A field model on fibre bundles can be extended in a standard way to the BRS-invariant SUSY field model which possesses the Lie supergroup $\text{ISp}(2)$ of symmetries.

1 Introduction

Generalizing the BRS mechanics of E.Gozzi and M.Reuter [1-4], we show that any field model (independently of its physical symmetries) can be extended in a standard way to a SUSY field model. In comparison with the SUSY field theory in Ref. [5,6], this extension is formulated in terms of simple graded manifolds. From the physical viewpoint, the SUSY-extended field theory may describe odd deviations of physical fields, e.g., of a Higgs field.

We follow the conventional geometric formulation of field theory where classical fields are represented by sections of a smooth fibre bundle $Y \to X$ coordinatized by $(x^\lambda, y^i)$. A first order Lagrangian $L$ of field theory is defined as a horizontal density

$$L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda) \omega : J^1Y \to \bigwedge^n T^*X, \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad n = \dim X,$$

on the first order jet manifold $J^1Y \to Y$ of sections of $Y \to X$ [7-9]. The jet manifold $J^1Y$ coordinatized by $(x^\lambda, y^i, y^i_\lambda)$ plays the role of a finite-dimensional configuration space of fields. The corresponding Euler–Lagrange equations take the coordinate form

$$\delta_i L = (\partial_i - d_\lambda \partial^\lambda_i) \mathcal{L} = 0, \quad d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_\lambda \mu \partial^\mu_i.$$

Every Lagrangian $L$ (1) yields the Legendre map

$$\tilde{L} : J^1Y \to \Pi, \quad p^\lambda_i \circ \tilde{L} = \pi^\lambda_i = \partial^\lambda_i \mathcal{L},$$

of $J^1Y$ to the Legendre bundle

$$\Pi = \bigwedge^n T^*Y \otimes V^*Y \otimes T^*X,$$

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where $V^*Y$ is the vertical cotangent bundle of $Y \rightarrow X$. The Legendre bundle $\Pi$ is equipped with the holonomic coordinates $(x^\lambda, y^i, p^\lambda_i)$. It is seen as a momentum phase space of covariant Hamiltonian field theory where canonical momenta correspond to derivatives of fields with respect to all space-time coordinates [8-10]. Hamiltonian dynamics on $\Pi$ is phrased in terms of Hamiltonian forms
\[
H = p^\lambda_i dy^i \wedge \omega_\lambda - \mathcal{H}(x^\lambda, y^i, p^\lambda_i)\omega, \quad \omega_\lambda = \partial_\lambda]\omega.
\] (4)
The corresponding covariant Hamilton equations read
\[
y^i_\lambda = \partial^i_\lambda \mathcal{H}, \quad p^\lambda_i = -\partial_i \mathcal{H}.
\] (5)
Note that, if $X = \mathbb{R}$, covariant Hamiltonian formalism provides the adequate Hamiltonian formulation of time-dependent mechanics [4, 11]. This fact enables us to extend the above-mentioned BRS mechanics of E.Gozzi and M.Reuter to field theory.

A preliminary step toward the desired SUSY extension of field theory is its extension to field theory on the vertical tangent bundle $VY$ of $Y \rightarrow X$, coordinated by $(x^\lambda, y^i, \dot{y}^i)$ [9]. Coordinates $\dot{y}^i$ describe linear deviations of fields. The configuration and phase spaces of field theory on $VY$ are
\[
J^1VY \cong VJ^1Y, \quad (x^\lambda, y^i, \dot{y}^i, \dot{y}^i_\lambda),
\] (6)
\[
\Pi_{VY} \cong V\Pi, \quad (x^\lambda, y^i, p^\lambda_i, \dot{y}^i, \dot{p}^\lambda_i).
\] (7)
Due to the isomorphisms (6) – (7), the corresponding vertical prolongation of the Lagrangian $L$ (1) reads
\[
L_V = (\dot{y}^i \partial_i + \dot{y}^i_\lambda \partial^\lambda_i)L,
\] (8)
while that of the Hamiltonian form $H$ (4) is
\[
H_V = (\dot{p}^\lambda_i dy^i + p^\lambda_i d\dot{y}^i) \wedge \omega_\lambda - \mathcal{H}_V\omega, \quad \mathcal{H}_V = (\dot{y}^i \partial_i + \dot{p}^\lambda_i \partial^\lambda_i)\mathcal{H}.
\] (9)
The corresponding Euler–Lagrange and Hamilton equations describe the Jacobi fields of solutions of the Euler–Lagrange equations (2) and the Hamilton equations (5) of the initial field model on $Y$.

The SUSY-extended field theory is formulated in terms of simple graded manifolds whose characteristic vector bundles are the vector bundles $VJ^1VY \rightarrow J^1VY$ in Lagrangian formalism and $V\Pi_{VY} \rightarrow \Pi_{VY}$ in Hamiltonian formalism [12]. The SUSY extension adds to $(y^i, \dot{y}^i)$ the odd variables $(c^i, \bar{c}^i)$. The corresponding SUSY extensions of the Lagrangian $L$ (1) and the Hamiltonian form $H$ (4) are constructed in order to be invariant under the BRS transformation
\[
u_Q = c^i \partial_i + iy^i \frac{\partial}{\partial c^i}.
\] (10)
Moreover, let us consider fibre bundles $Y \to X$ characterized by affine transition functions of bundle coordinates $y^i$ (they are not necessarily affine bundles). Almost all field models are of this type. In this case, the transition functions of holonomic coordinates $\dot{y}^i$ on $VY$ are independent of $y^i$, and $(c^i)$ and $(\bar{c}^i)$ have the same transition functions. Therefore, BRS-invariant Lagrangians and Hamiltonian forms of the SUSY-extended field theory are also invariant under the Lie supergroup $\text{ISp}(2)$ with the generators $u_{\bar{Q}}$ (10) and

$$u_{\bar{Q}} = \bar{c}^i \partial_i - i\dot{y}^i \partial c^i, \quad u_K = \dot{c}^i \partial c^i, \quad u_C = \dot{c}^i \partial c^i - \bar{c}^i \partial \bar{c}^i.$$ (11)

Note that, since the SUSY-extended field theory, by construction, is BRS-invariant, one hopes that its quantization may be free from some divergences.

2 Technical preliminaries

Given a fibre bundle $Z \to X$ coordinated by $(x^\lambda, z^i)$, the $k$-order jet manifold $J^kZ$ is endowed with the adapted coordinates $(x^\lambda, z^i_\Lambda)$, $0 \leq |\Lambda| \leq k$, where $\Lambda$ is a symmetric multi-index $(\lambda_m...\lambda_1)$, $|\Lambda| = m$. Recall the canonical morphism

$$\lambda = dx^\lambda \otimes (\partial_\lambda + z^i_\Lambda \partial_i) : J^1Z \hookrightarrow \mathcal{J}^*X \otimes TZ.$$ (12)

Exterior forms $\phi$ on a jet manifold $J^kZ$, $k = 0, 1, \ldots$, are naturally identified with their pull-backs onto $J^{k+1}Z$. There is the exterior algebra homomorphism

$$h_0 : \phi_\lambda dx^\lambda + \phi_i dz^i \mapsto (\phi_\lambda + \phi_\lambda^i y^i_\lambda) dx^\lambda,$$ (12)

called the horizontal projection, which sends exterior forms on $J^kZ$ onto the horizontal forms on $J^{k+1}Z \to X$. Recall the operator of the total derivative

$$d_\lambda = \partial_\lambda + z^i_\lambda \partial_i + z^i_\mu \partial_\mu^i + \cdots.$$ (12)

A connection on a fibre bundle $Z \to X$ is regarded as a global section

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^i \partial_i)$$

of the affine jet bundle $J^1Z \to Z$. One says that a section $s : X \to Z$ is an integral section of $\Gamma$ if $\Gamma \circ s = J^1s$, where $J^1s$ is the jet prolongation of $s$ to a section of the jet bundle $J^1Z \to X$.

Given the vertical tangent bundle $VZ$ of a fibre bundle $Z \to X$, we will use the notation

$$\dot{z}^i = \frac{\partial}{\partial \dot{z}^i}, \quad \partial_\nu = \dot{z}^i \partial_i.$$ (13)
3 The covariant Hamiltonian field theory

Let us summarize briefly the basics of the covariant Hamiltonian field theory on the Legendre bundle $\Pi$ (3) [8-10].

Let $Y \to X$ be a fibre bundle and $L$ a Lagrangian (1) on $J^1Y$. The associated Poincaré–Cartan form

$$ H_L = \pi^\lambda_i dy^i \wedge \omega_\lambda - (\pi^\lambda_i y^i_\lambda - \mathcal{L}) \omega $$

is defined as the horizontal Lepagean equivalent of $L$ on $J^1Y \to Y$, i.e. $L = h_0(H_L)$, where $h_0$ is the horizontal projection (12). Every Poincaré–Cartan form $H_L$ yields the bundle morphism of $J^1Y$ to the homogeneous Legendre bundle

$$ Z_Y = J^1*Y = T^*Y \wedge \bigwedge^{n-1} (\wedge T^*X) $$

equipped with the holonomic coordinates $(x^\lambda, y^i, p^\lambda_i, p)$. Because of the canonical isomorphism $\Pi \cong V^*Y \bigwedge\bigwedge^{n-1} T^*X$, we have the 1-dimensional affine bundle

$$ \pi_{Z\Pi} : Z_Y \to \Pi. $$

The homogeneous Legendre bundle $Z_Y$ is provided with the canonical exterior $n$-form

$$ \Xi_Y = p\omega + p^\lambda_i dy^i \omega_\lambda. $$

Then a Hamiltonian form $H$ (4) on the Legendre bundle $\Pi$ is defined in an intrinsic way as the pull-back $H = h^*\Xi_Y$ of this canonical form by some section $h$ of the fibre bundle (14). It is readily observed that the Hamiltonian form $H$ (4) is the Poincaré–Cartan form of the Lagrangian

$$ L_H = h_0(H) = (p^\lambda_i y^i_\lambda - \mathcal{H}) \omega $$

on the jet manifold $J^1\Pi$, and the Hamilton equations (4) coincide with the Euler–Lagrange equations for this Lagrangian. Note that the Lagrangian $L_H$ plays a prominent role in the path integral formulation of Hamiltonian mechanics and field theory.

The Legendre bundle $\Pi$ (3) is provided with the canonical polysymplectic form

$$ \Omega_Y = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial_\lambda. $$

Given a Hamiltonian form $H$ (4), a connection

$$ \gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial_i + \gamma^\mu_\lambda \partial^\mu) $$

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on $\Pi \to X$ is called a Hamiltonian connection for $H$ if it obeys the condition
\[
\gamma \lfloor \Omega_Y = dH, \\
\gamma^i_\lambda = \partial^i_\lambda H, \quad \gamma^\lambda_{\lambda i} = -\partial^\lambda_i H.
\] (19)

Every integral section $r : X \to \Pi$ of the Hamiltonian connection $\gamma$ (19) is a solution of the Hamilton equations (5). Any Hamiltonian form $H$ admits a Hamiltonian connection $\gamma_H$ (19). A Hamiltonian form $H$ (4) defines the Hamiltonian map
\[
\hat{H} : \Pi \to J^1 \Pi \to J^1 Y, \quad y^i_\lambda = \partial^i_\lambda H,
\]
which is the same for all Hamiltonian connections associated with $H$.

In the case of hyperregular Lagrangians, Lagrangian and covariant Hamiltonian formalisms are equivalent. For any hyperregular Lagrangian $L$, there exists a unique Hamiltonian form $H$ such that the Euler–Lagrange equations (2) for $L$ are equivalent to the Hamilton equations (5) for $H$. The case of degenerate Lagrangians is more intricate. One can state certain relations between solutions of Euler–Lagrange and Hamilton equations if a Hamiltonian form $H$ satisfies the relations
\[
\hat{L} \circ \hat{H} \circ \hat{L} = \hat{L}, \\
p^i_\lambda \partial^i_\lambda H - \mathcal{H} = \mathcal{L} \circ \hat{H}
\] (20a, 20b)
[8-10]. It is called associated with a Lagrangian $L$. If the relation (20b) takes place on the Lagrangian constraint space $\hat{L}(J^1 Y) \subset \Pi$, one says that $H$ is weakly associated with $L$. We will show that, if $L$ and $H$ are associated, then their vertical and SUSY prolongations are weakly associated.

### 4 The vertical extension of field theory

As was mentioned above, the vertical extension of field theory on a fibre bundle $Y \to X$ to the vertical tangent bundle $VY \to X$ describes linear deviations of fields.

Let $L$ be a Lagrangian on the configuration space $J^1 Y$. Due to the isomorphism (6), its prolongation (8) onto the vertical configuration space $J^1 VY$ can be defined as the vertical tangent morphism
\[
L_V = \text{pr}_2 \circ V L : V J^1 Y \to \tilde{\mathcal{L}} T^* X, \\
\mathcal{L}_V = \partial_V \mathcal{L} = (\dot{y}^i \partial_i + \dot{y}^i_\lambda \partial^i_\lambda) \mathcal{L},
\] (21)
to the morphism $L$ (1). The corresponding Euler–Lagrange equations read
\[
\delta_i \mathcal{L}_V = \delta_i \mathcal{L} = 0, \\
\delta_i \mathcal{L}_V = \partial_V \delta_i \mathcal{L} = 0, \quad \partial_V = \dot{y}^i \partial_i + \dot{y}^i_\lambda \partial^i_\lambda + \dot{y}^i_\mu \partial^i_\mu.
\] (22a, 22b)
The equations (22a) are exactly the Euler–Lagrange equations (2) for the Lagrangian $L$. In order to clarify the physical meaning of the equations (22b), let us suppose that $Y \rightarrow X$ is a vector bundle. Given a solution $s$ of the Euler–Lagrange equations (2), let $\delta s$ be a Jacobi field, i.e., $s + \varepsilon \delta s$ is also a solution of the same Euler–Lagrange equations modulo terms of order $> 1$ in a parameter $\varepsilon$. Then it is readily observed that the Jacobi field $\delta s$ satisfies the Euler–Lagrange equations (22b).

The Lagrangian (21) yields the vertical Legendre map

$$
\hat{L}_V = VJ^1Y \rightarrow V\Pi,
$$

$$
p^i_\lambda = \partial^i_\lambda L = \pi^i_\lambda, \quad \hat{p}^i_\lambda = \partial_V \pi^i_\lambda.
$$

Due to the isomorphism (7), the vertical tangent bundle $V\Pi$ of $\Pi \rightarrow X$ plays the role of the momentum phase space of field theory on $VY$, where the canonical conjugate pairs are $(y^i, \hat{p}^i_\lambda)$ and $(\dot{y}^i, p^i_\lambda)$. In particular, $V\Pi$ is endowed with the canonical polysymplectic form (17) which reads

$$
\Omega_{VY} = \partial_V \Omega_Y = [d\hat{p}^i_\lambda \wedge dy^i + dp^i_\lambda \wedge d\dot{y}^i] \wedge \omega \otimes \partial_\lambda.
$$

Let $Z_{VY}$ be the homogeneous Legendre bundle (13) over $VY$ with the corresponding coordinates $(x^\lambda, y^i, \dot{y}^i, p^\lambda_i, q^\lambda_i, p)$. It can be endowed with the canonical form $\Xi_{VY}$ (15). Sections of the affine bundle

$$
Z_{VY} \rightarrow V\Pi,
$$

by definition, provide Hamiltonian forms on $V\Pi$. Let us consider the following particular case of these forms which are related to those on the Legendre bundle $\Pi$. Due to the fibre bundle

$$
\zeta : VZ_Y \rightarrow Z_{VY},
$$

$$(x^\lambda, y^i, \dot{y}^i, p^\lambda_i, q^\lambda_i, p) \circ \zeta = (x^\lambda, y^i, \dot{y}^i, \hat{p}^\lambda_i, p^i_\lambda, \hat{p}),
$$

the vertical tangent bundle $VZ_Y$ of $Z_Y \rightarrow X$ is provided with the exterior form

$$
\Xi_V = \zeta^* \Xi_{VY} = \hat{p}\omega + (\hat{p}^\lambda_i dy^i + p^i_\lambda d\dot{y}^i) \wedge \omega_\lambda.
$$

Given the affine bundle $Z_Y \rightarrow \Pi$ (14), we have the fibre bundle

$$
V\pi_{Z\Pi} : VZ_Y \rightarrow V\Pi,
$$

where $V\pi_{Z\Pi}$ is the vertical tangent map to $\pi_{Z\Pi}$. The fibre bundles (26), (27) and (28) form the commutative diagram. Let $h$ be a section of the affine bundle $Z_Y \rightarrow \Pi$ and $H = h^*\Xi$ the corresponding Hamiltonian form (4) on $\Pi$. Then the section $Vh$ of the
fibre bundle (28) and the corresponding section \( \zeta \circ Vh \) of the affine bundle (26) defines the Hamiltonian form \( H_V = (Vh)^* \Xi_V \) (9) on \( \Pi \). One can think of this form as being a vertical extension of \( H \). In particular, let \( \gamma \) (18) be a Hamiltonian connection on \( \Pi \) for the Hamiltonian form \( H \). Then its vertical prolongation

\[
V\gamma = \gamma + dx^\mu \otimes [\partial_V \gamma^i_\mu \partial_i + \partial_V \gamma^\lambda_\mu \partial^i_\lambda].
\]

on \( \Pi \rightarrow X \) is a Hamiltonian connection for the vertical Hamiltonian form \( H_V \) (9) with respect to the polysymplectic form \( \Omega_{VY} \) (25).

The Hamiltonian form \( H_V \) (9) defines the Lagrangian \( L_{H_V} \) (16) on \( J^1 \Pi \), which takes the form

\[
L_{H_V} = h_0(H_V) = [\dot{p}_i^\lambda(y_i^\lambda - \partial_i^\lambda H) - \dot{y}_i^\lambda(p_i^\lambda + \partial_i^\lambda H) + d_\lambda(p_i^\lambda \dot{y}_i)]\omega.
\]

The corresponding Hamilton equations contain the Hamilton equations (5) and the equations

\[
\dot{y}_i^\lambda = \partial_\lambda \partial_i^\lambda H = \partial_V \partial_i^\lambda H, \quad \dot{p}_i^\lambda = -\partial_i^\lambda H = -\partial_V \partial_i^\lambda H
\]

for Jacobi fields \( \delta y^i = \varepsilon \dot{y}_i^\lambda, \delta p_i^\lambda = \varepsilon \dot{p}_i^\lambda \).

The Hamiltonian form \( H_V \) (9) on \( \Pi \) yields the vertical Hamiltonian map

\[
\widehat{H}_V = V\widehat{H} : \Pi \rightarrow VJ^1 Y, \quad (29)
\]

\[
y_i^\lambda = \partial_\lambda H_V = \partial_i^\lambda H, \quad \dot{y}_i^\lambda = \partial_V \partial_i^\lambda H. \quad (30)
\]

Let \( H \) be associated with a Lagrangian \( L \). Then \( H_V \) is weakly associated with \( L_V \). Indeed, if the morphisms \( \widehat{H} \) and \( \widehat{L} \) obey the relation (20a), then the corresponding vertical tangent morphisms satisfy the relation

\[
V\widehat{L} \circ V\widehat{H} \circ V\widehat{L} = V\widehat{L}.
\]

The condition (20b) for \( H_V \) reduces to the equality

\[
\partial_i^\lambda H(p) = -(\partial_i^\lambda \mathcal{L} \circ \widehat{H})(p), \quad p \in \widehat{L}(J^1 Y),
\]

which is fulfilled if \( H \) is associated with \( L \) [8-10].

5 Geometry of simple graded manifolds

Following the BRS extension of Hamiltonian mechanics, we formulate the SUSY-extended field theory in terms of simple graded manifolds.
Let \( E \to Z \) be a vector bundle with an \( m \)-dimensional typical fibre \( V \) and \( E^* \to Z \) the dual of \( E \). Let us consider the exterior bundle
\[
\wedge E^* = \mathbb{R} \oplus (\bigoplus_{k=1}^{m} \wedge^k E^*)
\]
whose typical fibre is the finite Grassmann algebra \( \wedge V^* \). By \( A_E \) is meant the sheaf of its sections. The pair \((Z, A_E)\) is a graded manifold with the body manifold \( Z \) coordinated by \( (z^A) \) and the structure sheaf \( A_E \) [13-15]. We agree to call it a simple graded manifold with the characteristic vector bundle \( E \). This is not the terminology of [16] where this term is applied to all graded manifolds of finite rank in connection with Batchelor’s theorem. In accordance with this theorem, any graded manifold is isomorphic to a simple graded manifold though this isomorphism is not canonical [14, 17]. To keep the structure of a simple graded manifold, we will restrict transformations of \((Z, A_E)\) to those induced by bundle automorphisms of \( E \to Z \).

Global sections of the exterior bundle (31) are called graded functions. They make up the \( \mathbb{Z}_2 \)-graded ring \( A_E(Z) \). Let \( \{c^a\} \) be a basis for \( E^* \to Z \) with respect to some bundle atlas with transition functions \( \{\rho^a_b\} \), i.e., \( c^a = \rho^a_b(z)c^b \). We will call \((z^A, c^a)\) the local basis for the simple graded manifold \((Z, A_E)\). With respect to this basis, graded functions read
\[
f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1...a_k} c^{a_1} \cdots c^{a_k},
\]
where \( f_{a_1...a_k} \) are local functions on \( Z \), and we omit the symbol of the exterior product of elements \( c \). The coordinate transformation law of graded functions (32) is obvious. We will use the notation \([\cdot]\) of the Grassmann parity.

Given a simple graded manifold \((Z, A_E)\), by the sheaf \( \text{Der} A_E \) of graded derivations of \( A_E \) is meant a subsheaf of endomorphisms of \( A_E \) such that any its section \( u \) over an open subset \( U \subset Z \) is a graded derivation of the graded algebra \( A_E(U) \) of local sections of the exterior bundle \( \wedge E^*|_U \), i.e.,
\[
u(f f') = u(f)f' + (-1)^{|u||f|}fu(f')
\]
for the homogeneous elements \( u \in (\text{Der} A_E)(U) \) and \( f, f' \in A_E(U) \). Graded derivations are called graded vector fields on a graded manifold \((Z, A_E)\) (or simply on \( Z \) if there is no danger of confusion). They can be represented by sections of some vector bundle as follows.

Due to the canonical splitting \( VE \cong E \times E \), the vertical tangent bundle \( VE \to E \) can be provided with the fibre basis \( \{\partial/\partial c^a\} \) dual of \( \{c^a\} \). This is the fibre basis for \( \text{pr}_2VE \cong E \). Then a graded vector field on a trivialization domain \( U \) reads
\[
u = u^A \partial_A + u^a \frac{\partial}{\partial c^a},
\]
where \( u^A, u^a \) are local graded functions. It yields a derivation of \( A_E(U) \) by the rule

\[
u(f_{a...b}c^a \cdots c^b) = u^A \partial_A (f_{a...b}) c^a \cdots c^b + u^d f_{a...b} \frac{\partial}{\partial c^d} (c^a \cdots c^b).
\]

(33)

This rule implies the corresponding coordinate transformation law

\[
u^A = u^A, \quad \nu^a = \rho^a_j u^j + u^A \partial_A (\rho^a_j) c^j
\]

of graded vector fields. It follows that graded vector fields on \( Z \) can be represented by sections of the vector bundle \( \mathcal{V}_E \rightarrow Z \) which is locally isomorphic to the vector bundle

\[\mathcal{V}_E \mid_U \approx \wedge^* E^* \otimes Z (\text{pr}_2 V E \oplus TZ) \mid_U,\]

and has the transition functions

\[
z^{iA}_{i_1...i_k} = \rho^{-1}_{a_1 i_1} \cdots \rho^{-1}_{a_k i_k} z^A_{a_1...a_k},
\]

\[
v^{i}_{j_1...j_k} = \rho^{-1}_{b_1 j_1} \cdots \rho^{-1}_{b_k j_k} \left[ \rho^a_i v^{j}_{b_1...b_k} + \frac{k!}{(k-1)!} \rho^a_i (\partial_A \rho^a_{b_k}) \right]
\]

of the bundle coordinates \((z^A_{a_1...a_k}, v^i_{b_1...b_k})\), \( k = 0, \ldots, m \). These transition functions fulfill the cocycle relations. Graded vector fields on \( Z \) constitute a Lie superalgebra with respect to the bracket

\[[u, u'] = uu' + (-1)^{|u||u'|+1} u'u.
\]

There is the exact sequence over \( Z \) of vector bundles

\[0 \rightarrow \wedge^* E^* \otimes \text{pr}_2 V E \rightarrow \mathcal{V}_E \rightarrow \wedge^*(E^* \otimes TZ) \rightarrow 0.
\]

Its splitting

\[
\tilde{\gamma} : \hat{z}^A \partial_A \mapsto \hat{z}^A (\partial_A + \tilde{\gamma}^a_A \frac{\partial}{\partial c^a})
\]

(34)

is represented by a section

\[
\tilde{\gamma} = dz^A \otimes (\partial_A + \tilde{\gamma}^a_A \frac{\partial}{\partial c^a})
\]

(35)

of the vector bundle \( T^* Z \otimes \mathcal{V}_E \rightarrow Z \) such that the composition

\[Z \rightarrow T^* Z \otimes \mathcal{V}_E \rightarrow T^* Z \otimes (\wedge^* E^* \otimes TZ) \rightarrow T^* Z \otimes TZ
\]
is the canonical form $dz^A \otimes \partial_A$ on $Z$. The splitting (34) transforms every vector field $\tau$ on $Z$ into a graded vector field

$$\tau = \tau^A \partial_A \mapsto \nabla_\tau = \tau^A(\partial_A + \tilde{\gamma}_A^a \frac{\partial}{\partial c^a}),$$

(36)

which is a graded derivation of $\mathcal{A}_E$ satisfying the Leibniz rule

$$\nabla_\tau (sf) = (\tau^r s) f + s \nabla_\tau (f), \quad f \in \mathcal{A}_E(U), \quad s \in C^\infty(Z), \quad \forall U \subset Z.$$

Therefore, one can think of the graded derivation $\nabla_\tau$ (36) and, consequently, of the splitting (34) as being a graded connection on the simple graded manifold $(Z, \mathcal{A}_E)$. In particular, this connection provides the corresponding decomposition

$$u = u^A \partial_A + u^a \frac{\partial}{\partial c^a} = u^A(\partial_A + \tilde{\gamma}_A^a \frac{\partial}{\partial c^a}) + (u^a - u^A \tilde{\gamma}_A^a) \frac{\partial}{\partial c^a}$$

of graded vector fields on $Z$. Note that this notion of a graded connection differs from that of connections on graded fibre bundles in Ref. [18].

In accordance with the well-known theorem on a splitting of an exact sequence of vector bundles, graded connections always exist. For instance, every linear connection

$$\gamma = dz^A \otimes (\partial_A + \gamma_A^a b^b \partial_a)$$

on the vector bundle $E \to Z$ yields the graded connection

$$\gamma_S = dz^A \otimes (\partial_A + \gamma_A^a b^b \partial_a)$$

(37)

such that, for any vector field $\tau$ on $Z$ and any graded function $f$, the graded derivation $\nabla_\tau (f)$ with respect to the connection (37) is exactly the covariant derivative of $f$ relative to the connection $\gamma$.

Let now $Z \to X$ be a fibre bundle coordinated by $(x^\lambda, z^i)$, and let

$$\gamma = \Gamma + \gamma_\lambda^a b^b dx^\lambda \otimes \partial_a$$

be a connection on $E \to X$ which is a linear morphism over a connection $\Gamma$ on $Z \to X$. Then we have the bundle monomorphism

$$\gamma_S : \wedge^r E^\ast \otimes TX \ni u^\lambda \partial_\lambda \mapsto u^\lambda (\partial_\lambda + \Gamma_i^j \partial_i + \gamma_\lambda^a b^b \partial_a) \in V_E$$

over $Z$, called a composite graded connection on $Z \to X$. It is represented by a section

$$\gamma_S = \Gamma + \gamma_\lambda^a b^b dx^\lambda \otimes \frac{\partial}{\partial c^a}$$

(38)
of the fibre bundle $T^*X \otimes \mathcal{V}_E \to Z$ such that the composition

$$Z \xrightarrow{\gamma} T^*X \otimes \mathcal{V}_E \to T^*X \otimes (\wedge^E \otimes TZ) \to T^*X \otimes TX$$

is the pull-back onto $Z$ of the canonical form $dx^\lambda \otimes \partial_\lambda$ on $X$.

Given a graded manifold $(Z, \mathcal{A}_E)$, the dual of the sheaf $\text{Der}_\mathcal{A}_E$ is the sheaf $\text{Der}^*\mathcal{A}_E$ generated by the $\mathcal{A}_E$-module morphisms

$$\phi : \text{Der}(\mathcal{A}_E(U)) \to \mathcal{A}_E(U).$$

One can think of its sections as being graded exterior 1-forms on the graded manifold $(Z, \mathcal{A})$. They are represented by sections of the vector bundle $\mathcal{V}_E^* \to Z$ which is the $\wedge^E$-dual of $\mathcal{V}_E$. This vector bundle is locally isomorphic to the vector bundle

$$\mathcal{V}_E^* \mid_U \approx \wedge^E \otimes (\text{pr}_2 V E^* \oplus T^*Z) \mid_U,$$

and has the transition functions

$$u_{j_1 \ldots j_k} = \rho_{j_1}^{-1} \cdots \rho_{j_k}^{-1} v_{a_1 \ldots a_k u},$$

$$z_{i_1 \ldots i_k A} = \rho_{i_1}^{-1} \cdots \rho_{i_k}^{-1} b_{i_1 \ldots i_k A} + \frac{k!}{(k-1)!} v_{b_1 \ldots b_{k-1} j} \partial_A \rho_{b_k}.$$

of the bundle coordinates $(z_{a_1 \ldots a_k A}, v_{b_1 \ldots b_k j})$, $k = 0, \ldots, m$, with respect to the dual bases \{dz^A\} for $T^*Z$ and \{dc^b\} for $\text{pr}_2 V E = E^*$. Graded exterior 1-forms read

$$\phi = \phi_A dz^A + \phi_a dc^a.$$

They have the coordinate transformation law

$$\phi'_a = \rho_{-1}^a \phi_b, \quad \phi'_A = \phi_A + \rho_{-1}^b \partial_A (\rho^a) \phi_b.$$

Then the morphism (39) can be seen as the interior product

$$u \mid_\phi = u^A \phi_A + (-1)^{\phi_a} u^a \phi_a.$$

There is the exact sequence

$$0 \to \wedge^E \otimes T^*Z \to \mathcal{V}_E^* \to \wedge^E \otimes \text{pr}_2 V E^* \to 0$$

of vector bundles. Any graded connection $\tilde{\gamma}$ (35) yields the splitting of this exact sequence, and defines the corresponding decomposition of graded 1-forms

$$\phi = \phi_A dz^A + \phi_a dc^a = (\phi_A + \phi_a \tilde{\gamma}_A^a) dz^A + \phi_a (dc^a - \tilde{\gamma}_A^a dz^A).$$
Graded $k$-forms $\phi$ are defined as sections of the graded exterior bundle $\bigwedge^k V_E^*$ such that
\[ \phi \wedge \sigma = (-1)^{|\phi||\sigma|+|\phi|} \sigma \wedge \phi, \]
where $|.|$ denotes the form degree. The interior product (40) is extended to higher degree graded forms by the rule
\[ u | (\phi \wedge \sigma) = (u | \phi) \wedge \sigma + (-1)^{|\phi|+|u|} \phi \wedge (u | \sigma). \]

The graded exterior differential $d$ of graded functions is introduced in accordance with the condition $u | df = u(f)$ for an arbitrary graded vector field $u$, and is extended uniquely to higher degree graded forms by the rules
\[ d(\phi \wedge \sigma) = (d\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (d\sigma), \quad d \circ d = 0. \]

It takes the coordinate form
\[ d\phi = dz^A \wedge \partial_A(\phi) + dc^a \wedge \frac{\partial}{\partial c^a}(\phi), \]
where the left derivatives $\partial_A$, $\partial/\partial c^a$ act on the coefficients of graded exterior forms by the rule (33), and they are graded commutative with the forms $dz^A$, $dc^a$. The Lie derivative of a graded exterior form $\phi$ along a graded vector field $u$ is given by the familiar formula
\[ L_u \phi = u | d\phi + d(u | \phi). \]

Let $E \to Z$ and $E' \to Z'$ be vector bundles and $\Phi : E \to E'$ a linear bundle morphism over a morphism $\zeta : Z \to Z'$. Then every section $s^*$ of the dual bundle $E'^* \to Z'$ defines the pull-back section $\Phi^* s^*$ of the dual bundle $E^* \to Z$ by the law
\[ v_z | \Phi^* s^*(z) = \Phi(v_z) | s^*(\zeta(z)), \quad \forall v_z \in E_z. \]

It follows that a linear bundle morphism $\Phi$ yields a morphism
\[ S\Phi : (Z, A_E) \to (Z', A_{E'}) \]
(41) of simple graded manifolds seen as locally ringed spaces [14]. This is the pair $(\zeta, \zeta_* \circ \Phi^*)$ of the morphism $\zeta$ of the body manifolds and the composition of the pull-back $A_{E'} \ni f \mapsto \Phi^* f \in A_E$ of graded functions and the direct image $\zeta_*$ of the sheaf $A_E$ onto $Z'$. With respect to local bases $(z^A, c^a)$ and $(z'^A, c'^a)$ for $(Z, A_E)$ and $(Z', A_{E'})$, the morphism (41) reads
\[ S\Phi(z) = \zeta(z), \quad S\Phi(c'^a) = \Phi_b^c(z)c^b. \]

Accordingly, the pull-back onto $Z$ of graded exterior forms on $Z'$ is defined.
6 SUSY-extended Lagrangian formalism

The SUSY-extended field theory is constructed as the BRS-generalization of the vertical extension of field theory on the fibre bundle \( VY \rightarrow X \) in Section 4.

Let us consider the vertical tangent bundle \( VVY \rightarrow VY \) of \( VY \rightarrow X \) and the simple graded manifold \((VY, A_{VVY})\) whose body manifold is \( VY \) and the characteristic vector bundle is \( VVY \rightarrow VY \). Its local basis is \((x^\lambda, y^i, \dot{y}^i, c^i, \bar{c}^\lambda)\) where \(\{c^i, \bar{c}^\lambda\}\) is the fibre basis for \(V^*VY\) dual of the holonomic fibre basis \(\{\partial_i, \partial^i\}\) for \(VY \rightarrow VY\). Graded vector fields and graded exterior 1-forms are introduced on \(VY\) as sections of the vector bundles \(V_{VVY}\) and \(V_{VVY}^*\), respectively. Let us complexify these bundles as \(\mathbb{C} \otimes V_{VVY}\) and \(\mathbb{C} \otimes V_{VVY}^*\). By the BRS operator on graded functions on \(VY\) is meant the complex graded vector field \(u_Q(10)\). It satisfies the nilpotency rule \(u_Q^2 = 0\).

The configuration space of the SUSY-extended field theory is the simple graded manifold \((V^{J^1}Y, A_{VV^{J^1}Y})\) whose characteristic vector bundle is the vertical tangent bundle \(VV^{J^1}Y \rightarrow V^{J^1}Y\) of \(V^{J^1}Y \rightarrow X\). Its local basis is \((x^\lambda, y^i, \dot{y}^i, y^i_\Lambda, \dot{y}^i_\Lambda, c^i, c^i_\ Lambda, \bar{c}^\lambda, \bar{c}^\lambda_\Lambda)\), where \(\{c^i, \bar{c}^\lambda, c^i_\Lambda, \bar{c}^\lambda_\Lambda\}\) is the fibre basis for \(V^*V^{J^1}Y\) dual of the holonomic fibre basis \(\{\partial_i, \partial^i, \partial^\Lambda, \partial^{\Lambda}\}\) for \(V^{J^1}Y \rightarrow V^{J^1}Y\). The affine fibration \(\pi^0: V^{J^1}Y \rightarrow VY\) and the corresponding vertical tangent morphism \(V\pi^0: VV^{J^1}Y \rightarrow VVY\) yields the associated morphism of graded manifolds \((V^{J^1}Y, A_{VV^{J^1}Y}) \rightarrow (VY, A_{VVY})\) (41).

Let us introduce the operator of the total derivative

\[ d_\lambda = \partial_\lambda + y^{i_\Lambda} \partial_i^i + \dot{y}^{i_\Lambda} \dot{\partial}_i^i + c^i_\Lambda \frac{\partial}{\partial c^i} + \bar{c}^{\Lambda} \frac{\partial}{\partial \bar{c}^{\Lambda}}. \]

With this operator, the coordinate transformation laws of \(c^i_\Lambda\) and \(\bar{c}^{\Lambda}\) read

\[ c^i_\Lambda' = d_\lambda c^i_\Lambda, \quad \bar{c}^{\Lambda'} = d_\lambda \bar{c}^{\Lambda}. \] (42)

Then one can treat \(c^i_\Lambda\) and \(\bar{c}^{\Lambda}\) as the jets of \(c^i\) and \(\bar{c}^{\Lambda}\). Note that this is not the notion of jets of graded bundles in [19]. The transformation laws (42) show that the BRS operator \(u_Q(10)\) on \(VY\) can give rise to the complex graded vector field

\[ Ju_Q = u_Q + c^i_\Lambda \partial_i^i + iy^i_\Lambda \frac{\partial}{\partial c^i_\Lambda} \] (43)

on the \(V^{J^1}Y\). It satisfies the nilpotency rule \((Ju_Q)^2 = 0\).

In a similar way, the simple graded manifold with the characteristic vector bundle \(VV^{J^k}Y \rightarrow V^{J^k}Y\) can be defined. Its local basis is the collection

\[(x^\lambda, y^i, y^i_\Lambda, c^i, \bar{c}^{\Lambda}, c^i_\Lambda, \bar{c}^{\Lambda}_\Lambda), \quad 0 < |\Lambda| \leq k.\]
Let us introduce the operators
\[ \partial_c = c^i \partial_i + c^i \lambda \frac{\partial}{\partial c^i} + \cdots, \quad \partial_c = c^i \partial_i + c^i \lambda \frac{\partial}{\partial c^i} + \cdots, \]
\[ d_\lambda = \partial_\lambda + y_i^\lambda \partial_i + c^i \lambda \frac{\partial}{\partial c^i} + \cdots. \]

It is easily verified that
\[ d_\lambda \partial_c = \partial_c d_\lambda, \quad d_\lambda \partial_c = \partial_c d_\lambda. \] (44)

As in the BRS mechanics [1-4], the main criterion of the SUSY extension of Lagrangian formalism is its invariance under the BRS transformation (43). The BRS-invariant extension of the vertical Lagrangian \( L_V \) (21) is the graded \( n \)-form
\[ L_S = L_V + i \partial_c \partial_c \omega \] (45)
such that \( L_J u_\omega L_S = 0 \). The corresponding Euler–Lagrange equations are defined as the kernel of the Euler–Lagrange operator
\[ \mathcal{E}_{L_S} = \left( dy^i \delta_i + dy^i \delta_i + dc^i \frac{\delta}{\delta c^i} + d\bar{c}^i \frac{\delta}{\delta \bar{c}^i} \right) L_S \wedge \omega. \]

They read
\[ \delta_i L_S = \delta_i \mathcal{L} = 0, \] (46a)
\[ \delta_i L_S = \delta_i L_V + i \partial_c \partial_c \delta_i L = 0, \] (46b)
\[ \frac{\delta}{\delta c^i} L_S = -i \partial_c \delta_i \mathcal{L} = 0, \] (46c)
\[ \frac{\delta}{\delta \bar{c}^i} L_S = i \partial_c \delta_i \mathcal{L} = 0, \] (46d)

where the relations (44) are used. The equations (46a) are the Euler–Lagrange equations for the initial Lagrangian \( L \), while (46b) - (46d) can be seen as the equations for a Jacobi field \( \delta y^i = \pi \varepsilon + \bar{\pi} \bar{\varepsilon} + i \varepsilon \varepsilon y^i \) modulo terms of order \( > 2 \) in the odd parameters \( \varepsilon \) and \( \bar{\varepsilon} \).

7 SUSY-extended Hamiltonian formalism

A momentum phase space of the SUSY-extended field theory is the complexified simple graded manifold \((V, \mathcal{A}_{V\Pi})\) whose characteristic vector bundle is \( V\Pi \rightarrow \Pi \) [12]. Its local basis is
\[ (x^\lambda, y^i, p^\lambda_i, \bar{y}\bar{i}, p^\lambda_i, c^i, \bar{c}^i, c^\lambda_i, \bar{c}^\lambda_i). \]
where \( c^i_\lambda \) and \( \overline{c}^i_\lambda \) have the same transformation laws as \( p^i_\lambda \) and \( \dot{p}^i_\lambda \), respectively. The corresponding graded vector fields and graded 1-forms are introduced on \( \mathcal{V} \) as sections of the vector bundles \( C_X \otimes \mathcal{V}_{\mathcal{V}^{\mathcal{V}} I} \) and \( C_X \otimes \mathcal{V}^{\mathcal{V}}_{\mathcal{V}^{\mathcal{V}} I} \), respectively.

In accordance with the above mentioned transformation laws of \( c^i_\lambda \) and \( \overline{c}^i_\lambda \), the BRS operator \( u_Q \) on \( \mathcal{V} \) can give rise to the complex graded vector field

\[
\tilde{u}_Q = \partial_c + i\dot{y}^i \frac{\partial}{\partial \overline{c}^i} + i\dot{p}^i_\lambda \frac{\partial}{\partial c^i_\lambda} \tag{47}
\]
on \( \mathcal{V} \). The BRS-invariant extension of the polysymplectic form \( \Omega_{\mathcal{V} \mathcal{Y}} \) on \( \mathcal{V} \) is the \( TX \)-valued graded form

\[
\Omega_S = [d\dot{p}^i_\lambda \wedge dy^i + dp^i_\lambda \wedge d\dot{y}^i + i(dc^i_\lambda \wedge dc^i - dc^i \wedge dc^i_\lambda)] \wedge \omega \wedge \partial_{\lambda},
\]

where \((\epsilon^i, -ic^i_\lambda)\) and \((\epsilon^i, ict^i_\lambda)\) are the conjugate pairs. Let \( \gamma \) be a Hamiltonian connection for a Hamiltonian form \( H \) on \( \Pi \). Its double vertical prolongation \( VV\gamma \) on \( \mathcal{V} \) is a linear morphism over the vertical connection \( V\gamma \) on \( \mathcal{V} \), and so defines the composite graded connection

\[
(VV\gamma)_S = V\gamma + dx^\mu \otimes [\overline{g}^i_\mu \frac{\partial}{\partial \overline{c}^i} + \overline{g}^i_\lambda \frac{\partial}{\partial c^i_\lambda} + g^i_\mu \frac{\partial}{\partial \epsilon^i} + g^i_\lambda \frac{\partial}{\partial \epsilon^i_\lambda}] \tag{38}
\]
on \( \mathcal{V} \), whose components \( g \) and \( \overline{g} \) are given by the expressions

\[
\overline{g}^i_\lambda = \partial_c \epsilon^i_\lambda \mathcal{H}, \quad \overline{g}^i_\lambda = -\partial_c \partial_i \mathcal{H}, \quad g^i_\lambda = \partial_c \partial_i \mathcal{H}, \quad g^i_\lambda = -\partial_c \partial_i \mathcal{H},
\]

\[
\partial_c = \epsilon^i \partial_i + \epsilon^i_\lambda \partial^i_\lambda, \quad \partial_\epsilon = \epsilon^i \partial_i + \epsilon^i_\lambda \partial^i_\lambda.
\]

This composite graded connection satisfies the relation

\[(VV\gamma)_S \Omega_S = -dH_S,
\]

and can be regarded as a Hamiltonian graded connection for the Hamiltonian graded form

\[
H_S = [\dot{p}^i_\lambda dy^i + p^i_\lambda d\dot{y}^i + i(\overline{c}^i_\lambda dc^i + dc^i \epsilon^i_\lambda)]\omega_\lambda - \mathcal{H}_S \omega, \tag{48}
\]
on \( \mathcal{V} \). It is readily observed that this graded form is BRS-invariant, i.e., \( L_{\tilde{u}_Q} H_S = 0 \).

Thus, it is the desired SUSY extension of the Hamiltonian form \( H \).

The Hamiltonian graded form \( H_S \) (48) defines the corresponding SUSY extension of the Lagrangian \( L_H \) (16) as follows. The fibration \( J^1 \mathcal{V} \to \mathcal{V} \) yields the pull-back of the
Hamiltonian graded form $H_S$ (48) onto $J^1\Pi$. Let us consider the graded generalization of the operator $h_0$ (12) such that
\[
h_0 : dc^i \mapsto c^i dx^\mu, \quad dc^\lambda_i \mapsto c^\lambda_i dx^\mu.
\]
Then the graded horizontal density
\[
L_{SH} = h_0(H_S) = (L_H)_S = L_{H_V} + i[(\tau^\lambda_i c^\lambda_i + \tau^\lambda_i c^\lambda_i) - \partial_{\lambda\mu} H]\omega = L_{H_V} + \nonumber
\]
\[
i[\tau^\lambda_i (c^\lambda_i - \partial_{\mu} \partial^\lambda_i H) + (\tau^\lambda_i - \partial_{\mu} \partial^\lambda_i H)c^\lambda_i + \tau^\lambda_i c^\mu_i \partial_{\lambda} \partial_{\mu} H - \tau^\lambda_i \partial_{\lambda} \partial_{\mu} H] \omega
\]
on $J^1\Pi \to X$ is the SUSY extension (45) of the Lagrangian $L_H$ (16). The Euler–Lagrange equations for $L_{SH}$ coincide with the Hamilton equations for $H_S$, and read
\[
y^\lambda_i = \dot{\partial}_{\lambda} \partial_{\lambda} H = \partial_{\lambda} \partial_{\lambda} H, \quad p^\lambda_i = -\partial_{\lambda} H = -\partial_{\lambda} H, \quad (50a)
\]
\[
y^\lambda_i = \partial_{\lambda} \partial_{\lambda} H = (\partial_{\lambda} + i\partial_{\lambda} \partial_{\lambda}) \partial_{\lambda} H, \quad \dot{p}^\lambda_i = -\partial_{\lambda} H = -(\partial_{\lambda} + i\partial_{\lambda} \partial_{\lambda}) \partial_{\lambda} H, \quad (50b)
\]
\[
c^\lambda_i = i\frac{\partial H}{\partial c^\lambda_i} = -\partial_{\lambda} \partial_{\lambda} H, \quad c^\lambda_i = i\frac{\partial H}{\partial c^\lambda_i} = -\partial_{\lambda} \partial_{\lambda} H, \quad (50c)
\]
\[
\tau^\lambda_i = -i\frac{\partial H}{\partial \tau^\lambda_i} = -\partial_{\lambda} \partial_{\lambda} H, \quad \tau^\lambda_i = -i\frac{\partial H}{\partial \tau^\lambda_i} = -\partial_{\lambda} \partial_{\lambda} H. \quad (50d)
\]
The equations (50a) are the Hamilton equations for the initial Hamiltonian form $H$, while (50b) – (50d) describe the Jacobi fields
\[
\delta y^i = \varepsilon c^i + \varepsilon \dot{c}^i, \quad \delta p^\lambda_i = \varepsilon c^\lambda_i + \varepsilon \dot{c}^i.
\]
Let us study the relationship between SUSY-extended Lagrangian and Hamiltonian formalisms. Given a Lagrangian $L$ on $J^1Y$, the vertical Legendre map $\tilde{L}_V$ (23) yields the corresponding morphism (41) of graded manifolds
\[
S\tilde{L}_V : (VJ^1Y, A_{VVJ^1Y}) \to (VII, A_{VVII})
\]
which is given by the relations (24) and
\[
c^\lambda_i = \partial_{\lambda} \pi^\lambda_i, \quad \tau^\lambda_i = \partial_{\lambda} \pi^\lambda_i.
\]
Let $H$ be a Hamiltonian form on $\Pi$. The vertical Hamiltonian map $\tilde{H}_V$ (29) yields the morphism of graded manifolds
\[
S\tilde{H}_V : (VII, A_{VVII}) \to (VJ^1Y, A_{VVJ^1Y})
\]
given by the relations (30) and
\[
c^\lambda_i = \partial_{\lambda} \partial^i \partial^i H, \quad \tau^\lambda_i = \partial_{\lambda} \partial^i \partial^i H.
\]
If a Hamiltonian form $H$ is associated with $L$, a direct computation shows that the Hamiltonian graded form $H_S$ (48) is weakly associated with the Lagrangian $L_S$ (45), i.e.,

\[ S\tilde{L} \circ S\tilde{H} \circ S\tilde{L} = S\tilde{L}, \]
\[ L_S \circ S\tilde{H} = (p^\lambda_i \partial^\lambda_i + \dot{p}^\lambda_i \dot{\partial}^\lambda_i + c^\lambda_i \partial c^\lambda_i + \dot{c}^\lambda_i \dot{\partial} c^\lambda_i)\mathcal{H}_S - \mathcal{H}_S, \]

where the second equality takes place at points of the Lagrangian constraint space $\tilde{L}(J^1Y)$.

### 8 The BRS-invariance

Now turn to the above mentioned case of fibre bundles $Y \to X$ with affine transition functions. Since transition functions of the holonomic coordinates $\dot{y}^i$ on $VY$ are independent of $y^i$, the transformation laws of the frames $\{\partial_i\}$ and $\{\dot{\partial}_i\}$ are the same, and so are the transformations laws of the coframes $\{c^i\}$ and $\{\dot{c}^i\}$. Then the graded vector fields (11) are globally defined on $VY$. The graded vector fields (10) and (11) constitute the above-mentioned Lie superalgebra of the supergroup $\text{ISp}(2):$

\[ [u^Q, u^Q] = [u^\bar{Q}, u^\bar{Q}] = [u^\bar{Q}, u^Q] = [u_K, u_Q] = [u_{\bar{K}}, u_{\bar{Q}}] = 0, \]
\[ [u_K, u_{\bar{Q}}] = u_{\bar{Q}}, \quad [u_{\bar{Q}}, u_Q] = u_{\bar{Q}}, \quad [u_K, u_{\bar{K}}] = u_C, \]
\[ [u_C, u_K] = 2u_K, \quad [u_C, u_{\bar{K}}] = -2u_{\bar{K}}. \quad (51) \]

Similarly to (43), let us consider the jet prolongation of the graded vector fields (11) onto $VJ^1Y$. Using the compact notation $u = u^a \partial_a$, we have the formula

\[ Ju = u + du^a \partial_a^\lambda \]

and, as a consequence, obtain

\[ Ju^Q = u^Q + c^\lambda_i \partial c^\lambda_i - i\dot{y}^i \partial^\lambda_i, \]
\[ Ju_K = u_K + c^\lambda_i \partial c^\lambda_i, \quad Ju_{\bar{Q}} = u_{\bar{Q}} + c^\lambda_i \partial c^\lambda_i, \quad (52) \]
\[ Ju_C = u_C + c^\lambda_i \partial c^\lambda_i - \dot{c}^\lambda_i \dot{\partial} c^\lambda_i. \]

It is readily observed that the SUSY-extended Lagrangian $L_S$ (45) is invariant under the transformations (52). The graded vector fields (43) and (52) make up the Lie superalgebra (51).
The graded vector fields (11) can give rise to VII by the formula

\[ \tilde{u} = u - (-1)^{[y^a]([p^a] + [u^a])} \partial_a u^b \rho^\lambda_a \frac{\partial}{\partial p^\lambda}. \]

We have

\[ \tilde{u}_Q = \partial_x - iy_i \frac{\partial}{\partial v^i} - i\rho^\lambda_i \frac{\partial}{\partial \lambda_i}, \]
\[ \tilde{u}_K = c_i \frac{\partial}{\partial c^i} + c^i \frac{\partial}{\partial \lambda^i}, \quad \tilde{u}_K = \tilde{c}_i \frac{\partial}{\partial c^i} + \tilde{c}^i \frac{\partial}{\partial \lambda^i}, \]
\[ \tilde{u}_C = c_i \frac{\partial}{\partial c^i} + c^i \frac{\partial}{\partial \lambda^i} - c_i \frac{\partial}{\partial c^i} - c^i \frac{\partial}{\partial \lambda^i}. \]

A direct computation shows that the BRS-extended Hamiltonian form \( H_S \) (48) is invariant under the transformations (53). Accordingly, the Lagrangian \( L_{SH} \) (49) is invariant under the jet prolongation \( J\tilde{u} \) of the graded vector fields (53). The graded vector fields (47) and (53) make up the Lie superalgebra (51).

With the graded vector fields (47) and (53), one can construct the corresponding graded currents \( J_u = \tilde{u} \upharpoonright H_S = u \upharpoonright H_S \). These are the graded \((n - 1)\)-forms

\[ Q = (c_i \rho^\lambda_i - y^i c^i) \omega_\lambda, \quad \overline{Q} = (\overline{c}_i \rho^\lambda_i - \overline{y}^i \overline{c}^i) \omega_\lambda, \]
\[ K = -ic_i c^i \omega_\lambda, \quad \overline{K} = ic^i \overline{c}^i \omega_\lambda, \quad C = i(c_i c^i - \overline{c}^i \overline{c}_i) \omega_\lambda \]
on VII. They form the Lie superalgebra (51) with respect to the product

\[ [J_u, J_{u'}] = J_{[u, u']}. \]

The following construction is similar to that in the SUSY and BRS mechanics. Given a function \( F \) on the Legendre bundle \( \Pi \), let us consider the operators

\[ F_\beta = e^{\beta F} \circ \tilde{u}_Q \circ e^{-\beta F} = \tilde{u}_Q - \beta \partial_x F, \quad \overline{F}_\beta = e^{-\beta F} \circ \tilde{u}_Q \circ e^{\beta F} = \tilde{u}_Q + \beta \partial_x F, \quad \beta > 0, \]
called the SUSY charges, which act on graded functions on VII. These operators are nilpotent, i.e.,

\[ F_\beta \circ F_\beta = 0, \quad \overline{F}_\beta \circ \overline{F}_\beta = 0. \]

(54)

By the BRS-invariant extension of a function \( F \) is meant the graded function

\[ F_S = -\frac{i}{\beta} (\overline{F}_\beta \circ F_\beta + F_\beta \circ \overline{F}_\beta). \]
We have the relations

\[ F_\beta \circ F_S - F_S \circ F_\beta = 0, \quad \overline{F}_\beta \circ F_S - F_S \circ \overline{F}_\beta = 0. \]

These relations together with the relations (54) provide the operators \( F_\beta, \overline{F}_\beta, \) and \( F_S \) with the structure of the Lie superalgebra \( \text{sl}(1/1) \) [20]. In particular, let \( F \) be a local function \( \mathcal{H} \) in the expression (4). Then

\[ \mathcal{H}_S = -i(F_1 \circ F_1 + F_1 \circ \overline{F}_1) \]

is exactly the local function \( H_S \) in the expression (48). The similar splitting of a super-Hamiltonian is the corner stone of the SUSY mechanics [21, 22].

References


