Nonperturbative dispersive sector in strong (quasi-)Abelian fields

G. Cvetič∗

Asia Pacific Center for Theoretical Physics, Seoul 130-012, Korea

Ji-Young Yu†

Department of Physics, Dortmund University, 44221 Dortmund, Germany

Abstract

In strong (quasi-)Abelian fields, quantum fluctuations of fermions induce an effective Lagrangian density whose imaginary (absorptive) part is purely nonperturbative and known to be responsible for the fermion–antifermion pair creation. On the other hand, the induced real (dispersive) part has perturbative and nonperturbative contributions. We argue how to separate the two contributions from each other for any strength of the field. We show numerically that the nonperturbative contributions are in general comparable with or larger than the induced perturbative ones. We arrive at qualitatively similar conclusions also for the induced energy density. Further, we investigate numerically the quasianalytic continuation of the perturbative results into the nonperturbative sector, by employing (modified) Borel–Padé. It turns out that in the case at hand, we have to integrate over renormalon singularities, but there is no renormalon ambiguity involved.

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I. INTRODUCTION

It has been well known for some time that the effects of the fermionic quantum fluctuations in space–time uniform Abelian gauge fields can be effectively integrated out, resulting in a one–loop effective action [1]–[7]. The results have been formulated also for the covariantly homogeneous, thus quasi–Abelian, fields of the SU(2) gauge group [8], and for specific nonhomogeneous magnetic field configurations [9]. All the results can be extended to the case of the quantum fluctuations of scalar particles. The problems arising when genuinely non–Abelian fields with translationally invariant gauge–invariants are present [e.g., in SU(3)c] were discussed, e.g., in Refs. [10,11].

∗e-mail: cvetic@apctp.kaist.ac.kr
†e-mail: yu@dilbert.physik.uni-dortmund.de
The quantum fluctuations of the strong gauge field itself (photons, or gluons) modify additionally those Lagrangian densities induced by the fermionic quantum fluctuations. Such two-loop effects have been successfully derived by Ritus [12] for homogeneous Abelian fields, and further discussed by others [13,14]. In QED, such two–loop effects in the coupling constant change the one–loop result by at most a few per cent ($\pi\alpha$). We will neglect them in our numerical investigation.

There are basically two classes of phenomena associated with the presence of intense gauge fields.

Firstly, they can produce pairs of particles. For the Abelian case of strong homogeneous electric field, Sauter [15] showed this by investigating solutions of the Dirac equation in the corresponding potential, and Schwinger [3] by using methods of action integral, Green’s functions and proper time. Differential probabilities for pair creation were investigated in Refs. [17] and [18]. In the latter Reference, the quasi–Abelian model was applied to investigation of the quark pair production in chromoelectric flux tubes. Experimental evidence related with the pair production in a strong QED (laser) field was reported in Ref. [19].

The pair production has its origin in the imaginary (absorptive) part of the effective Lagrangian induced by the fermionic quantum fluctuations in the strong field. That part is entirely nonperturbative in nature, because the production effects are $\sim \exp(-\text{const.}/g)$ and thus cannot be expanded in positive powers of the field–to–fermion coupling parameter $g$.

On the other hand, the other class of phenomena is associated with the real (dispersive) part of the induced effective Lagrangian. In QED, this class includes the following phenomena that affect a low energy ($\omega \ll m_e$) photon wave entering the region of the strong background field: photon splitting, change of the photon speed, and birefringence. Works on the theoretical aspects of these phenomena include Refs. [20]– [24]. The experimental aspects of birefringence in strong magnetic fields are discussed in [25]– [26]. The dispersive part of the induced action leads in principle to those corrections of the classical Maxwell equations which originate from the (fermionic) quantum fluctuations.

The aim of the present paper, while dealing with the dispersive part of the induced action, is somewhat different from these works. We concentrate on the concept of separating the nonperturbative from the perturbative contributions in the induced dispersive action when the product of the (quasi)electric field $E$ and the coupling constant $g$ is large: $gE/m^2 \gtrsim 1$, where $m$ is the fermion mass. Subsequently, we numerically investigate the two contributions. Afterwards, we use the discussed quantities as a “laboratory” for testing and investigating the efficiency of methods of quasianalytic continuation. The latter methods, involving the (modified) Borel-Padé approximants, allow us to obtain approximately the nonperturbative contributions from the approximate knowledge of the perturbative contributions alone. These considerations can give us insights into the problems of extraction of nonperturbative physics from the knowledge of perturbative physics in gauge theories, in particular in various versions of QCD (high–flavor, low–flavor).

In Section II, we argue how to perform the mentioned separation into the perturbative and nonperturbative contributions. After identifying the two contributions, we investigate

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1A related problem was first considered even earlier by Klein [16] who investigated solutions of the Dirac equation with a high vertical barrier potential (Klein’s paradox).
numerically their values for various values of the field parameter $\tilde{a} \sim g\mathcal{E}/m^2$. In Section III we then carry out an analogous analysis for the induced energy density, the latter being in principle observable. In Section IV we then numerically investigate, for the induced Lagrangian and energy densities, (quasi)analytic continuation from the perturbative into the nonperturbative sectors, employing the method of Borel–Padé for the induced Lagrangian and a modified Borel–Padé for the induced energy density. We encounter integration over renormalon poles, whose origin is nonperturbative, and we show how to carry it out. Section V summarizes our results and conclusions.

II. INDUCED DISPERSIVE LAGRANGIAN DENSITY

We start by considering the Euler–Heisenberg Lagrangian density [1] which is the part induced by the quantum fluctuations of the fermions in an arbitrarily strong (quasi–)Abelian homogeneous field

$$\delta L = \frac{1}{8\pi^2} \int_{-i\epsilon}^{i\epsilon} ds \frac{d}{ds} \exp[-is(m^2 + i\epsilon)] \left[ g^2 ab \coth(ags) \cot(bgs) - \frac{g^2}{3}(a^2 - b^2) - 1/s^2 \right]. \quad (1)$$

Here, $g$ is the field–to–fermion coupling parameter (in electromagnetism it is the positron charge $e_0$), $m$ is the mass of the (lightest) fermion, and the parameters $a$ and $b$ are Lorentz–invariant expressions characterizing the (quasi)electric and the (quasi)magnetic fields $\mathcal{E}$ and $\mathcal{B}$, respectively

$$a = \left[ +\mathcal{E}^2 - \mathcal{B}^2 + \sqrt{\left(\mathcal{E}^2 - \mathcal{B}^2\right)^2 + 4(\mathcal{E} \cdot \mathcal{B})^2} \right]^{1/2} / \sqrt{2}, \quad (2)$$

$$b = \left[ -\mathcal{E}^2 + \mathcal{B}^2 + \sqrt{\left(\mathcal{E}^2 - \mathcal{B}^2\right)^2 + 4(\mathcal{E} \cdot \mathcal{B})^2} \right]^{1/2} / \sqrt{2}. \quad (3)$$

We note that $ab = |\mathcal{E}\mathcal{B}|$, and $a^2 - b^2 = \mathcal{E}^2 - \mathcal{B}^2$. Further, $a \to |\mathcal{E}|$ when $|\mathcal{B}| \to 0$, and $b \to |\mathcal{B}|$ when $|\mathcal{E}| \to 0$. In the Lorentz frame where $\mathcal{B} \parallel \mathcal{E}$, we simply have: $a = |\mathcal{E}|$ and $b = |\mathcal{B}|$. Expression (1) can be obtained, for example, directly by integrating out the fermionic degrees of freedom in the path integral expression of the full effective action, and employing the proper–time integral representation for the difference of logarithms.

Performing a contour integration in the fourth quadrant of the complex proper–time $s$–plane (cf. Fig. 1), expression (1) can be rewritten as

$$\delta L = -\frac{1}{8\pi^2} \int_{+\epsilon}^{\infty} \frac{dz}{(z + i\epsilon)} e^{-zm^2} \left[ g^2 ab \cot(ags + i\epsilon) \coth(bgs + i\epsilon) + \frac{g^2}{3}(a^2 - b^2) - \frac{1}{(z + i\epsilon)^2} \right], \quad (4)$$

where $s = -iz + \epsilon$ now runs along the negative imaginary axis. In (1) and (4), the familiar [1] counterterm $\propto (a^2 - b^2) \left[ (\mathcal{E}^2 - \mathcal{B}^2) \right]$ is included which makes the integral finite. This divergent term leads to the renormalization of the field in the leading Lagrangian density $\mathcal{L}^{(0)} = (\mathcal{E}^2 - \mathcal{B}^2)/2$. We now divide the integration region into intervals for the integration variable $agz$: $i_0 = [0, \pi/2]$, $i_1 = [\pi/2, 3\pi/2]$, ..., $i_n = [(n-1)/2\pi, (n+1)/2\pi]$, ... Each interval,
except $i_0$, contains in its middle one pole of the integrand. The corresponding series for the real (dispersive) part of the Lagrangian density is

$$\text{Re} \delta \mathcal{L} = \text{Re} \delta \mathcal{L}_0 + \sum_{n=1}^{\infty} \text{Re} \delta \mathcal{L}_n ,$$

(5)

$$\text{Re} \delta \mathcal{L}_0 = -\int_0^{\pi/2} dw \exp \left( -\frac{w}{a} \right) \left[ p \cot(w) \coth(pw) + \frac{1}{3}(1-p^2) - \frac{1}{w^2} \right] ,$$

(6)

$$\text{Re} \delta \mathcal{L}_n = -\exp \left( -\frac{n\pi}{a} \right) \left\{ \int_{-\pi/2}^{\pi/2} dw \exp \left( -\frac{w}{a} \right) \left[ p \cot(w) \coth(p(w+n\pi)) - \frac{p \coth(pn\pi)}{w n\pi} \right] \right. \left. + \mathcal{P} \int_{-\pi/2}^{\pi/2} dw \exp \left( -\frac{w}{a} \right) \left( \frac{p \coth(pn\pi)}{n\pi} \right) \right\} , \quad (n \geq 1) .$$

(7)

Here, we used the notation

$$\tilde{a} \equiv \frac{ga}{m^2} , \quad \tilde{b} \equiv \frac{gb}{m^2} , \quad \frac{b}{a} \equiv \frac{\tilde{b}}{\tilde{a}} , \quad \delta \mathcal{L} \equiv \delta \mathcal{L}/\left( \frac{m^4 a^2}{8\pi^2} \right) ,$$

(8)

and we introduced the dimensionless integration variable $w \equiv agz$ when $agz$ is in the interval $i_0$, and $w \equiv agz+n\pi$ when $agz$ is in the interval $i_n \ (n \geq 1)$. In (7), we separated the integrand into a part that is entirely nonsingular in the integration region, and a part that is singular but gives a finite value of integration since the Cauchy principal ($\mathcal{P}$) value has to be taken. From a formal point of view, we note that $\delta \mathcal{L}_0$ doesn’t “feel” the poles of the integrand as depicted in Fig. 1, while $\delta \mathcal{L}_n \ (n \geq 1)$ “feels” the pole $s = -in\pi/(ag)$ via the principal value part in (7) that is proportional to

$$\mathcal{P} \int_{-\pi/2}^{\pi/2} dw \exp \left( -\frac{w}{a} \right) \frac{1}{(w+i\epsilon)} \left[ \frac{p \coth(pn\pi)}{n\pi} \right] = -e^x x^{-1} \left. \left[ \frac{1}{x^3} + \frac{1}{x^5} + \cdots \right] \right|_{x=\pi/(2\tilde{a})} .$$

(9)

We note that the dispersive part of the induced Lagrangian density as normalized here (5)–(8) depends only on two dimensionless parameters – on parameter $p \equiv b/a$ which characterizes in a Lorentz–invariant manner the ratio of the strengths of the (quasi)electric and magnetic fields [cf. (2)–(3)], and on parameter $\tilde{a} \equiv (ga)/m^2$ which characterizes the combined strengths of the (quasi)electric field parameter $a$ and the field–to–fermion coupling $g$. In the perturbative weak–field limit, $\tilde{a}$ is small. In this case, when reintroducing in (6)

$$z \equiv w/(ag)$$

we can see that the real part of expression (4) is approximately reproduced, since formally $\pi/(2ag) \to \infty$. In this case, the conventional perturbative expansion of the dispersive Lagrangian density in powers of $g^2$ (i.e., inverse powers of $x$) can be performed (cf. [1], [3])
\[
\delta \hat{L}^{\text{pert.}} = \left( c_1 \tilde{a}^2 + c_3 3 \tilde{a}^4 + c_5 5 \tilde{a}^6 + \cdots \right),
\]
where the expansion coefficients are
\[
c_1 = \frac{1}{45} \left[ (1-p^2)^2 + 7 p^2 \right], \quad c_3 = \frac{1}{945} \left[ 2(1-p^2)^3 + 13p^2(1-p^2) \right],
\]
\[
c_5 = \frac{1}{14175} \left[ 3(1-p^2)^4 + 22p^2(1-p^2)^2 + 19p^4 \right], \quad \text{etc.}
\]
Expression (11) can be derived alternatively by purely perturbative methods – the terms \(\sim \tilde{a}^{2n}\) can be obtained by calculating the one–fermion–loop Feynman diagram with 2\(n\) photon external legs of zero momenta. Expression (11) is a divergent asymptotic series and it gives the usual perturbative corrections to the Maxwell equations. When taking a few first terms in (11), the relative difference between (11) and \(\text{Re} \delta \hat{L}_0\) of (6) [or equivalently: (10)] disappears when \(\tilde{a} \to 0\). Formally, the difference between (11) and (6) is \(\sim \exp(-\text{const.}/\tilde{a})\), which vanishes faster than any positive power of \(\tilde{a}\) when \(\tilde{a} \to 0\). Its origin is the finite, though large, cutoff for the Euclidean proper time \(z\) in (10). Since that cutoff is well below the first pole of the integrand \([z_1 = \pi/(ag)]\), we see that the difference is \(\textit{not}\) of nonperturbative origin, but rather due to an infrared proper–time cutoff \(z \leq 1/\Lambda^2_{IR}\) \((\Lambda^2_{IR} \sim m^2 \tilde{a})\).\(^2\) These cutoff effects, manifested via the upper integration bound \(w \equiv \tilde{a} zm^2 = \pi/2\) in (6), thus only serve to regularize the otherwise badly behaved perturbative expansion (11) when we approach the nonperturbative case, i.e., when \(\tilde{a}\) increases. We can thus conclude from these considerations that \(\text{Re} \delta \hat{L}_0\) (6), which is always finite and well distanced from the poles of the integrand, is a good representative of the perturbative contributions to the induced dispersive part of the Lagrangian density, for arbitrary value \(\tilde{a}\).

On the other hand, the densities \(\text{Re} \delta \hat{L}_n\) \((n \geq 1)\) of (7) represent the nonperturbative part of the induced dispersive part of the density. This is so because the integration over the proper–time \(z\) runs here in the vicinity (in fact, across) the \(n\)’th pole of the integrand of (4). The poles of the integrand are in the nonperturbative regions. We recall that these poles are also the source of the nonzero imaginary (absorptive) part of the density leading to the fermion-antifermion pair creation, a clearly nonperturbative phenomenon. From a more formal perspective, we can check the following: Each of the two integrals in the curly brackets of (7) behaves as \(\sim \tilde{a} \exp[\pi/(2\tilde{a})]\) when \(\tilde{a} \to +0\), and thus the entire \(\text{Re} \delta \hat{L}_n\) behaves as \(\sim \tilde{a} \exp[-(n-1/2)\pi/\tilde{a}]\) \((n \geq 1)\) in this limit.\(^3\) However, this behavior alone, without the pole structure, would not suffice to argue that these contributions are nonperturbative.

From a somewhat different perspective, we can imagine transforming a truncated perturbative expansion for \(\text{Re} \delta \hat{L}^{\text{pert.}}/\tilde{a}\) of (11) (with several terms) via the Borel–Padé approx-

\(^2\)We note that the energy cutoff \(\Lambda_{IR} \sim m \sqrt{\tilde{a}}\) becomes low in the case where the perturbative effects dominate (i.e., at \(\tilde{a} < 1\)), but has to be higher in the case when the nonperturbative effects are significant (at \(\tilde{a} > 1\)). This agrees with the usual intuition – the nonperturbative effects reside in the infrared (IR) sector of (fermionic) momenta \(q < \Lambda_{IR}\), and the effective contributing size of this sector gets larger when \(\tilde{a}\) grows.

\(^3\)If we did not take the principal Cauchy value in (7), but some other prescription (which in the case at hand would be wrong), \(\text{Re} \delta \hat{L}_n\) would behave as \(\sim \exp[-n\pi/\tilde{a}]\).
imation. The resulting integrand approximately reproduces the integrand of (4), including the poles structure. Thus the integration over the $n$'th pole, contained in $\text{Re}\delta\tilde{L}_n$ of (7), can be interpreted as the $n$'th renormalon in the density, i.e., a nonperturbative quantity. We will return to this point later in this paper.

Thus, the densities (6) and (7) result in the fermion–induced perturbative and nonperturbative contributions, respectively, to the Maxwell equations. The fields were taken, strictly speaking, to be homogeneous in space and time. In practical terms, this means that they are not allowed to change significantly on the distance and time scales of the Compton wavelength of the fermion $1/m$. For electro–magnetic fields, $m$ is the electron mass, and $1/m$ is about $4 \cdot 10^{-13}$ m, and $1.3 \cdot 10^{-21}$ s.

An indication of the relative size of the perturbative and nonperturbative fermion–induced corrections to the Maxwell equations can be obtained by comparing the corresponding contributions to the induced Lagrangian density. This is done in Figs. 2-3. Figures 2 (a), (b) show the dimensionless perturbative (6) and nonperturbative (7) induced Lagrangian densities, respectively, as functions of the (quasi)electric field parameter $\tilde{a}$ (8), at four different fixed values $p \equiv \tilde{b}/\tilde{a}$ of the magnetic–to–electric field ratio. The case of the pure (quasi)magnetic field (p.m.f.) is also included in the Figures, as function of $\tilde{b}$. For the p.m.f. case, we normalized the Lagrangian density in analogy with (8), i.e., $\delta\tilde{L}$ is obtained in that case by dividing $\delta\mathcal{L}$ by $m^4\tilde{a}^2/(8\pi^2)$. The separation between the perturbative and the nonperturbative part was performed in the p.m.f. case analogously, i.e., the proper–time $z < \pi/(2bg)$ contributions were defined to be perturbative, and those from $z > \pi/(2bg)$ nonperturbative. We point out, however, that such a separation in the p.m.f. case may not be well motivated since the integrand does not have any poles. In Fig. 3, the corresponding ratios of the nonperturbative and perturbative induced densities are presented. When moving beyond the perturbative region (i.e., when $\tilde{a} \gg 1$), we see from these Figures that the nonperturbative parts in general become relatively significant and often even dominant.

Once we come into the nonperturbative region ($\tilde{a} \approx 1$), however, we must keep in mind that the pair creation, originating from the large absorptive part, will become so strong as to render the solutions of the corrected Maxwell equations unstable. We will quantify somewhat this fact in the next Section in the case of the induced energy density in QED.

In Figs. 2 (a), (b), the densities were normalized according to (8), so that the tree–level reference values for the densities are

$$\tilde{\mathcal{L}}^{(0)} = \mathcal{L}^{(0)} / \left( \frac{m^4\tilde{a}^2}{8\pi^2} \right) = \frac{4\pi^2}{g^2} (1 - p^2) .$$

Therefore, increasing only the coupling parameter $g$, while leaving the (quasi)electric field $a$ unchanged, results in correspondingly larger relative corrections originating from the induced parts, both nonperturbative and perturbative. In the special case of QED, on the other hand, $g = e_0$ is small $[\alpha = e_0^2/(4\pi) \approx 1/137]$, and the overall induced Lagrangian density accounts usually for less than 0.5 per mille of the total Lagrangian density when $\tilde{a} \leq 1$.

4The total induced dispersive Lagrangian densities and values of the truncated perturbative series (11) (including $\sim \tilde{a}^8$) are included in Figs. 6 in Section IV.
This indicates that strong fields in QED, unless they are really huge ($\tilde{a} \sim 10^2$), change the Maxwell equations insignificantly (see the next Section on related points).

### III. INDUCED ENERGY DENSITY

In this Section, we discuss the induced energy densities. Energy density is in principle a measurable quantity. It is not Lorentz–invariant. If the (quasi)electric and (quasi)magnetic fields are mutually parallel, the various induced energy densities can be obtained directly from the corresponding induced Lagrangian densities

$$\delta \mathcal{U}_k = a \frac{\partial \text{Re} \delta \mathcal{L}_k}{\partial a} \bigg| _b - \text{Re} \delta \mathcal{L}_k$$

$$\Rightarrow \delta \tilde{\mathcal{U}}_k = \tilde{a} \frac{\partial \text{Re} \tilde{\mathcal{L}}_k}{\partial \tilde{a}} \bigg| _b + \text{Re} \tilde{\mathcal{L}}_k \quad (k = 0, 1, 2, \ldots) , \quad (14)$$

where we denoted, in analogy with (8)

$$\delta \tilde{\mathcal{U}}_{(k)} \equiv \delta \mathcal{U}_{(k)} / \left( \frac{m^4 \tilde{a}^2}{8 \pi^2} \right) . \quad (15)$$

With the restriction to parallel fields $\vec{E} \parallel \vec{B}$ (i.e., $|\vec{E}| = a$ and $|\vec{B}| = b$) we do not lose the generality since, for any configuration of $\vec{E}$ and $\vec{B}$, there always exists a Lorentz boost, perpendicular to the plane of the fields, so that in the boosted frame the two fields are parallel. The corresponding perturbative and nonperturbative parts of the energy densities in such frames are

$$\text{Re} \delta \mathcal{U}_0 = - \int_0^{\pi/2} \frac{dw}{w} \exp \left( -\frac{w}{a} \right) \left\{ \left( \frac{w}{a} - 1 \right) \left[ p \cot(w) \coth(pw) + \frac{1}{3} \left( 1 - p^2 \right) - \frac{1}{w^2} \right] \right.$$

$$+ \left[ p \cot(w) \coth(pw) + \frac{p^2 w \cot(w)}{\sinh^2(pw)} + \frac{2}{3} - \frac{2}{w^2} \right] \right\} , \quad (16)$$

$$\text{Re} \delta \mathcal{U}_n = - \exp \left( -\frac{n \pi}{a} \right) \int_{-\pi/2}^{\pi/2} \frac{dw}{w} \exp \left( -\frac{w}{a} \right) \left[ \left( \frac{w + n \pi}{a} \right) - 1 \right] \left[ p \cot(w) \coth(p(w + n \pi)) \right. \frac{(w + n \pi)}{w} \]

$$- \frac{p}{w} \left[ \frac{\coth(pn \pi)}{n \pi} + \frac{p}{\sinh^2(pn \pi)} \right] + \frac{2}{3} \left( \frac{1}{w + n \pi} \right) - \frac{2}{(w + n \pi)^2} \right]$$

$$- \frac{p}{a} \left[ \frac{\coth(pn \pi)}{n \pi} + \frac{p}{\sinh^2(pn \pi)} \right] \left[ -E_1 \left( \frac{\pi}{2 \tilde{a}} \right) - \text{Ei} \left( \frac{\pi}{2 \tilde{a}} \right) \right]$$

$$+ \frac{2p}{n \pi} \coth(pn \pi) \sinh \left( \frac{\pi}{2 \tilde{a}} \right) \right\} , \quad (n \geq 1) . \quad (17)$$

The tree–level density in the normalization convention used [cf. (8)] is
\begin{equation}
\dot{U}^{(0)} \equiv U^{(0)} / \left( \frac{m^4 \tilde{a}^2}{8\pi^2} \right) = \frac{4\pi^2}{g^2} (1 + p^2). \tag{18}
\end{equation}

The perturbative power expansion of the induced energy density $\delta \tilde{U}$ is

$$
\delta \hat{U}^{\text{pert}} = \left( d_1 \tilde{a}^2 + d_3 3! \tilde{a}^4 + d_5 5! \tilde{a}^6 + \cdots \right), \tag{19}
$$

where the expansion coefficients are

$$
d_1 = \frac{1}{45} \left[ 3 + 5p^2 - p^4 \right], \quad d_3 = \frac{1}{945} \left[ 10 + 21p^2 - 7p^4 + 2p^6 \right],
$$

$$
d_5 = \frac{1}{14175} \left[ 21 + 50p^2 - 21p^4 + 10p^6 - 3p^8 \right], \text{ etc.} \quad \text{(20)}
$$

The results for the induced perturbative (16) and nonperturbative parts (17), and their ratios, are presented in Figs. 4 (a)–(b) and 5, respectively, in analogy with Figs. 2 (a)–(b) and 3. The case of the pure (quasi)magnetic field is not included in Figs. 4–5, because in this case $\delta \tilde{U} = -\delta \tilde{L}$ and thus the relevant information is already contained in Figs. 2–3. The behavior of the induced energy densities is, in broad qualitative terms, similar to that of the induced Lagrangian densities.

In the special case of QED, similarly as for the Lagrangian densities in the previous Section, the total induced energy densities account for a very small part of the total energy density (0.2–0.3 per mille when $\tilde{a} \approx 1$) and can become significant only when the field becomes exceedingly large ($\tilde{a} \gtrsim 10^2$). The same is true also for the dielectric permeability tensor $\varepsilon_{ij}$: In the direction of the fields, we have $\delta \varepsilon_{\parallel} \equiv \varepsilon_{\parallel} - 1 = a \partial (\text{Real} \tilde{L}) / \partial a$, i.e., by (14)–(15) we have $\delta \varepsilon_{\parallel} = (\delta \tilde{U} + \text{Real} \tilde{L}) \alpha / (2\pi)$, which is about $10^{-3}$ for $\tilde{a} \approx 1$ and $p = 1$. Therefore, the effective coupling parameter along the field direction $\alpha_{\parallel} = \alpha / \varepsilon_{\parallel}$ changes by about one per mille, while $\alpha_{\perp} = \alpha / \varepsilon_{\perp}$ remains unchanged since $\varepsilon_{\perp} = 1$. Therefore, in QED, any quantity which can be expanded in powers of the coupling parameter alone (without fields) remains a perturbative quantity. QED then remains a perturbative theory despite such strong fields – cf. also Ref. [27] on that point.

The energy density is not stable in time when $\tilde{a} \neq 0$, due to the energy losses to pair creation of fermions of mass $m$. It decreases by about 50 percent in the time $t_{1/2}$

$$
t_{1/2} \approx \frac{\pi^2}{8\alpha} \exp \left( + \frac{\pi}{\tilde{a}} \right) \left[ \frac{(1 + p^2)}{\pi \coth(p\pi)} \right] \frac{1}{m}, \tag{21}
$$

where $\alpha \equiv g^2 / (4\pi)$. The factor in the square brackets, appearing due to the presence of the (quasi)magnetic field, is usually not essential in the estimates since it is $\sim 1$ for $p \leq 5$. In the case of QED and with $p = 0$, $t_{1/2}$ is about $0.9 \cdot 10^5 m_e^{-1}$, $0.4 \cdot 10^4 m_e^{-1}$ and $0.3 \cdot 10^8 m_e^{-1}$ for $\tilde{a} = 0.5$, 1, and 5, respectively. Here, $m_e^{-1} \approx 1.3 \cdot 10^{-21}$ s is the electron Compton time.

The total induced energy densities and values of the truncated perturbative series (19) that include terms $\sim \tilde{a}^8$ are included in Figs. 7 in Section IV.
IV. QUASIANALYTIC CONTINUATION INTO THE NONPERTURBATIVE SECTOR

In this Section, we use the discussed induced densities as an example on which to test and get some insights into methods of approximate analytic (i.e., quasianalytic) continuation. In various physical contexts, such methods allow one to extract all or part of the information on the nonperturbative sector from the knowledge of the perturbative sector alone. We will use the method of Borel–Padé transformation, or a modification thereof.

One may ask whether the perturbative expansions (11) and (19) allow us to obtain the full, including the nonperturbative, information about the corresponding densities. The answer for the Lagrangian density is yes, but under the condition that we take in the corresponding Borel–Padé approximants the Cauchy principal values when integrating over the positive poles of the Padé integrand in the inverse Borel transformation. This rule is connected with the causality of the fermionic propagators in the (quasi–)Abelian field, reflected in the terms $i\epsilon$ in the denominators of the integrands of (4) and/or (7). More specifically, we first Borel–transform (B) the perturbative series (11)

$$B \left[ \frac{\delta \tilde{L}_{\text{pert}}(\tilde{a};p)}{\tilde{a}} \right] = c_1(p)\tilde{a} + c_3(p)\tilde{a}^3 + c_5(p)\tilde{a}^5 + \cdots ,$$ (22)

then construct an $[N/M]_B(\tilde{a};p)$ Padé approximant to B of (22), and then apply the inverse Borel transformation

$$BP^{[N/M]} \left[ \delta \tilde{L}_{\text{pert}} \right](\tilde{a};p) = \int_0^\infty dw \exp \left( -\frac{w}{\tilde{a}} \right) [N/M]_B(w;p).$$ (23)

On the other hand, the real part of the actual density (4) can also be written as a Borel–type integral, when introducing $w \equiv agz$ in (4) and normalizing the density according to (8)

$$\text{Re} \delta \tilde{L} = \text{Re} \int_0^\infty dw \exp \left( -\frac{w}{\tilde{a}} \right) \left[ \frac{p \cos(w)}{\sin(w+i\epsilon)} \coth(pw) + \frac{1}{3}(1-p^2) - \frac{1}{w^2} \right] ,$$ (24)

The expansion of the integrand of (24), excluding the exponential, in powers of $w$ is identical with the Borel transform (22) with $\tilde{a} \mapsto w$, as it should be. Comparing (23) with the exact result (24), we see that the Borel–Padé method (23) will be efficient in (quasi)analytic continuation if Padé approximants $[N/M]_B(w)$ approach the integrand of (24) in an increasingly wide integration interval of $w$ when the Padé order indices $N$ and $M$ ($\approx N$) increase. This in fact happens, since the integrand in (24) is a meromorphic function in the complex plane whose poles structure on the positive axis is especially simple – there are only single (not multiple) poles, located at $w = \pi, 2\pi, 3\pi$. Padé approximants to power expansions of such functions are known to approximate such functions increasingly better when the Padé order

$^[N/M](\tilde{a})$ Padé approximant to (22) is defined by two properties: 1. it is a ratio of two polynomials in $\tilde{a}$, the nominator polynomial having the highest power $\tilde{a}^N$ and the denominator $\tilde{a}^M$; 2. when expanded in powers of $\tilde{a}$, it reproduces the coefficients at the terms $\tilde{a}^n$ in (22) for $n \leq N+M$; it is based solely on the knowledge of these latter coefficients $c_n$ ($n \leq N+M$).
indices $N \approx M$ increase [28]. Furthermore, the causality of the fermionic propagator in the strong (quasi–)Abelian field is reflected in the terms $i\epsilon'$ in the denominators of the integrand of (24). Thus, near the poles $w \approx n\pi$ the integrand behaves as $\sim (w - n\pi + i\epsilon')^{-1}$. Hence, for obtaining the real (dispersive) part of the density, the Borel integration over the poles has to be taken with the Cauchy principal value prescription – not just in the exact expression (24), but also in the approximate expression (23). Thus, the Borel integration in (23) over the $n$'th pole, i.e., the $n$'th renormalon contribution, has in the case at hand no renormalon ambiguity. As the Padé order indices $N \approx M$ are increased, we thus systematically approach the exact $\text{Re} \delta \tilde{\mathcal{L}}$ via the Cauchy principal part values of (23). This means that in the case at hand [strong (quasi–)Abelian fields with fermionic fluctuations included], the full induced Lagrangian density can be obtained uniquely solely on the basis of the knowledge of perturbative expansion (11) for weak fields. The more terms in (11) [and thus in (22)] we know, the higher Padé order indices $N \approx M$ we can have, and hence the closer to the full Lagrangian density we can come via (23).

On the basis of the knowledge of the first four nonzero perturbative terms in (11) and correspondingly in (22), we can construct the following Padé approximants of the perturbative Borel transform (22): $[1/2]_B, [1/4]_B, [3/4]_B$. Then we can calculate the corresponding Borel-Padé transforms via the Cauchy principal value prescription in (23). The corresponding results of the approximants for the full induced density $\text{Re} \delta \tilde{\mathcal{L}}$ are presented in Fig. 6, together with the exact numerical values calculated by (24) in Section II. The curves are given as functions of the (quasi)electric strength parameter $\tilde{a}$ for four fixed values of the electric–to–magnetic field ratio $p$, and Fig. 6 (d) is for the case of the pure (quasi)magnetic field ($\tilde{a} = 0$). We see that the highest order ([3/4]) Padé–Borel results agree well with the exact results over the entire depicted region of $\tilde{a}$. When the Padé order indices $N$ and $M$ ($\sim N$) increase, the region of agreement includes increasingly large values of $\tilde{a}$. For comparison, we also included the results of the truncated perturbative series (TPS) made up of the first four nonzero terms of (11) [in Fig. (d): for the corresponding p.m.f. case], i.e., those perturbative terms which the presented Borel–Padé transforms are based on.

If we apply the very same procedure in the case of the energy density – Borel–transforming the series $\delta \tilde{U}^{\text{pert.}}/\tilde{a}$ of (19), constructing Padé approximants, and carrying out the inverse Borel transformation by using the Cauchy principal value prescription – the results are disappointing. It turns out that increasing the Padé order indices $N$ and $M$ ($\sim N$) does not generally result in a better precision. For example, for $p \approx 0.5$ and $\tilde{a} \approx 0.5$, the Borel–Padé transforms of the order $[3/4]$ and $[3/6]$ give significantly worse results than those of the lower order $[1/4]$. The reason for this erratic behavior of the Borel–Padé approximants in this case lies in the more complicated poles structure of the Borel–Padé transforms. This can be seen if we rewrite $\delta \tilde{U}$ in the Borel–integral form analogous to (24), obtained from (24) by applying relation (14)

$$\text{Re} \delta \tilde{U} = \text{Re} \int_0^\infty dw \exp \left( -\frac{w}{\tilde{a}} \right) \frac{(-1)}{w} \left[ -\frac{pw}{\sin^2(w + i\epsilon')} \coth(pw) + \frac{1}{3} (1 + p^2) + \frac{1}{w^2} \right]. \quad (25)$$

The expansion in powers of $w$ of the integrand in (25), excluding the exponential, gives of course the exact Borel transform of the perturbative series (19) divided by $\tilde{a}$ (and replacing $\tilde{a} \mapsto w$).
\[- \frac{1}{w} \left( - \frac{pw}{\sin^2(w + ie')} \coth(pw) + \frac{1}{3}(1 - p^2) + \frac{1}{w^2} \right) \]

\[= d_1(p)w + d_3(p)w^3 + d_5(p)w^5 + \cdots \equiv B \left[ \frac{\delta U^\text{pert.}(w; p)}{w} \right]. \tag{26} \]

However, we now see that this integrand has a double poles structure on the positive \( w \) axis, the double poles located at \( w = \pi, 2\pi, 3\pi, \ldots \). The Padé approximants to the power series (26) have great trouble simulating this double poles structure adequately. When they do it by creating one single or two nearby real poles, say near \( w = \pi \), then it turns out that the inverse Borel transformation via the Cauchy principal value prescription often gives good results. However, when the Padé approximants try to simulate the double pole near \( w = \pi \) by creating two mutually complex–conjugate poles \( a \pm ib \) \((a \approx \pi, |b| \ll \pi)\), the inverse Borel transformation gives very unsatisfactory results. This occurs, for example, in Padé approximants \([3/4](w; p)\) and \([3/6](w; p)\) for \( p \leq 0.5 \). Heuristically we can understand that such a simulation is bad, because the structure of the integrand in (25) suggests that a double pole at \( a - ib \) alone, just below the real axis, would do a better job, but it is not allowed in the Padé approximants. The latter is true because the perturbative expansion (26) is explicitly real for real \( w \)’s, and this property is hence shared also by the Padé approximants, enforcing for each complex pole another pole which is complex–conjugate.

To overcome this problem, the idea is to modify the Borel transformation of the perturbative series (19) in such a way that the resulting transformed series is represented by a (meromorphic) function without any double poles on the real positive axis, in contrast to the Borel–transformed series (26). This, in fact, can be implemented in the easiest way by using the following modification of the Borel transformation (MB):

\[
\frac{\partial MB}{\partial w} \left[ \frac{\delta U^\text{pert.}}{w} \right](w; p) = \frac{\delta U^\text{pert.}(w; p)}{w} = d_1(p)w + d_3w^3 + d_5w^5 + \cdots
\]

\[
\Rightarrow MB \left[ \frac{\delta U^\text{pert.}}{w} \right](w; p) = d_1(p)\frac{w^2}{2} + d_3\frac{w^4}{4} + d_5\frac{w^6}{6} + \cdots. \tag{27} \]

This trick changes every double pole in the B into the corresponding single pole in the MB. Then we apply Padé approximants \([N/M]_{MB}(w)\) to the MB series (27), and carry out the corresponding inverse modified Borel transformation

\[
MBP^{[N/M]} \left[ \frac{\delta U^\text{pert.}}{w} \right] (\tilde{a}; p) = \frac{1}{\tilde{a}} \int_0^\infty dw \exp \left( -\frac{w}{\tilde{a}} \right) \left[ N/M \right]_{MB}(w; p) \tag{28} \]

with the Cauchy principal value (CPV) prescription when integrating over the single poles. This CPV prescription originates again from the \( e' \) terms in the double poles structure \((w - n\pi + ie')^{-2}\) of the Borel–transform (B) integrand of (25) that is now changed to the single poles structure \((w - n\pi + ie')^{-1}\) in the modified Borel–transform (MB) of (27) whose Padé approximants \([N/M]_{MB}(w; p)\) appear in (28).

The numerics clearly confirm that these MBP’s (28) are well behaved, i.e., they approximate well the actual full induced energy density \( \delta U(\tilde{a}; p) \) in the region of \( \tilde{a} \) which is getting wider when the Padé order indices \( N \) and \( M \) \((\approx N)\) increase. The results are presented in Fig. 7, where the MBP’s for the first three possible Padé order indices \([2/2], [2/4] \) and \([4/4], \)
along with the exact numerical results, are shown as functions of $\tilde{a}$, at four fixed values of $p \equiv b/\tilde{a}$. Another reason why the results now behave better than those of the usual BP transforms lies in the fact that the Padé approximants ([2/2], [2/4] and [4/4]) are now more diagonal than earlier ([1/2], [1/4] and [3/4]). This is due to one additional power of $w$ in the MB series (27), as compared with the usual B series. The diagonal and near-diagonal Padé approximants are known to behave better than the (far) off-diagonal ones [28]. In fact, Figs. 7 suggest that clear improvement – extension of the $\tilde{a}$ range of agreement with the exact results – sets in when we switch from [2/2] to [4/4] MBP, while the off-diagonal [2/4] MBP may even be slightly worse than [2/2]. For comparison, we also included the results of the truncated perturbative series (TPS) made up of the first four nonzero terms (up to $\sim \tilde{a}^8$) of (19), i.e., the terms on which the presented Borel–Padé transforms are based. The case of the pure (quasi)magnetic field was not included in these Figures because in this case because then $\delta U = -\delta \tilde{L}$ and thus the information on this case is contained in Fig. 6 (d).

This application of Borel–Padé transformations and their modification may give us some insights into how the (quasi)analytic continuation from the perturbative (small $\tilde{a}$) into the nonperturbative (large $\tilde{a}$) regions can be carried out in other theories whose exact behavior in the latter region is still theoretically unknown. One such example is the perturbative QCD (pQCD), where some observables are known at the next-to-next-to-leading order (N$^2$LO). The coupling parameter in that case [$\tilde{a} \mapsto \alpha_s(Q^2)$] can be quite large when the relevant energies of the process are low ($Q \sim 1$ GeV), thus rendering the direct evaluation of the N$^2$LO TPS unreasonable or at best unreliable. When applying Borel–Padé transformations or modifications thereof to such series, we are faced with two major problems:

- The first problem is of a more technical nature. Since only very few, at most two, coefficients beyond the leading order are known, the Padé approximants associated with the (modified) Borel transform of the series have low order indices ($N, M \leq 2$) and thus do not necessarily reproduce the location of the leading poles on the positive axis, if they exist, adequately.

- The second problem is of a deeper theoretical nature. Since we do not know the exact nonperturbative contributions to the observable represented by the available TPS, we do not know how to integrate over the possible positive poles in the inverse (modified) Borel transformation – this can be termed the infrared renormalon ambiguity [29].

In the discussed case of integrated fermionic fluctuations in strong (quasi–)Abelian fields – for the Lagrangian and energy densities – we do not face any of the two afore–mentioned problems since the “exact” solution is known. It is the causal structure of the fermion propagators that eliminates the renormalon ambiguity in this case – we have to apply the Cauchy principal value prescription in the integration of the Borel–Padé transform of the induced dispersive Lagrangian density, and in the modified Borel–Padé transform of the induced energy density. The knowledge of the full theoretical solution in the latter case also tells us that the poles structure of the usual Borel transform of the induced energy density is more complicated (double poles), so that we have to apply a modified Borel-Padé transform which changes the double poles into a single poles structure.

We point out that the positive poles – renormalons – discussed in the present work cannot be directly identified with the usual infrared (ultraviolet) renormalons in QCD (QED). The latter renormalons, as defined in the literature [29], are interpreted in the perturbative
language as originating from renormalon chains at low (high) momenta \( k \). The renormalon chains are momentum–\( k \) gluon (photon) propagators with \( n \) chained one–loop insertions, where \( n \) can be arbitrarily large. In the model at hand, however, only quantum fluctuations of ferms, in the slowly–varying strong fields, are considered; the effects of the quantum fluctuations of propagating gluons (photons) were not included in the discussed effective model. The positive poles, i.e. renormalons, in the present model originate from a collective effect of arbitrarily many very soft gluons (photons) coupling to a fermion loop or to a fermion propagator – cf. [30]. The relevant parameter of the effective coupling of these soft gauge bosons to the fermions, appearing in the induced effective action, is \( \tilde{a} = ga/m^2 \) and it can be large due to the strong field \( a \) and/or due to the strong coupling \( g \). These nonperturbative contributions are then roughly \( \sim \exp(-\text{const.}/\tilde{a}) = \exp[-\text{const}.m^2/(ga)] \) – cf. (23), (28). This is similar, but not identical, to the infrared renormalon contributions in QCD \( \sim \exp(-\text{const}'.g^2) \). We may be tempted to term the renormalons discussed in the present paper as infrared renormalons due to their nonperturbative origin, although this name is reserved for the afore–mentioned QCD–type renormalons.

V. CONCLUSIONS

We introduced the concept of separation of the induced dispersive action into the nonperturbative and perturbative parts. We then investigated numerically the nonperturbative contributions to the dispersive (real) part of the Lagrangian density and to the real energy density, induced by quantum fluctuations of fermions in the strong (quasi–)Abelian fields that don’t change significantly in space–time over the typical fermionic Compton wave-lengths \( 1/m \). There are only nonperturbative contributions in the absorptive (imaginary) part of the strong field Lagrangian density, the latter part being responsible for the fermion–antifermion pair creation. On the other hand, the nonperturbative contributions in the real (dispersive) sector are in general also significant and can often even dominate over the perturbative induced contributions there. The induced dispersive Lagrangian density modifies the Maxwell equations for strong fields. The induced energy density is in principle an observable quantity. When the (quasi)electric fields are strong, however, these densities decay fast (in \( \sim 10^3 \) Compton times, for \( \tilde{a} \sim 1 \)). These two induced densities lead to a change in the dielectric permeability tensor of the vacuum. In the special case of QED, all these induced effects are below one per cent unless the fields are huge (\( \tilde{a} \sim 10^2 \)).

We then used the discussed induced quantities as a “laboratory” to test and investigate the efficiency of specific methods of quasianalytic continuation from the perturbative region (weak fields) into the nonperturbative region (strong fields). We employed the method of Borel–Padé for the induced dispersive Lagrangian density, since the function represented by the Borel transform series has only simple poles. For the induced energy density, we had to employ a modified Borel–Padé transformation since the function represented by the (nonmodified) Borel transform series has double poles. We found out numerically that such quasianalytic continuations become precise over an increasing region of the effective expansion parameter \( \tilde{a} \) when the number of available terms in the perturbative expansion increases. This means that the quasianalytic continuation gradually becomes the analytic (exact) continuation when the number of the perturbative expansion terms accounted for increases. Further, in the case at hand, the Borel integration over positive poles (renor-
malons) is necessary, and it turns out that it involves no (renormalon) ambiguity – due to the causal structure of the fermionic propagators in strong gauge fields. Such analyses could give us some insight into the problems faced in QCD when nonperturbative contributions to observables are investigated either on the basis of the perturbative results themselves or by using other models [31] that are at least partly motivated by perturbative methods.

Our analysis shows that the analytic continuation in the discussed case of strong background gauge fields has a unique solution (no ambiguity). This is in agreement with the conclusions of Ref. [32] which were obtained from quite different considerations involving the renormalization group – that the vacuum polarization induced by the intense gauge fields is in principle determined by the information on the behavior of the theory in the perturbative region. The situation in QCD is not so clear. The uniqueness of the analytic continuation from the perturbative into the nonperturbative region in QCD is possible if a nontrivial infrared stable fixed point for the running strong coupling parameter exists. Such an infrared stable fixed point, however, seems to exist only if the number of the quark flavors is high $\left( N_f > 9 \right)$ [33]. For the real (low–$N_f$) QCD, a phase transition takes place, and methods of analytic continuation have probably only a limited range of applicability. Stated differently, in this case the full knowledge of the perturbative sector probably does not allow us to obtain information on the deep nonperturbative sector. The renormalon ambiguity in the low-flavor perturbative QCD (pQCD) is then probably an intrinsic ambiguity that cannot be eliminated within the pQCD framework alone.

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FIG. 1. The contour integration, in the complex $s$–plane, needed to rewrite (1) in the form (4). The location of the poles is denoted explicitly.
FIG. 2. (a) Perturbative and (b) nonperturbative induced dispersive (Euler–Heisenberg) Lagrangian densities [cf. (6) and (7)] as functions of the (quasi)electric field parameter $\tilde{a}$ (8), at various fixed values of the electric–to–magnetic field ratio $p = \tilde{b}/\tilde{a}$ (8). The actual values of the curves for $p \approx 0$, $p = 1.0$ and $p = 5.0$ have been multiplied here by factors 10, 5 and 1/2, respectively, for better visibility. Included is also the case of the pure (quasi)magnetic field (p.m.f.), for which the x-axis represents $\tilde{b} = gb/m^2$.

FIG. 3. Ratios of the nonperturbative and the perturbative induced dispersive Lagrangian densities for the cases depicted in Figs. 2. For the p.m.f. case, the x-axis represents $\tilde{b} = gb/m^2$. 
FIG. 4. (a) Perturbative and (b) nonperturbative induced energy densities [cf. (16) and (17)] as functions of $\tilde{a}$ at various fixed values of $p = \tilde{b}/\tilde{a}$ (8). The actual values of the curves have been multiplied, for better visibility, by the denoted factors, just as in Figs. 2.

FIG. 5. Ratios of the nonperturbative and the perturbative induced energy densities for the cases depicted in Figs. 4. The ratio for $p = 1$ varies strongly for $\tilde{a} = 0.5$–1.5 because the perturbative induced density has a zero at $\tilde{a} \approx 1.1$. 
FIG. 6. Borel–Padé approximants (BP’s) to the induced dispersive Lagrangian density (8) as functions of $\tilde{a}$, for various values of $p = \tilde{b}/\tilde{a}$: (a) $p = 0$; (b) $p = 0.5$; (c) $p = 5.0$; (d) pure magnetic field ($\tilde{a} = 0$). Depicted are those BP’s (23) which are based on the Padé approximants $[1/2]$, $[1/4]$ and $[3/4]$ of the Borel–transform (22). The numerically exact curves [sum of curves of Figs. 2 (a) and (b)] are also included and they virtually agree with the $[3/4]$ BP results. Included are also the the results of the truncated perturbative series which include terms $\sim \tilde{a}^8$ [in (d): $\sim \tilde{b}^8$].
FIG. 7. Modified Borel–Padé approximants [MBP’s – cf. (28)] to the induced energy densities, based on the Padé approximants $[2/2]$, $[2/4]$ and $[4/4]$ for the MBP’s (27), as functions of $\tilde{a}$, at fixed values of $p=b/\tilde{a}$: (a) $p=0$; (b) $p=0.5$; (c) $p=1.0$; (d) $p=5.0$.