Thermal Renormalons in Scalar Field Theory

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Abstract
In the frame of the scalar theory $\lambda\phi^4$, we explore the occurrence of thermal renormalons, i.e. temperature dependent singularities in the Borel plane. Using Thermofield Dynamics, we found in fact a series of singularities of this kind, which are located to the right of the well known zero temperature pole, being therefore of a subleading character in the ambiguity of the Borel sum.
1 Introduction

During the last years, there have been an impressive amount of theoretical work on renormalons in different scenarios. For a recent review see [1]. One of the main motivations behind this effort concerns the non perturbative structure of Quantum Chromodynamics (QCD). On the other side, finite temperature effects have also called the attention of many authors [2], due to their crucial role in understanding new thermal aspects of the hadron dynamics, with special emphasis on the deconfining phase transition and the production of the quark gluon plasma [3]. It seems natural, therefore, to start a systematic study of the occurrence and also the possible phenomenological role of thermal renormalons. Here we discuss, as a first step in this direction, the scalar theory $\lambda \phi^4$. In spite of being a non realistic approach, this analysis will allow us to gain a first impression on this kind of effects.

In this paper we explore in detail the extension to the finite temperature scenario of one kind of relevant Feynman diagrams, see Fig. 1, that contribute to the two-point function. These diagrams are related to the existence of a pole in the Borel plane in the usual zero temperature situation. For that purpose we will make use of the machinery of Thermo Field Dynamics (TFD) [4], finding a whole series of new temperature dependent singularities, which are, however, of a subleading type with respect to the zero temperature singularity. The situation in QCD, where infrared renormalons are present, will be probably different, due to the long side thermal correlations which could be significant in this case.

The plan of this paper is as follows. In section 2 we briefly review how a zero temperature renormalon in $\lambda \phi^4$ theory appears, as a consequence of a particular kind of Feynman diagrams contributing to the two-point function. Section 3 is devoted to a brief discussion of how to handle the chain of bubbles shown in Fig. 2 [5]. This will be done at finite temperature in the deep euclidean region, i. e. $-p^2 \gg m^2$, and will be use in section 4 where we calculate at finite temperature our renormalon-type diagram (Fig. 1), finding, finally, the location and residues of the new thermal renormalons.
2 The zero temperature renormalon

In the present paper the diagram shown in Fig. 1 will be taken as a source for renormalons. Let us call the order \( k \) diagram by \( R_k(p) \) where \( k \) denotes the number of vertices. In order to fix our notation, first we will review the usual zero temperature calculation for the renormalon associated to this diagram [6].

![Figure 1: Renormalon type contribution to the two-point function.](image)

If we denote by \( B(q) \) the one loop correction to the four point function, the so called “fish” diagram [7], we have

\[
R_k(p) = \int \frac{d^4 q}{(2\pi)^4} \frac{i}{(p + q)^2 - m^2 + i\epsilon} \frac{1}{(-ig)^{k-2} [B(q)]^{k-1}}
\]

where the \((-ig)^{k-2}\) factor is due to the double counting of vertices in \( B(q) \). The relevant contribution to the integral comes from the deep euclidean region. It is easy to see that

\[
B(p) = \frac{(-ig)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k + p)^2 - m^2 + i\epsilon},
\]

\[\rightarrow -p^2 = -\frac{ig^2}{32\pi^2} \log(-p^2).\]  

(2)

(the argument of the logarithm is in mass units)

In this way we have

\[
R_k(p) = \frac{-ig^k}{(32\pi^2)^{k-1}} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(p + q)^2 + m^2 (\log(q^2))^{k-1}}.
\]

(3)

where a Wick rotation has been performed.
This expression is ultraviolet divergent. To make it convergent the propagator is expanded in powers of $1/q^2$ and the first two ultraviolet divergent terms are subtracted through the renormalization procedure. Due to this expansion, the dependence on $p$, the external momentum, disappears. So we find

$$R_k \propto -i \left( \frac{g}{32\pi^2} \right)^k \int dq \frac{1}{q^3} (\log(q^2))^{k-1}.$$  \hspace{1cm} (4)

It is not difficult to see that the main contribution to $R_k$, for large values of $k$, comes from the large $q$ region. Introducing the new variable $q = e^t$, $R_k$ becomes proportional to the gamma function.

$$R_k \propto -i \left( \frac{g}{32\pi^2} \right)^k \Gamma(k).$$  \hspace{1cm} (5)

This $k!$ behavior induces a pole in the real axis of the Borel plane

$$B[b] = \sum \left( \frac{R_k}{g^k} \right) \frac{b^{k-1}}{(k-1)!},$$

$$\propto -i \left( \frac{1}{1 - b/32\pi^2} \right).$$  \hspace{1cm} (6)

3 Chain of bubbles at finite temperature

In this section we revise the finite temperature calculation for the chain of bubbles shown in Fig. 2, which contribute to the four-point function. The original discussion can be found in [5]. Later we will consider an appropriate approximation for our case. Let us denote by $I_k$ the sum of all diagrams of the form shown in Fig. 2, with $k$ bubbles and fixed external vertices of type one, including a global imaginary factor $i$. The sum is over all possible combinations of internal type of vertices, which, as we know, can be of first or second type, according to the Feynman rules in TFD.

As it was shown in [5], and after correcting some missprints, $I_k$ can be expressed as a function of $I_1$ according to
\[ \text{Re } I_k = \frac{g}{2} (\alpha^k + \gamma^k). \quad (7) \]
\[ \text{Im } I_k = -\frac{ig}{2} \left( \frac{e^{\beta|p_0|} + 1}{e^{\beta|p_0|} - 1} \right) (\alpha^k - \gamma^k). \quad (8) \]

where
\[ \alpha(\gamma) = \frac{A \pm iB}{g}. \quad (9) \]

with
\[ A = \text{Re } I_1. \quad (10) \]
\[ B = \frac{e^{\beta|p_0|} - 1}{e^{\beta|p_0|} + 1} \text{Im } I_1. \quad (11) \]

where \( I_1 \) corresponds to
\[ I_1 = i(-ig)^2 \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{i}{k^2 - m^2 + i\epsilon} + \frac{2\pi\delta(k^2 - m^2)}{e^{\beta|k_0|} - 1} \right\} \]
\[ \times \left\{ \frac{i}{(k + p)^2 - m^2 + i\epsilon} + \frac{2\pi\delta((k + p)^2 - m^2)}{e^{\beta|k_0 + p_0|} - 1} \right\}. \quad (12) \]

It is convenient at this point to give the explicit expressions for the real and imaginary part of \( I_{1_0} \) and \( I_{1_\beta} \), zero and finite temperature parts, respectively, with \( I_1 = I_{1_0} + I_{1_\beta} \).
\[ Re I_{10} = \frac{g^2}{32\pi^2} \sqrt{\frac{|4m^2 - p^2|}{|p^2|}} \log \left( \frac{\sqrt{|4m^2 - p^2|} + \sqrt{|p^2|}}{\sqrt{|4m^2 - p^2|} - \sqrt{|p^2|}} \right). \quad (13) \]

\[ Im I_{10} = \theta(p^2 - 4m^2) \left( \frac{-g^2}{32\pi} \right) \sqrt{1 - \frac{4m^2}{p^2}}. \quad (14) \]

\[ Re I_{1\beta} = \frac{g^2}{16\pi|p|} \int_0^\infty dk \frac{k}{E(e^{\beta E} - 1)} \log \left( \frac{(2p_0E)^2 - (2k|\vec{p}| + p^2)^2}{(2p_0E)^2 - (2k|\vec{p}| - p^2)^2} \right). \quad (15) \]

\[ Im I_{1\beta} = -\frac{g^2}{8} \int d^3k (2\pi)^2 E_1 E_2 \left( \delta(p_0 + E_1 + E_2) + \delta(p_0 - E_1 - E_2) \right. \]

\[ + \left. \delta(p_0 + E_1 - E_2) + \delta(p_0 - E_1 + E_2) \right) \times \left( \frac{1}{e^{\beta E_1} - 1} + \frac{1}{e^{\beta E_2} - 1} + \frac{2}{(e^{\beta E_1} - 1)(e^{\beta E_2} - 1)} \right). \quad (16) \]

Discontinuities of Green functions at finite temperature are discussed in [8]. In the previous expressions \( E_1 = \sqrt{\vec{k}^2 + m^2} \) and \( E_2 = \sqrt{(\vec{k} + \vec{p})^2 + m^2} \). Since \( \alpha \) and \( \gamma \) are conjugated to each other, we can rewrite

\[ Re I_k = ga^k \cos(k\theta) \quad (17) \]

\[ Im I_k = ga^k \sin(k\theta) \left( \frac{e^{\beta|p_0|} + 1}{e^{\beta|p_0|} - 1} \right) \quad (18) \]

where we have introduced

\[ a = \frac{\sqrt{A^2 + B^2}}{g}. \quad (19) \]

\[ \theta = \arctan \left( \frac{B}{A} \right). \quad (20) \]

From the zero temperature calculation we know that the renormalon type contribution given by the diagram shown in Fig. 1 comes from the deep euclidean region. Specifically from \( \sqrt{-p^2} \approx e^{k/2} \) where \( k \) is the number of bubbles in the chain. We will examine the chain of bubbles in the deep euclidean region, at finite temperature, assuming \( \sqrt{-p^2} \approx e^{k/2} \) as the relevant
momentum. Later, by replacing the explicit expression for our chain of bubbles shown in Fig. 1, we will check if the previous condition, \( \sqrt{-p^2} \approx e^{k/2} \), is actually realized.

First we calculate \( \text{Im} I_{1,\beta} \) and \( \text{Re} I_{1,\beta} \) for \( T \neq 0 \). From the four \( \delta \)'s that appear in \( \text{Im} I_{1,\beta} \), it is easy to see that only those whose arguments included energy differences survive in the deep euclidean limit. So we have

\[
\text{Im} I_{1,\beta} \approx -\frac{g^2}{8} \int \frac{d^3k}{(2\pi)^2E_1E_2} 2\delta(E_1 - E_2) \left( \frac{2}{e^{\beta E_1} - 1} + \frac{2}{(e^{\beta E_1} - 1)^2} \right). \tag{21}
\]

By integrating and considering that \( e^{\beta \vec{p}/2} \gg 1 \), we have finally

\[
\text{Im} I_{1,\beta} \approx -\frac{g^2}{8\pi e^{\beta \vec{p}/2}}. \tag{22}
\]

Turning to the real part of \( I_{1,\beta} \), note that the argument of the logarithm in the deep euclidean limit can be simplified in such a way that the integral can be written as

\[
\text{Re} I_{1,\beta} \approx \frac{g^2}{16\pi |\vec{p}|} \int_0^q dk \frac{k E(1 - 4k|\vec{p}|)}{E(\beta E - 1)} 2 \log \left( \frac{1}{(-p^2)} \right) \tag{23}
\]

where \( -p^2 \) is sufficiently big so that we can reach the main contribution to the integral in the region where the logarithm can be expanded in powers of \( 4k|\vec{p}|/(-p^2) \). In the previous expression, \( q \) denotes a certain bound for the integration in \( k \) such that for values of \( k \) bigger than \( q \), the contribution to the integral turns out to be negligible due to the exponential supression. By expanding the logarithm, at the end we can take \( q \rightarrow \infty \). The real part, then, can be written as a series in powers of \( 1/\sqrt{-p^2} \)

\[
\text{Re} I_{1,\beta} \approx -\frac{g^2}{32\pi^2} \sum_{n=2}^{\infty} f_n(\beta) \left( \frac{1}{-p^2} \right)^{n/2} \tag{24}
\]

where the coefficients, which in our limit turn out to be essentially independent of the external momentum, are given by

\[
f_n(\beta) \approx -4\pi \int_0^{\infty} dk \frac{k E(\beta E - 1)}{n - 1}. \tag{25}
\]

Using the previous results, it is not difficult to see that
where, once again, we have taken the assumption $\sqrt{-p^2} \approx e^{k/2}$. This result allows us to conclude, from eqs. 17 and 18 that $I_k \approx ga^k$. Since $B$ in eq. 20 can be neglected, it is also easy to see that $I_k \approx g(A/g)^k$. Therefore, using our expression for the real part of $I_1$, finally we get

$$I_k \approx g\frac{g}{32 \pi^2} \left[ (\log(-p^2))^k + k(\log(-p^2))^{k-1} \frac{f_2(\beta)}{-p^2} + \cdots \right].$$

(27)

We can find an approximated expression for the coefficients $f_n(\beta)$ in the low temperature region, where $\beta m \gg 1$

$$f_n(\beta) \approx -4n\pi \int_0^\infty dx \frac{x^{(n-1)/2}}{\sqrt{x + 1}} e^{-\beta \sqrt{x+1}}$$

(28)

which can be evaluated exactly, see [9]

$$f_n(\beta) \approx -4n\sqrt{\pi} \left(\frac{2}{\beta}\right)^{n/2} \Gamma\left(\frac{n + 1}{2}\right) K_{-n/2}(\beta)$$

(29)

and where $K_\nu$ are Bessel type functions.

4 Renormalons at finite temperature

Using eq. 27 for the chain of bubbles, the renormalon diagram shown in Fig. 1 will be calculated in the frame of TFD. At finite temperature, $R_k(T)$ denotes the sum of all diagrams of the shape shown in Fig. 1 with $k$ vertices, being the external vertices of the first type. As it was the case in the chain of bubbles, the sum is over all possible combinations of internal type of vertices. In order to use the result of the previous section, this sum must be performed before the integral over the internal momentum that circulates through the chain of bubbles. Here we will obtain an expression for $R_k(T)$, finding the location of the induced poles in the Borel plane and the corresponding residues.

The expression we have to deal with is
\[ R_k(p) = \int \frac{d^4q}{(2\pi)^4} \left( \frac{i}{(p+q)^2 - m^2 + i\epsilon} + \frac{2\pi\delta((p+q)^2 - m^2)}{e^{\beta|p_0+q_0|} - 1} \right) I_{k-1}(q) \frac{1}{i}. \] (30)

The zero temperature contribution arises from the zero temperature part of the propagator times the zero temperature part of \( I_{k-1} \). We would like now to discuss the following two cases: a) the thermal part of the propagator times the zero temperature part of \( I_{k-1} \), \( R^a_k \), and b) \( R^b_k \) the usual zero temperature propagator times the first thermal correction to \( I_{k-1} \), \( R^b_k \), according to the series given in eq. 27.

Let us start with the case a). We have

\[ R^a_k(p) = \int \frac{d^4q}{(2\pi)^4} \left( \frac{2\pi\delta((p+q)^2 - m^2)}{e^{\beta|p+q_0|} - 1} \right) \left( \frac{-ig^k}{(32\pi^2)^{k-1}} \right) \left( \log(-q^2) \right)^{k-1}. \] (31)

Although the delta function forbids the deep euclidean region, after integrating in \( q^0 \), we can examine the limit when \( |\vec{q}| \gg m \) finding that case a) turns out to be proportional to

\[ R^a_k(p) \propto -i \left( \frac{g}{32\pi^2} \right)^k \int dq \frac{q}{e^{\beta q} - 1} (\log q)^{k-1}. \] (32)

The sum over \( k \) in this expression is Borel summable and, therefore, it does not imply any new renormalon.

The second case, case b), including also the zero temperature contribution, \( R^{b,0}_k \), corresponds to

\[ R^{b,0}_k(p) = \int \frac{d^4q}{(2\pi)^4} \left( \frac{1}{(p+q)^2 - m^2 + i\epsilon} \right) \left( \frac{g^k}{32\pi^2} \right)^{k-1} \times \left[ (\log(-q^2))^{k-1} + (k-1)(\log(-q^2))^{k-2} \frac{f_2(\beta)}{-q^2} \right]. \] (33)

After doing the Wick rotation and subtracting the divergent terms as we did in section 2, we find that the last expression is proportional to

\[ R^{b,0}_k \propto -i \left( \frac{g}{32\pi^2} \right)^k \int dt e^{-2t} \left[ (2t)^{k-1} + (k-1)(2t)^{k-2} e^{-2t} f_2(\beta) \right]. \] (34)
where the maxima of the first and second terms are reached at \( t = \log(\sqrt{-q^2}) \approx k/2 \) and \( t \approx k/4 \) respectively. Note that the assumption we made in the previous section, \( \sqrt{-p^2} \approx e^{k/2} \), changes for the second term, due to the factor 4 instead of 2. This fact does not affect our approximations.

By doing the integration, we see that \( R_k^{k,0} \) is proportional to

\[
R_k^{k,0} \propto -i \left( \frac{g}{32\pi^2} \right)^k \Gamma(k) - 2i \int f_2(\beta) \left( \frac{g}{64\pi^2} \right)^k \Gamma(k).
\]

(35)

We see that those terms proportional to the product of the temperature dependent part of the propagator times the whole series of \( I_k \) are Borel summable, since the leading term in the series of \( I_k \), case a), is already Borel summable. On the contrary, the zero temperature part of the propagator times \( I_k \) gives us a series of terms that behave like \( k! \) being, therefore, non Borel summable. We have calculated explicitly the first temperature dependent term of this series associated to a \( k! \) behavior.

If we perform the Borel transform \( B[b] \) of the sum \( \sum R_k \), by taking into account only those non-summable Borel terms, we find:

\[
B[b] = \sum \left( \frac{R_k}{g^k} \right)^{b-1} (k-1)!
\]

\[
\propto -i \left[ \frac{1}{1 - b/32\pi^2} + 2f_2(\beta) \frac{1}{1 - b/64\pi^2} + \cdots \right].
\]

(36)

This expression shows us the first thermal singularity in the Borel plane, and the dots denote the other new singularities, a whole series to the right of the first one. This is shown in Fig. 3.

Summarizing, in this paper we have shown the existence of a series of new singularities in the Borel plane, located to the right with respect to the normal, zero temperature leading pole. Note that the location of these singularities does not depend on temperature. Actually, the position of the singularities is related to the expansion of the logarithms coming from the bubbles in powers of \( 1/\sqrt{-p^2} \). The residues, on the contrary, have an explicit dependence on temperature through the factors \( f_n(\beta) \) which vanish when \( T \to 0 \).
Figure 3: Singularities in the Borel plane.

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