A small but nonzero cosmological constant

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Abstract

Recent astrophysical observations seem to indicate that the cosmological constant is small but nonzero and positive. The old cosmological constant problem asks why it is so small; we must now ask, in addition, why it is nonzero, and why it is positive. In this essay, we try to kill these three metaphorical birds with one stone. That stone is the unimodular theory of gravity, which is the canonical theory of gravity, except for the way the cosmological constant arises in the theory.

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Until recent years, there used to be only one well-known problem [1] with the cosmological constant, viz., why it is so small — some 120 orders of magnitude smaller than what we naively think it should be. If it is that small, it must be zero, so some of us thought. Now we know better. The recent astrophysical observations indicate that, quite likely, the cosmological constant is not zero, though small, and positive, giving rise to cosmic repulsion. [2] We are automatically invited to ask these additional questions: Why is the cosmological constant not zero? Why does it have the observed magnitude, contributing to the energy density of the observable universe about twice as much as matter?

In this essay, we will attempt to present a qualitative solution to these three problems of the cosmological constant. We will do so in the framework of unimodular gravity [3–7] which, as we will show, is nothing but the canonical theory of gravity — except for one curious twist which has to do with the way the cosmological constant arises in the theory.

First let us reiterate the cosmological constant problems and put them in a form that will be useful later in the essay. From the Einstein-Hilbert action of gravity, we know that the cosmological constant $\Lambda$ has units of the reciprocal of length squared. Until recent years, all galactic observations had failed to detect any spacetime distortions that one can attribute to a nonzero cosmological constant out to the farthest distance, about $10^{28}$ cm., in the observable universe. Denote the 4-volume of the observable universe by $V$, then the empirical observations give the bound $\Lambda \lesssim V^{-1/2}$. But theoretical expectations would predict a much larger value: $\Lambda \sim l_P^{-2}$ with $l_P \sim 10^{-33}$ cm being the Planck length. This vast discrepancy by 122 orders of magnitude constitutes the old cosmological constant problem: why is $\Lambda$ so small? Recent observations [8] (supernovae 1a, cosmic microwave background, cluster density and evolution etc) are consistent with a geometrically flat universe and they indicate that the cosmological constant contributes about 70% of the energy density; hence

$$\Lambda \sim + \frac{1}{\sqrt{V}},$$

the cosmological constant is non-zero (and positive) after all.

Two observations are now in order. First, it is not surprising that $\Lambda$ is non-zero since
setting $\Lambda = 0$ does not enhance the existing symmetry of the gravitation theory. Second, the (old) cosmological constant problem is insensitive to the non-renormalizability of quantized general relativity as the problem occurs well below the Planck scale (even the relatively small vacuum energy density in QCD yields a discrepancy of about 42 orders of magnitude). Thus it seems reasonable that one can adequately address the cosmological constant problems in the framework of a gravity theory whose classical limit resembles general relativity. In the following, we consider the unimodular theory of gravity.

Unimodular gravity is actually very well motivated on physical grounds. Following Wigner [9] for a proper quantum description of the massless spin-two graviton, the mediator in gravitational interactions, we naturally arrive at the concept of gauge transformations. Without loss of generality, we can choose the graviton’s two polarization tensors to be traceless (and symmetric). But since the trace of the polarization states is preserved by all the transformations, it is natural to demand that the graviton states be described by traceless symmetric tensor fields. The strong field generalization of the traceless tensor field is a metric tensor $g_{\mu\nu}$ that has unit determinant: $-\det g_{\mu\nu} \equiv g = 1$, hence the name ”unimodular gravity.”

At first sight, the unimodular constraint has greatly changed the gravitational field equation, since now only the traceless combinations appear:

$$R^{\mu\nu} - 1/4g^{\mu\nu}R = -8\pi G(T^{\mu\nu} - 1/4g^{\mu\nu}T^{\lambda}_{\lambda}),$$

where $T^{\mu\nu}$ is the conserved matter stress tensor. But in conjunction with the Bianchi identity for the covariant derivative of the Einstein tensor, the field equation yields $D^{\nu}(R - 8\pi GT^{\lambda}_{\lambda}) = 0$. Denoting that constant of integration by $-4\Lambda$, we find

$$R^{\mu\nu} - 1/2g^{\mu\nu}R = \Lambda g^{\mu\nu} - 8\pi GT^{\mu\nu},$$

the familiar Einstein’s equation. The only difference from the canonical theory is in the way $\Lambda$ arises in the theory — it is an (arbitrary) integration constant, unrelated to any parameter in the original action. There are two other differences [3] that are worth mentioning.
Unlike the canonical theory, the Lagrangian for unimodular gravity can be expressed as a polynomial of the metric field. Conformal transformations $g_{\mu\nu} = C^2 g'_{\mu\nu}$ in the unimodular theory of gravity are very simple, the unimodular constraint fixes the conformal factor $C$ to be 1.

Since $\Lambda$ arises as an arbitrary constant of integration, it has no preferred value classically. In the corresponding quantum theory, we expect the state vector of the universe to be given by a superposition of states with different values of $\Lambda$ and the quantum vacuum functional to receive contributions from all different values of $\Lambda$. For the quantum theory, it is advantageous to start with a generalized version of the classical unimodular theory given above, that is generally covariant while preserving locality. We will use the version of unimodular gravity given by the Henneaux and Teitelboim action [6]

$$S_{\text{unimod}} = -\frac{1}{16\pi G} \int \left[ \sqrt{g} (R + 2\Lambda) - 2\Lambda \partial_\mu T^\mu \right] (d^3x) dt. \quad (4)$$

One of its equations of motion is $\sqrt{g} = \partial_\mu T^\mu$, the generalized unimodular condition. Note that, in this theory, $\Lambda$ plays the role of "momentum" conjugate to the "coordinate" $\int d^3x T_0$ which can be identified, with the aid of the generalized unimodular condition, as the spacetime volume $V$ [10]. Hence $\Lambda$ and $V$ are conjugate to each other.

We are ready to argue why the observed cosmological constant is so small. The argument [4] makes crucial use of quantum mechanics. Consider the vacuum functional for unimodular gravity given by path integrations over $T^\mu$, $g_{\mu\nu}$, the matter fields (to be represented by $\phi$), and $\Lambda$:

$$Z_{\text{Minkowskii}} = \int d\mu(\Lambda) \int d[\phi] d[g_{\mu\nu}] \int d[T^\mu] \exp \left\{ -i[S_{\text{unimod}} + S_M(\phi, g_{\mu\nu})] \right\}, \quad (5)$$

where $S_M$ stands for the contribution from matter fields (and $d\mu(\Lambda)$ denotes the measure of the $\Lambda$ integration). The integration over $T^\mu$ yields $\delta(\partial_\mu \Lambda)$, which implies that $\Lambda$ is spacetime-independent (befiting its role as the cosmological constant). A Wick rotation now allows us to study the Euclidean vacuum functional $Z$. The integrations over $g_{\mu\nu}$ and $\phi$ give $\exp[-S_\Lambda(\bar{g}_{\mu\nu}, \bar{\phi})]$ where $\bar{g}_{\mu\nu}$ and $\bar{\phi}$ are the background fields which minimize the effective
action $S_\Lambda$. A curvature expansion for $S_\Lambda$ yields a Lagrangian whose first two terms are the Einstein-Hilbert terms $\sqrt{g}(R+2\Lambda)$ where $\Lambda$ now denotes the fully renormalized cosmological constant. We can make a change of variable from the original (bare) $\Lambda$ to the renormalized $\Lambda$. Let us assume that for the present cosmic era, $\phi$ is essentially in the ground state, then it is reasonable to neglect the effects of $\phi$. To continue, we follow Baum [11] and Hawking [12] to evaluate $S_\Lambda(g_{\mu\nu},0)$. For negative $\Lambda$, $S_\Lambda$ is positive; for positive $\Lambda$, one finds $S_\Lambda(g_{\mu\nu},0) = -3\pi/G\Lambda$, so that

$$Z = \int d\mu(\Lambda)\exp(3\pi/G\Lambda).$$

The essential singularity of the integrand at $\Lambda = 0+$ means that the overwhelmingly most probable configuration is the one with $\Lambda = 0$, and this in turn implies that the observed cosmological constant in the present era is essentially zero.

There is one serious shortcoming in the above argument involving the Wick rotation to Euclidean space. It is well-known that the Euclidean formulation of quantum gravity is plagued by the conformal factor problem, due to divergent path-integrals. In our defense, we want to point out that we have used the effective action in the Euclidean formulation at its stationary point only. We should also recall that the conformal factor problem is arguably rather benign in the original version of unimodular gravity (as pointed out above), so perhaps it is not that serious even in the generalized version that we have just employed. There is another cause for concern. Since part of the above argument bears some resemblance to Coleman’s wormhole approach [13], one may worry that some of the objections to Coleman’s argument (on top of the conformal factor problem) may also apply here. Fortunately, it appears that they do not. [14] In any case, to the extent that our argument is valid, we have understood why the observed cosmological constant is so small and why, if the cosmological constant is not exactly zero, it is positive.

In the above argument, we have assumed that for the present cosmic era, the matter fields are in their ground states so that their effects on the effective action can be neglected; and the end result is that the observed $\Lambda$ is zero. Plausible as this assumption is, it is not
entirely correct. So, we do expect a non-vanishing (but small) cosmological constant for the present era, and $\Lambda$ goes to zero only asymptotically as the universe expands. Regrettably, we have not been able to calculate the small but not-entirely-negligible effects of the matter fields on the effective action. We will adopt the attitude that the above result is valid to the lowest order of approximation for which $\Lambda$ is zero. We will now borrow an argument due to Sorkin [7] to make an order of magnitude estimate of the cosmological constant (to the next leading order). [15]

There are two ingredients to Sorkin’s argument. First, from unimodular gravity he takes the idea that $\Lambda$ is in some sense conjugate to the spacetime volume $V$. Hence their fluctuations obey a Heisenberg-type uncertainty principle,

$$\delta V \delta \Lambda \sim 1,$$  \hspace{1cm} (7)

where we have used the natural units ($\hbar = 1$, $G = 1$). The second ingredient to Sorkin’s argument does not seem to be related to the unimodular theory of gravity. It is drawn from the causal-set theory [16], which stipulates that continuous geometries in classical gravity should be replaced by "causal-sets", the discrete substratum of spacetime. The fluctuation in the number of elements $N$ making up the set is of the Poisson type, i.e., $\delta N \sim \sqrt{N}$. For a causal set, the spacetime volume $V$ becomes $N$. It follows that

$$\delta V \sim \delta N \sim \sqrt{N} \sim \sqrt{V}.$$  \hspace{1cm} (8)

Putting Eqs. (7) and (8) together yields a minimum uncertainty in $\Lambda$ of $\delta \Lambda \sim V^{-1/2}$. [17] But we have already argued that $\Lambda$ vanishes to the lowest order of approximation and that it is positive if it is not zero. So we conclude that $\Lambda$ fluctuates about zero with a magnitude of $V^{-1/2}$ and it is positive:

$$\Lambda \sim + \frac{1}{\sqrt{V}},$$  \hspace{1cm} (9)

which, lo and behold, is Eq. (1)! The cosmological constant is small, but non-zero and positive, and has the correct order of magnitude as observed. In other words, $\Lambda$ contributes
to the energy of the universe an amount on the order of the critical density. As a side remark, we note that if we now appeal to the inflationary universe scenario, we can also understand why matter contributes a comparable amount. [18]

In summary, we have proposed to understand why the cosmological constant is so small but non-zero and positive in the framework of the unimodular theory of gravity. This theory is well motivated, based on the quantum description of massless spin-two particles. It leads to a theory of gravity in which the cosmological constant is freed to become dynamical — it evolves with time as $1/\sqrt{V}$ with $V$ being the spacetime volume. [19] Admittedly, our argument above is not completely satisfactory; part of it may even be flawed. But hopefully, our scenario is correct. May this essay provoke further productive works towards the solution of the vexing cosmological constant problems and the physics beyond.

This essay is based on the talk given by one of us (YJN) in the 1999 DESY Theory Workshop. We thank H. Nielsen for encouraging us to write it up. We also thank R. D. Sorkin for a useful correspondence and discussion. The work reported here was supported in part by the U.S. Department of Energy under #DF-FC02-94ER40818 and #DE-FG05-85ER-40219, and by the Bahnson Fund of the University of North Carolina at Chapel Hill. YJN was on leave of absence at MIT where this essay was written. He thanks the faculty at the Center for Theoretical Physics for their hospitality.
REFERENCES


[2] We will not discuss other ideas and mechanisms (like quintessence) that claim to explain the astrophysical data.


[8] For a review of the recent data, see N. A. Bahcall, J. P. Ostriker, S. Perlmutter, and P. J. Steinhardt, Science 284, 1481 (1999), and the references therein.

We have taken the additive constant term to be \( V = 0 \) at \( t = 0 \).


Since we have a simple (instead of Coleman’s double) exponential in the integrand for the vacuum functional in Eq. (6) in the text, Polchinski’s argument (J. Polchinski, Phys. Lett. B219, 251 (1989)) against the wormhole approach is not applicable here. Also, Duff’s argument (M. Duff, Phys. Lett. B226, 36 (1989)) against the Baum-Hawking approach is not applicable here because our argument does not involve the step that Duff (correctly) finds objectionable.

Sorkin’s formulation and interpretation of the unimodular theory differs somewhat from ours. See Ref. [7]. Here we merely make use of some of his ideas.

For an introduction to the causal-set theory, see, e.g., D. D. Reid, gr-qc/9909075.

Note that the fluctuations in the renormalized \( \Lambda \) are given entirely by those in the bare \( \Lambda \), since only the bare \( \Lambda \), as the conjugate to \( V \), undergoes this kind of quantum fluctuations. This result is consistent with Sorkin’s prediction (see Ref. [7]). On the other hand, we should point out that the dynamics of the causal-set theory is not yet fully understood; so the last part of the argument may not be on solid ground.

Thus we may have ameliorated the “cosmic coincidence” problem of why both matter and the cosmological constant contribute comparable amounts to the energy density of the universe in the present era.

How far back to the early Universe one can extrapolate this result is an interesting question.