Motional Squashed States

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We show that by using a feedback loop it is possible to reduce the fluctuations in one quadrature of the vibrational degree of freedom of a trapped ion below the quantum limit. The stationary state is not a proper squeezed state, but rather a squashed state, since the uncertainty in the orthogonal quadrature, which is larger than the standard quantum limit, is unaffected by the feedback action.

I. INTRODUCTION

In recent years there has been an increasing interest on trapping phenomena and related cooling techniques [1]. A number of recent theoretical and experimental papers have investigated the ability to coherently control or “engineer” atomic quantum states. Experiments on trapped ions, where the zero point of motion was closely approached through laser cooling [2], already showed the effects of nonclassical motion in the absorption spectrum [2,3]. More recent experiments report the generation of Fock, squeezed and Schrödinger cat states [4,5]. These states appear to be of fundamental physical interest and possibly of use for sensitive detection of small forces [6]. Moreover, the possibility to synthesize nonclassical motional states gave rise to new models in quantum computation [7].

In this paper we present a way to reach a stationary nonclassical motional state for a trapped particle [8], which is able to give a significant uncertainty contraction in one phase-space direction. The scheme can be then applied to control the vibrational motion against the heating processes responsible for decoherence [9]. This could be important to obtain high fidelity in quantum logic operations [9].

The basic idea of the scheme is to realize an effective and continuous measurement of a vibrational quadrature for the trapped particle and then apply a feedback loop able to control, i.e. to reduce, its fluctuations even below the quantum limit.

The fact that a feedback loop may reduce the fluctuations in one quadrature of an in-loop field without increasing the fluctuations in the other has been known for a long time, and has recently been called “squashing” [10] as opposed to “squeezing” of a free field, in which the conjugate fluctuations are increased. In our scheme, the obtained stationary state results as a “squashed” state; however, the quadrature measurement increases the noise in the orthogonal quadrature well above the standard quantum limit and the squashing comes with respect to the state one has in presence of the measurement process and without the feedback action. In this way the uncertainty principle is not violated.

The paper is organized as follows. In section II we show how to realize the indirect continuous measurement of a vibrational quadrature for the trapped particle and then apply a feedback loop able to control, i.e. to reduce, its fluctuations even below the quantum limit.

II. CONTINUOUS MONITORING OF ATOMIC MOTION

We consider a generic particle trapped in an effective harmonic potential. For simplicity we shall consider the one-dimensional case, even if the method can be in principle generalized to the three-dimensional case. This particle can be an ion trapped by a rf-trap [9] or a neutral atom in an optical trap [11,12]. Our scheme however does not depend on the specific trapping method employed and therefore we shall always refer from now on to a generic trapped “atom”.

The trapped atom of mass \(m\), oscillating with frequency \(\omega_a\) along the \(\hat{x}\) direction and with position operator \(x = x_0(a + a^\dagger)\), with \(x_0 = (\hbar/2m\omega_a)^{1/2}\), is coupled to a standing wave in a cavity with frequency \(\omega_b\), wave-vector \(k\) along \(\hat{x}\) and annihilation operator \(b\). The standing wave is quasi-resonant with the transition between two internal atomic levels \(|\pm\rangle\) separated by \(\hbar\omega_0\). We consider also an external driving of the standing wave with a laser at frequency \(\omega_B\)
and of the atomic center-of-mass motion with a classical electric field along the \( \hat{x} \) direction, with frequency \( \omega_A \). The resulting Hamiltonian of the system is

\[
H = \frac{\hbar \omega_0}{2} \sigma_z + \hbar \omega_a a^\dagger a + \hbar \omega_b b^\dagger b + i \hbar \epsilon (\sigma_+ + \sigma_-)(b - b^\dagger) \sin (kx + \phi) \\
- qE_{0x}(a + a^\dagger) \sin (\omega_A t + \theta) + i \hbar \left( B e^{-i\omega_A t} b^\dagger - B^* e^{i\omega_A t} b \right),
\]

where \( \sigma_z = |+\rangle \langle +| - | -\rangle \langle -| \), \( \sigma_x = \frac{1}{2} \), and \( \epsilon \) is the coupling constant.

In the interaction representation with respect to \( H_0 = \hbar \omega_B (b^\dagger b + \frac{\omega_B^2}{2}) \), and making the rotating wave approximation, that is, neglecting terms rapidly oscillating at the driving laser frequency \( \omega_B \), this Hamiltonian becomes

\[
H = \hbar \Delta \sigma_z + \hbar \omega_a a^\dagger a + \hbar (\omega_b - \omega_B) b^\dagger b + i \hbar \epsilon (\sigma_+ - \sigma_- b^\dagger) \sin (kx + \phi) \\
- qE_{0x}(a + a^\dagger) \sin (\omega_A t + \theta) + i \hbar \left( B b^\dagger - B^* b \right),
\]

where \( \Delta = \omega_0 - \omega_B \) is the atomic detuning. This detuning can be set to be much larger than all the other parameters \( \Delta \gg \epsilon, \omega_b - \omega_B \), and in this case, the excited level can be adiabatically eliminated, so to get the following effective Hamiltonian for the vibrational motion of the atom and the standing wave mode alone \( [13] \)

\[
H = \hbar \left( \omega_b - \omega_B - G/2 \right) b^\dagger b + \hbar \omega_a a^\dagger a - \hbar \frac{\Delta^2}{2} b^\dagger b \sin^2 (kx + \phi) \\
- qE_{0x}(a + a^\dagger) \sin (\omega_A t + \theta) + i \hbar \left( B b^\dagger - B^* b \right),
\]

If we set the spatial phase \( \phi = 0 \), and assume the Lamb-Dicke regime, one can approximate \( \sin^2 (kx + \phi) \approx k^2 x^2 \) in Eq. (3). Then, in the interaction representation with respect to \( \hbar \omega_b a^\dagger a \) and making the rotating wave approximation, i.e., neglecting all the terms oscillating at \( \omega_A \) (which is of the order of 1 MHz) or faster because we are interested in the dynamics at much larger times, we finally get

\[
H = \hbar \left( \omega_b - \omega_B - G/2 \right) b^\dagger b + \hbar \omega_a a^\dagger a - \hbar \omega_B b^\dagger b a + i \hbar \left( A a^\dagger - A^* a \right) \sin (\omega_A t + \theta) + i \hbar \left( B b^\dagger - B^* b \right),
\]

where \( G = 2(kx_0)^2/\Delta \) and \( A = -qE_{0x}^2/2\hbar \). This Hamiltonian gives rise to a crossed Kerr-like effect which could be exploited to generate nonclassical states analogously to the all-optical case proposed in Ref. [14]. A similar approach was used for Schrödinger cat motional states of atoms in cavity QED [15]. Here, instead, we are looking for stationary nonclassical states. The evolution of the density matrix \( D \) of the whole system (vibrational degree of freedom plus the cavity mode) is determined by the Hamiltonian (4) and by the terms describing the photon leakage out of the cavity with decay rate \( \kappa \) and the coupling of the vibrational motion with the thermal environment \( L_{th}D \), that is

\[
\dot{D} = L_{th}D - \frac{i}{\hbar} [H, D] + \frac{\kappa}{2} \left( 2b D b^\dagger - b^\dagger b D - D b^\dagger b \right).
\]

For the determination of the damping term of the vibrational motion \( L_{th}D \), we note that it occurs at a frequency \( \omega_0 \) of the order of MHz and that the corresponding damping rate \( \gamma \) is usually much smaller [5]. It seems therefore reasonable to use the rotating wave approximation in the interaction between the atom center-of-mass and its reservoir, leading us to describe the damping of the vibrational degree of freedom in terms of the quantum optical master equation (at nonzero temperature) [20],

\[
L_{th} \rho = \frac{\gamma}{2} (n + 1) \left( 2a^\dagger a^2 \rho - a^2 \rho a^\dagger - a^\dagger a^2 \rho - a^2 \rho a^\dagger \right),
\]

where \( n = \left[ \exp (\hbar \nu/k_BT) - 1 \right]^{-1} \), is the number of thermal phonons (\( k_B \) is the Boltzmann constant and \( T \) the equilibrium temperature). An analogous treatment is considered in [9]. We have to remark, however, that the damping and heating mechanisms of a trapped atom are not yet well understood [9] and that different kinds of ion-reservoir interaction have been proposed [21].

The quantum Langevin equations [20] corresponding to the master equation (5) reads

\[
\dot{b} = -i(\omega_b - \omega_B - G/2)b + iG a^\dagger b - \frac{\kappa}{2} b + B + \sqrt{\kappa} \eta_{in}(t),
\]

\[
\dot{a} = -i(\omega_a - \omega_A)a + iG b^\dagger a \sin (\omega_A t + \theta) + \frac{\gamma}{2} a + \sqrt{\gamma} \eta_{in}(t),
\]

\[
\text{(7, 8)}
\]
where the input quantum noises $b_n(t)$ and $a_n(t)$ have zero mean and the following correlation functions

$$\langle b_n(t)b_n(t') \rangle = \langle b_n^\dagger(t)b_n(t') \rangle = 0$$

(9)

$$\langle b_n(t)b_n^\dagger(t') \rangle = \delta(t-t')$$

(10)

$$\langle a_n(t)a_n(t') \rangle = 0$$

(11)

$$\langle a_n(t)a_n^\dagger(t') \rangle = \delta(t-t')$$

(12)

$$\langle a_n(t)a_n^\dagger(t') \rangle = (n+1)\delta(t-t')$$

(13)

When the external driving terms described by $A$ and $B$ are sufficiently large, the stationary state of the system is quasi-classical, that is, the standing wave is approximately in a coherent state with a large amplitude $\beta \gg 1$, and the atomic vibrational motion along $\hat{k}_x$ is approximately in a coherent state with a large amplitude $\alpha \gg 1$. The values of $\alpha$ and $\beta$ are given by the solutions of the coupled nonlinear equations given by the semiclassical version of the quantum Langevin equations Eqs. (7) and (8):

$$0 = -i(\omega_b - \omega_B - G/2 - G|\alpha|^2) - \frac{\kappa}{2}\beta + B,$$

(14)

$$0 = -i(\omega_a - \omega_A - G|\beta|^2) - \frac{\kappa}{2}\alpha + A.$$  

(15)

Since $\alpha \simeq 2A/\gamma$ and $\beta \simeq 2B/\kappa$, the semiclassical condition for the steady state is satisfied when $A \gg \gamma$ and $B \gg \kappa$.

The fluctuations around this steady state are instead described by quantum mechanics and their dynamics can be obtained by appropriately shifting both modes, i.e., $b \rightarrow b + \beta$ and $a \rightarrow a + \alpha$. In the semiclassical limit $|\alpha|, |\beta| \gg 1$ it is reasonable to linearize the equations, and since it is always possible to tune the two driving frequencies $\omega_A$ and $\omega_B$ so to have zero detunings, i.e., $|\alpha|^2 = (\omega_a - \omega_B)/G - 1/2$, $|\beta|^2 = (\omega_b - \omega_B)/G$, the linearized quantum Langevin equations for the quantum fluctuations around the steady state can be written as

$$\dot{b} = iG\beta(\alpha^* a + aa^*) - \frac{\kappa}{2}b + \sqrt{\kappa}b_n(t),$$

(16)

$$\dot{a} = iG\alpha(\beta^* b + bb^*) - \frac{\gamma}{2}a + \sqrt{\gamma}a_n(t).$$

(17)

The effective linearized Hamiltonian leading to Eqs.(16), (17), can be written as

$$H = \hbar\chi Y,$$

(18)

where $\chi = -4G|\alpha||\beta|$, $Y = (be^{-i\phi_b} + b^\dagger e^{i\phi_b})/2$ is the standing wave field quadrature with phase $\phi_b$ equal to the phase of the classical amplitude $\beta$, and $X = (ae^{-i\phi_a} + a^\dagger e^{i\phi_a})/2$ is the vibrational quadrature with phase $\phi_a$ given by the phase of the classical amplitude $\alpha$. For the sake of simplicity we shall consider $\phi_a = \phi_b = 0$, i.e., the atomic position quadrature, from now on, even if the following considerations can be easily extended to the case of generic phases. Note that in order to remain in the Lamb-Dicke regime it is required that $kx_0/|\alpha| \ll 1$; however the linearisation is justified only when $|\alpha| \gg 1$, and therefore we need $kx_0 \ll \frac{1}{|\alpha|} \ll 1$.

Eq. (18) implies that an effective continuous, quantum non-demolition (QND) measurement of the phonon quadrature $X$ is provided by the homodyne measurement of the light going out from the cavity, which plays the role of the “meter”. In fact, the homodyne photocurrent is [16]

$$I(t) = 2\eta\kappa(\langle Y(t)\rangle)_c + \sqrt{\eta\kappa}\xi(t),$$

(19)

where $Y_c = (be^{-i\varphi} + b^\dagger e^{i\varphi})/2$ is the measured quadrature, the phase $\varphi$ is related to the local oscillator, and $\eta$ is the detection efficiency. The subscript $c$ in Eq. (19) denotes the fact that the average is performed on the state conditioned on the results of the previous measurements and $\xi(t)$ is a Gaussian white noise [16]. In fact, the continuous monitoring of the field mode performed through the homodyne measurement, modifies the time evolution of the whole system. The state conditioned on the result of measurement, described by a stochastic conditioned density matrix $\hat{D}_c$, evolves according to the following stochastic differential equation (considered in the Itô sense)

$$\dot{\hat{D}}_c = \mathcal{L}_{th}\hat{D}_c \frac{i}{\hbar}[\hat{H}, \hat{D}_c] + \kappa \left(2\hat{b}\hat{D}_c\hat{b}^\dagger - \hat{b}\hat{b}\hat{D}_c - \hat{D}_c\hat{b}\hat{b}^\dagger\right)$$

$$+ \sqrt{\eta\kappa}\xi(t)\left(e^{-i\varphi}\hat{b}\hat{D}_c + e^{i\varphi}\hat{D}_c\hat{b}^\dagger - 2\langle Y_c \rangle_\xi\hat{D}_c\right).$$

(20)

We note that by performing the average over the white noise $\xi(t)$, one gets the master equation of Eq. (5).
It is now reasonable to assume that the standing wave mode is highly damped, i.e. $\kappa \gg \chi$ (this does not conflict with the preceding assumptions, since the coupling constant $\chi = -8\epsilon^2(kx_0)^2[a\beta]/\Delta$ is usually smaller than the cavity decay rate). This means that the radiation field will almost always be in its lower state $|0\rangle_b$ (displaced by an amount $\beta$). This allows us to adiabatically eliminate the field and to perform a perturbative calculation in the small parameter $\chi/\kappa$, obtaining (see also Ref. [17]) the following expansion for the total conditioned density matrix $D_c$

$$D_c = \left(\rho_c - \frac{\chi^2}{\kappa^2} X \rho_c X\right) \otimes |0\rangle_b \langle 0| - \frac{\chi}{\kappa} (X \rho_c \otimes |1\rangle_b \langle 0| - \rho_c X \otimes |0\rangle_b \langle 1|)
+ \frac{\chi^2}{\kappa^2} X \rho_c X \otimes |1\rangle_b \langle 1| - \frac{\chi^2}{\kappa^2 \sqrt{2}} (X^2 \rho_c \otimes |2\rangle_b \langle 0| + \rho_c X^2 \otimes |0\rangle_b \langle 2|),$$

where $\rho = \text{Tr}_b D$ is the reduced density matrix for the vibrational motion. In the adiabatic regime, the internal dynamics instantaneously follows the vibrational one and therefore one gets information on $X$ by observing the quantity $Y_\varphi$. The relationship between the conditioned mean values follows from Eq. (21)

$$\langle Y_\varphi(t) \rangle_c = \frac{\chi}{\kappa} \langle X(t) \rangle_c \sin \varphi.$$

Moreover, if we adopt the perturbative expansion (21) for $D_c$ in (20) and perform the trace over the internal mode, we get an equation for the reduced density matrix $\rho_c$ conditioned to the result of the measurement of the observable $\langle Y_\varphi(t) \rangle_c$, and therefore $\langle X(t) \rangle_c$

$$\dot{\rho}_c = \mathcal{L}_{th} \rho_c - \frac{\chi^2}{2\kappa} [X, [X, \rho_c]] + \sqrt{\eta \chi^2/\kappa} \xi(t) \left(i e^{i\varphi} \rho_c X - i e^{-i\varphi} X \rho_c + 2 \sin \varphi \langle X(t) \rangle_c \rho_c\right).$$

This equation describes the stochastic evolution of the vibrational state of the trapped atom conditioned to the result of the continuous homodyne measurement of the light field. The double commutator with $X$ is typical of QND measurements.

### III. THE FEEDBACK LOOP

We are now able to use the continuous record of the atom phonon quadrature to control its motion through the application of a feedback loop. We shall use the continuous feedback theory proposed by Wiseman and Milburn [18].

One has to take part of the stochastic output homodyne photocurrent $I(t)$, obtained from the continuous monitoring of the meter mode, and feed it back to the vibrational dynamics (for example as a driving term) in order to modify the evolution of the mode $a$. To be more specific, the presence of feedback modifies the evolution of the conditioned state $\rho_c(t)$. It is reasonable to assume that the feedback effect can be described by an additional term in the master equation, linear in the photocurrent $I(t)$, i.e. [18]

$$[\dot{\rho}_c]_{fb} = \frac{I(t-t)}{\eta X} \mathcal{K} \rho_c(t),$$

where $\tau$ is the time delay in the feedback loop and $\mathcal{K}$ is a Liouvillian superoperator describing the way in which the feedback signal acts on the system of interest.

The feedback term (24) has to be considered in the Stratonovich sense, since Eq. (24) is introduced as limit of a real process, then it should be transformed in the Ito sense and added to the evolution equation (23). A successive average over the white noise $\xi(t)$ yields the master equation for the reduced density matrix $\rho = \text{Tr}_b D$ in the presence of feedback. In the general case of a nonzero feedback delay time, one gets a non-Markovian master equation which is very difficult to solve [18] (see however Ref. [19]). Most often however, the feedback delay time is much shorter than the characteristic time of the $a$ mode, which in the present case is given by the energy relaxation time $\gamma^{-1}$, and in this case the dynamics in the presence of feedback can be described by a Markovian master equation [18], which is given by

$$\dot{\rho} = \mathcal{L}_{th} \rho - \frac{\chi^2}{2\kappa} [X, [X, \rho]] + \mathcal{K} \left(i e^{i\varphi} \rho X - i e^{-i\varphi} X \rho\right) + \frac{\mathcal{K}^2}{2\eta X^2/\kappa} \rho.$$

The third term is the feedback term itself and the fourth term is a diffusion-like term, which is an unavoidable consequence of the noise introduced by the feedback itself.
Then, since the Liouville superoperator $\mathcal{K}$ can only be of Hamiltonian form \[18\], we choose it as $\mathcal{K}\rho = g \left[ a - a^\dagger, \rho \right]/2$ \[17\], which means feeding back the measured homodyne photocurrent to the vibrational oscillator with a driving term in the Hamiltonian involving the quadrature orthogonal to the measured one; $g$ is the feedback gain related to the practical way of realizing the loop. One could have chosen to feed the system with a generic phase-dependent quadrature, due to the homodyne current, however, it will turn out that the above choice gives the best and simplest result. Since the measured quadrature of the vibrational mode is its position, the feedback will act as a driving for the momentum. Using the above expressions in Eq. (25) and rearranging the terms in an appropriate way, we finally get the following master equation:

$$
\dot{\rho} = \frac{\Gamma}{2}(N + 1) \left( 2a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right) + \frac{\Gamma}{2} N \left( 2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger \right)
- \frac{\Gamma}{2} \left( 2a^\dagger \rho a^\dagger - a^\dagger a^\dagger \rho - \rho a^\dagger a^\dagger \right) - \frac{\Gamma}{2} M^* \left( 2a \rho a - a^2 \rho - \rho a^2 \right)
- \frac{g}{4} \sin \varphi \left[ a^2 - a^2 \right],
$$

(26)

where

$$
\Gamma = \gamma - g \sin \varphi;
$$

(27)

$$
N = \frac{1}{\Gamma} \left[ \gamma n + \frac{\lambda^2}{4\kappa} + \frac{g^2}{4\eta} + \frac{g}{2} \sin \varphi \right];
$$

(28)

$$
M = -\frac{1}{\Gamma} \left[ \frac{\lambda^2}{4\kappa} - \frac{g^2}{4\eta} - \frac{g}{2} \cos \varphi \right].
$$

(29)

Eq. (26) is very instructive because it clearly shows the effects of the feedback loop on the vibrational mode $a$. The proposed feedback mechanism, indeed, not only introduces a parametric driving term proportional to $g \sin \varphi$, but it also simulates the presence of a squeezed bath, characterized by an effective damping constant $\Gamma$ and by the coefficients $M$ and $N$, which are given in terms of the feedback parameters \[17\]. An interesting aspect of the effective bath described by the first four terms in the right hand side of (26) is that it is characterized by phase-sensitive fluctuations, depending upon the experimentally adjustable phase $\varphi$.

### IV. THE STATIONARY SOLUTION

Because of its linearity, the solution of Eq. (26) can be easily obtained by using the normally ordered characteristic function \[20\] $C(\lambda, \lambda^*, t)$. The partial differential equation corresponding to Eq. (26) is

$$
\left\{ \partial_t + \frac{\Gamma}{2} \lambda \partial_\lambda + \frac{\Gamma}{2} \lambda^* \partial_{\lambda^*} + \frac{g}{2} \sin \varphi (\lambda \partial_\lambda + \lambda^* \partial_{\lambda^*}) \right\} C(\lambda, \lambda^*, t)
= \left\{ -\Gamma N |\lambda|^2 + \left( \frac{\Gamma}{2} M + \frac{g}{4} \sin \varphi \right) (\lambda^*)^2 + \left( \frac{\Gamma}{2} M^* + \frac{g}{4} \sin \varphi \right) \lambda^2 \right\} C(\lambda, \lambda^*, t),
$$

(30)

The stationary state is reached only if the parameters satisfy the stability condition, i.e. $g \sin \varphi < \gamma$. In this case the stationary solution has the following form

$$
C(\lambda, \lambda^*, \infty) = \exp \left[ -\zeta |\lambda|^2 + \frac{1}{2} \mu (\lambda^*)^2 + \frac{1}{2} \mu^* \lambda^2 \right],
$$

(31)

where

$$
\zeta = \frac{\Gamma^2 + 2 g \sin \varphi (\Gamma \text{Re} \{M\} + 2 \nu \text{Im} \{M\}) + g^2 \sin^2 \varphi / 2}{\Gamma^2 - g^2 \sin^2 \varphi};
$$

(32)

$$
\mu = \frac{\Gamma (N + 1/2) g \sin \varphi + \Gamma \text{Re} \{M\}}{\Gamma^2 - g^2 \sin^2 \varphi} + i \frac{\Gamma (1/2 - g \sin \varphi) \text{Im} \{M\} \text{Re} \{e^{i\varphi}\}}{\Gamma^2 - g^2 \sin^2 \varphi}.
$$

(33)

Under the stability conditions and in the long time limit ($t \to \infty$) the variance of the generic quadrature operator $X_\theta = (a e^{i\theta} + a^\dagger e^{-i\theta})/2$ becomes

$$
\langle X_\theta^2 \rangle = \frac{1}{2} \left[ 1 + \zeta + \text{Re} \{\mu e^{2i\theta}\} \right].
$$

(34)
V. DISCUSSIONS AND CONCLUSIONS

For the $X$ quadrature Eq.(34) can be simply written as

$$\langle X^2 \rangle = \frac{1}{2} \left[ \frac{1}{2} + n_{\text{eff}} \right],$$

with $n_{\text{eff}} = \zeta + \text{Re}\{\mu\}$. In absence of feedback ($g = 0$) we have $n_{\text{eff}} \equiv n$, otherwise $n_{\text{eff}}$ can be smaller than $n$, providing a stochastic localisation in the $X$ quadrature. Depending on the external parameters, it can also be negative (but it is always $n_{\text{eff}} \geq -1/2$) accounting for the possibility of going beyond the standard quantum limit. This is a relevant result of the present feedback scheme since it is able to reduce not only the thermal fluctuations but even the quantum ones.

The potentiality of this feedback mechanism is clearly shown in Fig.1, where $n_{\text{eff}}$ goes well below zero for increasing values of $\chi$. Instead in Fig.2 we have sketched the phase space uncertainty contours obtained by cutting the Wigner function corresponding to Eq.(31) at $1/\sqrt{e}$ times its maximum height. We see that the state resulting from the feedback action (solid line) has a relevant contraction in the $X_{\theta=0}$ direction, but the same uncertainty in the $X_{\theta=\pi/2}$ direction with respect to the state of the system undergoing measurement without feedback (dashed line). We refer to this type of noise reduction produced by feedback loop as squashing, whereas squeezing refers to conventional quantum noise reduction [10].

Summarizing, we have proposed a feedback scheme based on an indirect (QND) measurement which is able not only to contain the heating of the vibrational motion of a trapped ion, but also to produce nonclassical motional states (squashed ones). Up to this point we have not discussed the specific way in which a particular feedback Hamiltonian could be implemented. In our case, it is important to be able to realize a term in the feedback Hamiltonian proportional to the quadrature orthogonal to $X$. This is not straightforward, but could be realized by using the feedback current to vary an external potential applied to the atom without altering the trapping potential [22]. In principle the model could be extended to the three dimensional case, one should only consider three orthogonal standing waves far from resonant transitions.

In conclusion, although the experimental implementation of the presented model may not be easy, it is certainly a promising experimental challenge, stimulated by the possibility of producing nonclassical states for trapped atoms and of controlling their heating to minimize decoherence effects, especially in quantum information processing [23].

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FIG. 1. The quantity $n_{eff}$ is plotted vs $g$ for different values of $\chi$; the values of the other parameters are $n = 0.5$, $\gamma = 10^{-2}$ s$^{-1}$, $\kappa = 10^2$ s$^{-1}$ and $\eta = 0.8$. The quantities $\chi$ and $g$ are expressed in s$^{-1}$. 


FIG. 2. The phase space uncertainty contours are represented for $g = 0$ (dashed line) and $g = 0.025 \text{ s}^{-1}$ (solid line), the values of the other parameters are $n = 0.5$, $\chi = 2.5 \text{ s}^{-1}$, $\gamma = 10^{-2} \text{ s}^{-1}$, $\kappa = 10^{2} \text{ s}^{-1}$ and $\eta = 0.8$. The dotted line represents the vacuum noise.