j}} K_{i\bar{j}} \bar{W} - \bar{W} \right] , \tag{42} 
\]

where we have restored explicit factors of \(\kappa_D\) for later convenience. Together, these equations imply the equations of motion for gravity in \(D\) dimensions coupled to a set of complex scalars with Kähler potential \(K\) and potential energy given by (3).

## 4 BPS Bounds and the Junction Energy

In this section we compute the energy of configurations described by the BPS equations that we have derived. We are interested in static configurations, for which the energy density is defined as the negative of the action:

\[
E = \int dx^{D-1} dx^{D-2} (I_{\text{grav}} + I_{\text{matter}}) , \tag{43} 
\]

where

\[
I_{\text{grav}} \equiv -\frac{1}{2} \sqrt{|g| R} = 2 \exp ((D - 2)A) \left[ 2\partial \bar{\partial}(B + (D - 2)A) + (D - 1)(D - 2)(\partial A)(\bar{\partial} A) \right] , \tag{44} 
\]

and

\[
I_{\text{matter}} = \exp ((D - 2)A) \left\{ 2K_{i\bar{j}}(\partial \phi^i)(\bar{\partial} \phi^j) + 2K_{i\bar{j}}(\partial \phi^i)(\bar{\partial} \phi^j) 
+ \frac{(D - 2)}{4} \exp (2B + K) \left[ (D - 2)K^{i\bar{j}} W_{i\bar{j}} - 2(D - 1)W \right] \right\} . \tag{45} 
\]
The gravitational piece of this can be rewritten in the form

\[
I_{\text{grav}} = (D - 2) \exp ((D - 2)A) \left\{ (\partial A)(\bar{\partial} K) + (\bar{\partial} A)(\partial K) - 2(\bar{\partial} A)(\partial N) - 2(\bar{\partial} A)(\bar{\partial} N^*) - 2(D - 3)(\partial A)(\bar{\partial} A) \right\} + \Sigma_1 , \tag{46}
\]

where \(\Sigma_1\) is a total derivative given by

\[
\Sigma_1 \equiv \partial \left[ \exp ((D - 2)A) (2(D - 2)\bar{\partial} A + 2\partial N - \bar{\partial} K) \right] + \bar{\partial} \left[ \exp ((D - 2)A) (2(D - 2)\partial A + 2\partial N^* - \partial K) \right] . \tag{47}
\]

Similarly, the matter piece of the energy density can be rewritten as

\[
I_{\text{matter}} = \exp ((D - 2)A) \left\{ 4K_{ij}(\partial \phi^i)(\bar{\partial} \phi^j) + (D - 2)(\partial A) \left[ 2K_{ij}(\bar{\partial} \phi^j) - (\bar{\partial} K) \right] + (D - 2)(\bar{\partial} A) \left[ 2K_{\bar{i}j}(\partial \phi^j) - (\partial K) \right] + \frac{(D - 2)}{4} \exp (2B + K) \left[ (D - 2)K^{ij} \bar{W}_{\bar{i}j} - 2(D - 1)W \bar{\bar{W}} \right] \right\} + \Sigma_2 , \tag{48}
\]

where \(\Sigma_2\) is a total derivative term given by

\[
\Sigma_2 \equiv \partial \left[ \exp ((D - 2)A) (K_{\bar{i}j}(\partial \phi^j) - K_{ij}(\bar{\partial} \phi^j)) \right] + \bar{\partial} \left[ \exp ((D - 2)A) (K_{\bar{j}j}(\bar{\partial} \phi^j) - K_{ij}(\partial \phi^j)) \right] . \tag{49}
\]

Adding (46) to (48), the resulting integrand is

\[
I_{\text{grav}} + I_{\text{matter}} = \exp ((D - 2)A) \left\{ 4K_{\bar{i}j}(B1)^i(B1)^{\bar{j}} + 2(D - 2)K_{\bar{i}j}(B1)^i(B2)^* + 2(D - 2)K_{\bar{j}j}(B2)^i(B1)^{\bar{j}} - 2(D - 2)(B2)^i(B3)^* - 2(D - 2)(B3)^* - 2(D - 2)(B2)^i(B2)^* \right\} + \Sigma_1 + \Sigma_2 + \Sigma_3 , \tag{50}
\]

where each of the terms \((B1)^i\), \((B2)^i\), \((B3)^*\) vanishes independently by virtue of BPS equations (40), (41), (42) respectively, and the third total derivative is

\[
\Sigma_3 \equiv (D - 2)\partial \left[ \exp (N + (D - 2)A) W \right] + (D - 2)\bar{\partial} \left[ \exp (N^* + (D - 2)A) \bar{W} \right] . \tag{51}
\]
We therefore find that the total energy as defined by (43), can be expressed for a BPS junction state as

\[ E_{\text{total}} = \int d^D x (\Sigma_1 + \Sigma_2 + \Sigma_3) . \]  

(52)

To interpret this expression, it is useful to consider the flat-space limit of BPS junctions, considered (in four dimensions) in \([13, 14]\). Setting \(\kappa_D = 0\) in (40) yields

\[ \partial \phi^i = (D - 2) \frac{1}{4} \bar{W}_j \bar{W}_j , \]  

(53)

which is precisely the flat-space BPS equation for wall junctions. Meanwhile, in this limit (41) simply becomes \(\partial A = 0\), so \(A\) is a constant that can be set to zero by a trivial coordinate transformation. Further, (36) implies that in the \(\kappa_D = 0\) limit \(B\) is a harmonic real function of \(z\) and \(\bar{z}\), and therefore can be brought to zero by a holomorphic coordinate transformation of \(z\) and \(\bar{z}\). Thus, we recover flat spacetime in this limit. However, we also find important information from the \(O(\kappa_D^1)\) term in the small-\(\kappa_D\) limit; since the \(O(\kappa_D^0)\) term in \(A\) can be set to zero, we can define \(A' \equiv -A/\kappa_D\), which in this limit satisfies

\[ \partial A' = \frac{1}{2} \bar{W} , \]  

(54)

which is precisely the equation satisfied by the preprofile for the superpotential for BPS junctions discussed in \([14]\). Thus, in this limit, the \(\Sigma_1\) term in (52) vanishes, while \(\Sigma_3\) becomes the sum of the energy of the individual walls and \(\Sigma_2\) becomes the tension associated with the junction itself. As shown in \([21]\), the junction tension is negative.

5 A graviton zero mode

Arkani-Hamed et al. \([6]\) have shown that the effective \((D - 2)\)-dimensional theory on a junction of infinitely thin walls should contain a localized graviton. In this section we check that our solitonic walls (of finite thickness) also contain a distinguished zero mode metric fluctuation which, for a well-behaved junction solution, will be localized to the hub. We find that there exists a closed form expression for this mode, and note that its existence appears to be a general feature of domain wall junctions in AdS, rather than a special feature that occurs as a result of supersymmetry. We will also examine the graviton zero mode for a single domain wall, and make an explicit
connection with the wave equation of [3] and its zero-eigenvalue solution in the long-distance limit.

We begin with a set of background fields $\phi^i(z, \bar{z})$ and background metric $g_{ab}$ satisfying the Bogolmonyi equations (40)-(42), and consider linearized equations for the metric perturbations by substituting $g_{ab} \rightarrow g_{ab} + \delta g_{ab}$ into the equations of motion (36)-(39). We find that there is a self-consistent solution without introducing perturbations in the scalar fields. We take the metric fluctuation $\delta g_{ab}$ to be of the form

$$\delta g_{zz} = \delta g_{\bar{z}z} = \delta g_{z\bar{z}} = \delta g_{\mu z} = 0,$$

$$\delta g_{\mu\nu} = e^{ip \cdot x} e^{2A} \Psi(z, \bar{z}) P_{\mu\nu},$$

where $\Psi(z, \bar{z})$ is the graviton wavefunction. Here $p \cdot x \equiv p_\mu x^\mu$, with $p_\mu$ some $(D - 2)$ vector, and $P_{\mu\nu} = P_{\nu\mu}$ is a polarization tensor satisfying

$$\partial_\alpha (p_\nu) = \partial_\alpha (P_{\nu\sigma}) = \eta^{\mu\nu} P_\mu P_\nu = \eta^{\mu\nu} P_{\mu\nu} = 0.$$

It is convenient to write the perturbed Einstein equation as

$$\delta R_{ab} - \delta \left[ \frac{1}{2} R g_{ab} + T_{ab} \right] = 0.$$

Straightforward computation then shows that the first term, the variation of the Ricci tensor, is

$$\delta R_{\mu\nu} = - e^{ip \cdot x} e^{2A-2B} \left\{ (D - 2) (\partial A) (\bar{\partial} \Psi) + (D - 2) (\bar{\partial} A) (\partial \Psi) + 2 \partial \bar{\partial} \Psi + 4 \left[ \partial \bar{\partial} A + (D - 2) (\partial A) (\bar{\partial} A) \right] \Psi \right\} P_{\mu\nu},$$

with all other components vanishing. The second term in (58) is

$$\frac{1}{2} \delta (R g_{\mu\nu}) + \delta T_{\mu\nu} = e^{ip \cdot x} e^{2A} \left( \frac{2V}{D - 2} \right) \Psi P_{\mu\nu}.$$

Therefore, the linearized equations of motion are equivalent to the Schrödinger equation

$$\left[ -2 \partial \bar{\partial} - (D - 2) (\partial A) \bar{\partial} - (D - 2) (\bar{\partial} A) \partial \right] \Psi + 4 \partial \bar{\partial} A + 4(D - 2) (\partial A) (\bar{\partial} A) + \frac{2V}{D - 2} \right] \Psi = 0.$$
However, using equations (40)-(42) and the definition (3) of the scalar potential, we see that the second term vanishes, and the distinguished solution to the massless Schrödinger equation is simply \( \Psi = \text{constant} \), or

\[
\delta g_{\mu\nu} \propto e^{ip \cdot x} e^{2A} P_{\mu\nu} .
\]  

Lacking an explicit solution to the BPS equations, we can nevertheless check that our graviton fluctuation reduces to that of [6] in the long-distance (thin-wall) limit. This serves to verify that the integral of the square norm of the wavefunction is indeed convergent far from the walls. For the case of a four-wall junction in two extra dimensions, the metric of [6] can be written

\[
ds^2 = \frac{1}{(k|y_1| + k|y_2| + 1)^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dy_1^2 + dy_2^2) ,
\]

where we are using \( y_i \) instead of \( \bar{z}_i \) to avoid confusion with our complex coordinates, and \( k \equiv (\sqrt{2}L)^{-1} \), with \( L \) the AdS curvature radius in the bulk. This is a special case of our metric (6), with

\[
e^A = e^B = (k|y_1| + k|y_2| + 1)^{-1} .
\]

The associated graviton zero-mode is

\[
\delta g_{\mu\nu} \propto e^{ip \cdot x} (k|y_1| + k|y_2| + 1)^{-2} P_{\mu\nu} ,
\]

which therefore agrees with the form given by (62). Thus, integrating the norm square of the wavefunction \( \Psi \) over the \( z-\bar{z} \) plane, we find that the norm is finite in this thin-wall limit. More precisely, since the metric may be singular near the walls, we have shown that there is a distinguished zero-frequency wavefunction whose lack of normalizability can only arise as a result of the singular behavior of the solution when curvatures become large and the supergravity approximation breaks down. However, since we know that under favorable conditions the wall solutions themselves can become nonsingular when embedded in higher-dimensional supergravity, we believe that the normalizable zero mode is likely to be a robust feature of the full supergravity (or string theory) state.
6 Discussion

The idea that the three observed dimensions of our universe may not be the only large spatial dimensions, and that our universe may exist as a distinguished 3-manifold embedded in a higher dimensional space, is an intriguing alternative to conventional compactification [3]. If the true universe is (4+1) dimensional, then the relevant 3-manifold is a domain wall, or 3-brane. However, for larger background spaces, the 3-manifold representing our universe may lie at an intersection of a number of such codimension one branes [6]. (For alternative proposals supporting more than one extra dimension, see [22], [23].) In this paper we have considered models of scalar fields coupled to gravity, which are higher dimensional generalizations of four dimensional supergravity. We were able to derive first-order BPS equations appropriate to junction configurations with (3 + 1)-dimensional Poincare invariance on the hub.

The question of the cosmological constant as measured by inhabitants of the brane world is a crucial one in Randall-Sundrum scenarios and their generalizations. In models where the branes are put in by hand as delta-function sources, the induced cosmological constant can be tuned to zero by appropriately balancing the brane tension against the bulk cosmological constant (either in the original single-brane scenario [3] or in junction models [6]). In models where the branes are solitons constructed in a field theory, there may be additional constraints on the induced brane geometry. In the single-brane case, Behrndt and Cvetic have argued that a BPS state in an actual supergravity theory will automatically have a flat induced geometry [19], although it may also necessarily be singular. The examples of Skenderis and Townsend [16] and DeWolfe et al. [17] demonstrate that it is possible to find non-singular BPS-like solutions with flat induced geometries, although not necessarily in theories that are truncations of supergravity.

In the case of junctions, there is an additional complication due to the tension of the junction itself; one might worry that there could not simultaneously be flat walls, a flat junction, and a nonvanishing junction tension. We proceeded with an ansatz satisfying (3 + 1)-dimensional Poincare invariance, implying a flat geometry on the junction itself, and were able to derive a consistent set of BPS-like equations in a theory of scalars coupled to gravity. Since our solutions feature a nonzero junction tension, the external geometry in the thin-wall limit will therefore resemble the AdS patches of [6] along
with extra global identifications representing a deficit angle due to the gravitational influence of the junction, as discussed in [8]. Our ability to derive the appropriate BPS equations is good evidence that such solutions exist (although they may be singular); it would be interesting to further explore the relationship between the tensions on the walls and the junction, the bulk cosmological constant, and the induced geometry. It is unclear, for example, whether preserving supersymmetry necessarily induces a flat metric on a junction.

In an infinitely thin brane junction universe, it has been argued that the graviton is confined to the junction [6]. Here we have examined this question for junctions of finite width, constructed as solitonic solutions to field theories. In this microphysical context we have shown that the junction state admits a graviton zero mode, normalizable away from the walls, and thus localizes gravity effectively. In addition, this result does not appear to depend crucially on supersymmetry. Rather it appears to be a general feature of domain wall junctions in an anti-de Sitter background.

The models considered in this paper are particle-physics realizations of a generalized Randall-Sundrum scenario, and as such should provide a testing ground for details of the theory. We have demonstrated how various aspects of the brane-world idea arise in the context of these models, and hope that further investigation will cast light on the implications of this scenario for the observable universe.

Acknowledgments

We would like to thank Oliver DeWolfe, Dan Freedman, Jeff Harvey, Neamanja Kaloper, Per Kraus, Rob Meyers, Joe Polchinski, Walter Polkosnik, and Raman Sundrum for valuable discussions. This work was supported in part by the National Science Foundation under grants PHY/94-07194 and PHY/97-22022, and by the U.S. Department of Energy (D.O.E.)

References


