Branes, AdS gravitons and Virasoro symmetry

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We consider travelling waves propagating on the anti-de Sitter (AdS) background. It is pointed out that for any dimension \(d\), this space of solutions has a Virasoro symmetry with a non-zero central charge. This result is a natural generalization to higher dimensions of the three-dimensional Brown-Henneaux symmetry.

We consider in this note travelling waves in \(d = 3 + n\) dimensions propagating on AdS spacetime described by the line element,

\[ ds^2 = \frac{r^2}{z^2} [dz^2 + H(z, x^i, u) du^2 + dv^2 + dx^i dx_i] . \]  \hspace{1cm} (1)

Here \(i = 1, 2, \ldots, n\) and all coordinates are dimensionless. This metric satisfies Einstein’s equations with a cosmological constant proportional to \(-1/l^2\), provided \(H\) satisfies the Siklos [1–3] equation,

\[ z^{d-2} \partial_i \left( \frac{1}{z^{d-2}} \partial_i H \right) + \nabla^2 H = 0. \]  \hspace{1cm} (2)

The solution (1) can also be obtained, for example, following the general technique developed in [4] using anti-de Sitter space as background geometry. The perturbation \(H du^2/z^2\) is of the general Kerr-Shild type \(\Phi(x) \xi_i \xi_i \, dx^i dx^n\) where \(\xi^i\) is a null, hypersurface orthogonal, Killing field. The dependence of \(H\) in the coordinate \(u\) is arbitrary and defines the profile of the travelling wave.

Solutions of the form (1), and their generalizations including other gauge fields, have been extensively studied in the string theory literature [5–8]. Recently, (1) has appeared in [9] as an exact non-linear version of the Randall and Sundrum model [10].

In the particular case with \(d = 3\) and no transverse dimensions \((n = 0)\), the metric (1) enjoys a conformal symmetry which follows from the analysis of [11]. There exists a transformation of coordinates (non vanishing at infinity) whose associated Noether charges obey the Virasoro algebra with a central charge \(c = 3l/2G_3\). Actually, in three dimensions, the solution (1) can be generalised to include an arbitrary function of \(v\) (see [13] for the explicit form of the metric). This yields two copies of the Virasoro algebra with the same central charge, and hence, the conformal group. This result has provided an elegant statistical description of (non-extreme) black hole entropy in three dimensions [12].

The goal of this note is to point out that the results of [11] can be extended in a natural way to higher dimensions. Indeed, we shall prove that the class of metrics (1) has a Virasoro symmetry with a central charge

\[ c = \frac{3l^{d-2}}{2G_d} , \]  \hspace{1cm} (3)

where \(G_d\) is the \(d\)-dimensional Newton constant.

The existence of an infinite dimensional symmetry \(u \rightarrow f(u)\) is signaled by the arbitrariness of \(H\) as a function of \(u\). This, together with the adS structure, yields the Virasoro algebra. In fact, the line element (1) is form-invariant under the following redefinition of the coordinates,

\[ u = f(u') , \]

\[ x^i = x'^i \sqrt{\partial f} , \]

\[ z = z' \sqrt{\partial f} , \]

\[ v = v' - \frac{1}{2} \left( x'^i x_i' + z'^2 \right) \frac{\partial^2 f}{\partial f} . \]  \hspace{1cm} (4)

where \(f\) is an arbitrary function of its argument. This transformation was already known to Siklos [2]. In the primed coordinates, the metric has the same form as (1) with
The transformation (4) is not a Killing symmetry of (1) because it changes the metric. However it does take solutions to solutions. In other words it acts on the infinite-dimensional symplectic manifold consisting of classical solutions of Einstein’s Equations of the form (1). When a group acts on a symplectic manifold preserving the symplectic form the associated Poisson algebra will in general be a central extension of the Lie algebra of the original group. In our case the group is that of one-dimensional diffeomorphisms and the central extension that of Virasoro.

To make contact with previous work we note that the transformation only changes the value of the charge at infinity (which is located in the open region \( z \to 0 \) and since the map \( u \to f(u) \) does not depend on \( z \), the transformation (4) does not go to the identity there. In fact, the transformation (4) is an asymptotic symmetry of the space of solutions (1) with a corresponding non-zero Noether charge.

By a straightforward application of the results presented in [15,11] and references therein, the Noether charge associated to (4) is

\[
J(\epsilon) = \lim_{z \to 0} \int \frac{d^nu}{2\pi} \epsilon(u) T(z,x',u).
\]

(7)

Here \( \epsilon(u) = f(u) - u \) is the infinitesimal parameter of the transformation, and \( T \) is related to \( H \) by

\[
T(z,x',u) = \frac{f'' - 2}{8Gd} \partial_z H(z,x',u).
\]

(8)

In the calculation of \( J \) we have assumed that the metric at infinity \( z \to 0 \) takes the form (1) for arbitrary values of \( H \).

The relevant properties of the transformation (4) are encoded in the Noether charge \( J \). If \( J \) is finite, then the transformation (4) represent a physical –not gauge– change of the state. This is analogous to a boost in an asymptotically flat space which induces momentum. Furthermore, the knowledge of the charge \( J \) and its transformation properties will enable us to determine the algebra of the generators of (4) in a straightforward way.

The function \( T \) appearing in (7) transforms as a Virasoro operator. Indeed, from (5) and (8) it is straightforward to prove (applying the operator \( \frac{1}{z} \partial_z \) at both sides of (5)) that \( T \) transforms as

\[
T'(z',x',u') = T(z,x,u)(\partial' f)^2 - \frac{c}{12}\{f,u'\},
\]

(9)

where the central charge is given in (3).

Let us now analyse the value of the Noether charge \( J \) on the space of solutions of the equation (2). The general solution to (2) can be written down and involves Bessel functions. Actually, the function \( T \), related to \( H \) in (8), satisfies a Bessel equation as well. Since in our discussion \( H \) is not relevant (the charge and Virasoro transformation only depend on \( T \)) we write the general solution directly in terms of \( T \),

\[
T(u,z,x') = \int d^n k z^{n/2}[A(k,u)J_{n/2}(kz)
+ B(k,u)Y_{n/2}(kz)]e^{ikx'},
\]

(10)

where \( k = \sqrt{k_0k} \), \( J_{n/2} \) and \( Y_{n/2} \) are Bessel functions, and \( A \) and \( B \) are arbitrary functions of their arguments.

Inserting (10) into (7) we find the formula for the Noether charge,

\[
J(\epsilon) = \lim_{z \to 0} \int \frac{du}{2\pi} \epsilon(u) \left[ A(0,u) + \frac{1}{z^n}B(0,u) \right],
\]

(11)

showing that, as usual, it only depends on the zero modes. In this formula we have absorbed some irrelevant coefficients into redefinitions of \( A \) and \( B \).

The sector \( A \neq 0, B = 0 \) has a finite Noether charge in the limit \( z \to 0 \). Note, however, that the central term in (9) does not depend on \( z \). Since the solution with \( A \neq 0 \) has a prefactor \( z^{n/2}J_{n/2}(kz) \sim z^n \) we conclude that the central piece in (9) is not carried by this sector. Since we are interested in the centrally extended Virasoro algebra we set, from now on, \( A = 0 \).

Next, we note that the central piece in (9) does not depend on \( x' \) either. This means that only the zero mode of \( B(k,u) \) will see it. Setting \( B(k,u) = B(u) \) we find the formula for the charge

\[
J(\epsilon) = \lim_{z \to 0} \int \frac{du}{2\pi} \epsilon(u) \left[ A(0,u) + \frac{1}{z^n}B(0,u) \right],
\]

(11)

Note, however, that (1) is not globally asymptotically AdS.

2We use the coordinate \( v \) as time, and we assume that \( u \) is compact with period \( 2\pi \), as in DLCGQ. See, for example, [14]. This yields a meaningful ADM decomposition provided \( H \neq 0 \).

3The components of the curvature tensor in an orthonormal frame, which control the existence of singularities [9], depend only on \( T \) as well.
where the prefactor $V_\perp := \int d^nx/z^n$ is the proper volume of transverse space (recall that $z^{-n}$ is the factor in the \( \text{adS} \) volume element). Furthermore, from (9) we derive the transformation law of \( B(u) \).

\[
B'(u') = B(u)(\delta'f)^2 - \frac{c}{12}\{f, u'\}. \tag{13}
\]

As usual when working with extended objects, the charge \( J_B \) diverges because \( V_\perp \to \infty \). We then define the density of charge per unit of proper volume,

\[
j_B(\epsilon) = \frac{J_B(\epsilon)}{V_\perp} = \int \frac{du}{2\pi} \epsilon(u)B(u), \tag{14}
\]

which is indeed finite.

The formula (13) is not enough to claim that \( B \) carries a central charge \( c \) because it depends on the normalization of \( B \). In order to fix the value of the central charge we need to prove that \( B \) satisfies the Virasoro algebra. This can be done using a general theorem proved in [15,11,16]. Let \( (M, g) \) be a Riemannian spacetime with a group of asymptotic Killing vectors fields \( \xi^\mu \) \((i = 1, \ldots, N)\), and let \( H[\xi] = \int \xi^\mu H_\mu + J[\xi] \) the corresponding canonical generators. By definition, \( J[\xi] \) is a boundary term that makes \( H \) differentiable and is equal to the Noether charge associated to the symmetry \( \xi \) [15]. Then, it follows,

\[
\delta \xi, J(\xi) = [J(\xi_i), J(\xi_j)]. \tag{15}
\]

This formula yields a powerful way to compute the algebra of global charges in gauge theories. We remark that the number of asymptotic symmetries need not to be finite. Indeed, in our case, as in [11], the asymptotic symmetries are parametrised by the infinite number of Fourier modes of \( f(u) \). Another example on which this procedure yields an infinite dimensional algebra is the affine Kac-Moody algebra in non-Abelian Chern-Simons theory [17].

We now apply this result to our situation. First, we notice that since we are working with a density of charge, the right hand side of (15) should be understood as a density of Poisson bracket, i.e., as the Poisson bracket divided by the proper volume \( V_\perp \). The function \( B(u) \) transforms as in (13). We then find,

\[
\int \frac{du}{2\pi} \epsilon \left( \gamma \partial B + 2B \partial \gamma - \frac{c}{12} \partial^2 \gamma \right) = \int \frac{d\epsilon}{2\pi} \int \frac{du}{2\pi} \epsilon(u) \gamma(u') [B(u), B(u')], \tag{16}
\]

which is indeed the Virasoro algebra with a central charge \( c \).

The Virasoro symmetries described here arise generically for all solutions describing travelling waves on \( \text{adS} \) backgrounds, or whose near horizon geometries are of the form \( \text{adS}_n \times N_{d-n} \). These include many examples relevant for the \( \text{adS}/\text{CFT} \) correspondence as, for example, D3 branes and the M2/M5 solution. Consider, in particular, the D1/D5 system consisting of a ten dimensional string theory compactified on \( S_4 \times T_4 \) with a non-zero RR field \( H_3 \) carrying both electric and magnetic charges \( Q_1 \) and \( Q_5 \). This system can be described by a two dimensional conformal field theory with a central charge \( c = 6Q_1Q_5 \) [18]. On the other hand, the near horizon geometry has the well-known structure \( \text{adS}_3 \times S_3 \times T^4 \) and we should then expect a realisation of the Brown-Henneaux conformal symmetry [11]. It was noticed in [12] that the effective three-dimensional Newton constant \( G_3 \) and cosmological constant \( \Lambda \) satisfy \( c = 3l^2/2G_3 = 6Q_1Q_5 \). Thus, the Brown-Henneaux Virasoro generators are indeed those of the CFT.

Travelling waves propagating on this background have been studied in [8]. It was already noticed in that reference that there exists a transformation of coordinates mapping the longitudinal travelling wave into the background geometry. This transformation is a particular case of the general Virasoro symmetry discussed here and corresponds to solving the differential equation \( T'(u') = T(u)(\partial'f)^2 - \frac{c}{12}\{f, u'\} = 0 \). Note that a function \( f(u) \) mapping \( T \neq 0 \) into \( T' = 0 \) can exist because of the inhomogenous term in the transformation law of \( T \). Let us now check that the central charge, appearing in the classical Poisson bracket of the Noether charges, is the correct one.

The (extreme) ten dimensional metric describing a longitudinal travelling wave is (see [8] and references therein)

\[
ds^2 = \left( 1 + \frac{r_0^2}{r^2} \right)^{-1} \left[ dudv + \frac{p(u)}{r^2} du^2 \right] + \left( 1 + \frac{r_0^2}{r^2} \right) (dr^2 + r^2 d\Omega_3) + dy^i dy^i \tag{17}
\]

where \( y^i \) are coordinates on \( T^4 \), whose volume is \( (2\pi)^4 V \). The electric and magnetic charges are given by

\[
Q_1 = V \frac{r_0^2}{g}, \quad Q_5 = \frac{r_0^2}{g}, \tag{18}
\]

where \( g \) is the string coupling related to Newton’s constant as \( G_{10} = 8\pi g^2 \). The near horizon geometry \( (r \sim 0) \) is \( \text{adS}_3 \times S^3 \times T^4 \) and we can then apply the transformations (4) adapted to these coordinates. The resulting Noether charge is

\[
J(\epsilon) = \frac{1}{16\pi G_{10}} \int d^4y \int r_0^2 d\Omega_3 \int r_0 du \epsilon(u) 2p(u). \tag{19}
\]

In this case the volume of transverse space is finite, in fact the problem is effectively three-dimensional. Inserting the values for \( G_{10}, V \) and \( r_0 \) we find
\[ J(\varepsilon) = Q_1 Q_5 \int \frac{du}{2\pi} \varepsilon(u) p(u) \]  

(20)

and the combination \( T := Q_1 Q_5 p(u) \) satisfies the Virasoro algebra with central charge \( c = 6Q_1 Q_5 \), as desired.

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