Vacuum solutions for multidimensional gravity on the principal bundle with the SU(2) structural group as the extra dimensions are found and discussed. This generalizes the results of Ref. [1] from U(1) to the SU(2) gauge group. The spherically symmetric solution with the off-diagonal components of the multidimensional metric is obtained. It is shown that two types of solutions exist: the first has a wormhole-like 4D base, the second is a gravitational flux tube with two color\(^1\) and electric charges. The solution depends only on two parameters: the values of the electric and magnetic fields at the origin. In the plane of these parameters there exists a curve separating the regions with different types of solutions. An analogy with the 5D solutions is discussed.

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\(^1\) Usually, this word refers to Quantum Chromodynamics and its SU(3) group and not to SU(2). We, nevertheless, use that word here to bring the close analogy met between the color degrees of freedom in both cases to attention.
I. INTRODUCTION

The presence of the off-diagonal components of the multidimensional (MD) metric can essentially change the properties of the appropriate MD gravitational equations. The physical reason for this is evident: these components of the MD metric are connected with the physical gauge fields (electromagnetic or Yang-Mills gauge fields) and excluding the physical fields can lead to this essential change. Let us repeat the following theorem \(^2\) [2], [3]:

Let \( G \) be a structural group of the principal bundle. Then there is a one-to-one correspondence between the \( G \)-invariant metrics

\[
ds^2 = \varphi(x^\alpha) \sum_{a=0}^{\text{dim}G} \left[ \sigma^a + A^a_\mu(x^\alpha)dx^\mu \right]^2 + g_{\mu\nu}(x^\alpha)dx^\mu dx^\nu
\]

on the total space \( \mathcal{X} \) and the triples \((g_{\mu\nu}, A^a_\mu, \varphi)\). Here \( g_{\mu\nu} \) is the 4D Einstein’s pseudo-Riemannian metric on the base; \( A^a_\mu \) are the gauge fields of the group \( G \) (the nondiagonal components of the multidimensional metric); \( \varphi_{ab} \) is the symmetric metric on the fibre \((\sigma^a = \gamma_{ab}\sigma^b, \gamma_{ab} = -\delta_{ab}, a = 5, \ldots, \text{dim} G \) is the index on the fibre and \( \mu = 0, 1, 2, 3 \) is the index on the base).

According to this theorem we have the following independent degrees of freedom: the scalar field \( \varphi(x^\alpha) \), the gauge fields \( A^a_\mu(x^\alpha) \) and the 4D metric \( g_{\mu\nu}(x^\alpha) \). Note that all fields in this MD gravity can depend only on the spacetime points (points on the base) as the total space is \( G \)-invariant. Such kind of MD gravity can easily solve the following problem: why the physical degrees of freedom do not depend on the extra dimensions (ED). In addition to this, a topological structure of the ED is given that leads to a decrease of the numbers of equations connected with the ED in comparison with an ordinary MD gravity with non-fixed ED. Usually, the number of these equations is too large: this results in essential problems with the compactification. Therefore, it is necessary to introduce some external fields in order to have a compactified ED. In our case the vacuum MD gravitational equations is sufficient to obtain the solutions with the compactified ED. In MD gravity on the principal bundle the preferable choice of the coordinate transformations \(^3\) is

\(^2\) which is useful for understanding how many degrees of freedom we have.

\(^3\) as the initial interpretation of Kaluza-Klein gravity
\[
y^a = y^a(y^a) + f^a(x^\mu), \quad (2)
x^\mu = x^\mu(x^\mu). \quad (3)
\]

Here \( y^a \) are the coordinates on the fibre and \( x^\mu \) are the coordinates on the base. The first term in (2) means that the choice of coordinate system on the fibre is arbitrary. The second term indicates that in addition we can move the origin of the coordinate system on each fibre on the value \( f^a(x^\mu) \). It is well known that such a transformation law (2) leads to a local gauge transformation for the appropriate non-Abelian field (see for overview [4]). That means, the coordinate transformation (2) and (3) are the most natural transformations for the MD gravitation on the principal bundle. Certainly we can perform the much more general coordinate transformations:

\[
y^a = y^a(y^a, x^\mu), \quad (4)
x^\mu = x^\mu(y^a, x^\mu). \quad (5)
\]

But in this case we will mix the points of the fibre 4 with the points of the base 5. Then the new coordinates \( y^a \) and \( x^\mu \) are not the coordinates along the fibre and the base of the given bundle.

We can introduce the new coordinates and kill the \( A_\mu^a \). But in this case the initial 4D metric can be changed very radical. For example, the initial static spherically symmetric 4D metric can become nonstatic and nondiagonal. This situation is very clear in the initial Kaluza interpretation of 5D gravity with the constant and nonvariable \( G_{55} \) component of the metric. In this case we have the ordinary vacuum electrogravity which is equivalent to the 5D gravity with the nondynamical \( G_{55} \) component of MD metric. And we can choose the new coordinates \( x^B \) so that the initial 5D metric:

\[
d s^2 = (d\chi + A_\mu dx^\mu)^2 + g_{tt} dt^2 + g_{rr} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (6)
\]

will be

\[
d s^2 = G_{55} d\chi^2 + g_{\mu\nu} dx^\mu dx^\nu \quad (7)
\]

4 which are the elements of some group.

5 which are the ordinary spacetimes points.
Evidently metric (7) will be nonstatic, nondiagonal and depend on the 5th coordinate. Of course, the new coordinate system \(x'^{A}(A = 0, 1, 2, 3, 5)\) is worse than the initial \(x^{A}\). This remark allow us to say that the MD metric with the off-diagonal components of the metric \(G^a_\mu\) can have unusual properties in comparison with the solutions where \(G^a_\mu = 0\).

II. 7D ANSATZ AND EQUATIONS

Here we will consider the MD gravity on the principal bundle with SU(2) structural group. In this case the extra dimensions is SU(2) group and the 4D physical spacetime is the base of this bundle. We will search a solution for the following 7D metric

\[
ds^2 = \frac{\Sigma^2(r)}{w^3(r)}dt^2 - dr^2 - a(r) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) - r_0^2 u(r) \left( \sigma^a + A^a_\mu dx^\mu \right)^2
\]

(8)

here \(r_0\) is some constant, \(\sigma^a (a = 5, 6, 7)\) are the Maurer-Cartan form with relation \(d\sigma^a = \epsilon^a_{bc} \sigma^b \sigma^c\)

\[
\sigma^1 = \frac{1}{2}(\sin \alpha d\beta - \sin \beta \cos \alpha d\gamma),
\]

(9)

\[
\sigma^2 = -\frac{1}{2}(\cos \alpha d\beta + \sin \beta \sin \alpha d\gamma),
\]

(10)

\[
\sigma^3 = \frac{1}{2}(d\alpha + \cos \beta d\gamma),
\]

(11)

where \(0 \leq \beta \leq \pi, 0 \leq \gamma \leq 2\pi, 0 \leq \alpha \leq 4\pi\) are the Euler angles. We choose the potential \(A^a_\mu\) in the ordinary monopole-like form

\[
A^a_\theta = \frac{1}{2}(1 - f(r)) \{ \sin \phi; - \cos \phi; 0 \},
\]

(12)

\[
A^a_\phi = \frac{1}{2}(1 - f(r)) \sin \theta \{ \cos \phi \cos \theta; \sin \phi \cos \theta; - \sin \theta \},
\]

(13)

\[
A^a_t = v(r) \{ \sin \theta \cos \phi; \sin \theta \sin \phi; \cos \theta \},
\]

(14)

Let us introduce the color electric \(E^a_i\) and magnetic \(H^a_i\) fields

\[
E^a_i = F^a_{ti},
\]

(15)

\[
H^a_i = \sqrt{\gamma} \epsilon_{ijk} \sqrt{g_{ij}} F^{a}_{jk}
\]

(16)

6 this is the gauge group of the weak interaction

7 topologically it means that the ED are the \(S^3\) sphere.
here the field strength components are defined via $F_{\mu\nu}^a = A_{\nu,\mu}^a - A_{\mu,\nu}^a + \epsilon_{\nu\mu}^a B_{\mu\nu}^a A_{\nu}^a$, $\gamma$ is the determinant of the 3D space matrix, $(i, j = 1, 2, 3)$ are the space index. In our case we have

$$E_r \propto v', \quad E_{\theta,\phi} \propto v f,$$

$$H_r \propto \sum \frac{1 - f^2}{a^{3/2}}, \quad H_{\theta,\phi} \propto f'$$

In order to have the wormhole-like (WH) solution we must demand that the functions $\Sigma(r), u(r), a(r), f(r)$ are even functions and $v(r)$ is an odd function. This means that at origin ($r = 0$) we have only the radial $E_r$ and $H_r$ fields that indicate the presence of a flux tube of color electric and magnetic fields across the throat of this WH-like solution. The substitution to the 7D gravitational equations

$$R_{\mu}^A = 0$$

$$R^5_5 + R^0_0 + R^2_2 = 0$$

leads to the following system of equations

$$-R_{22} - 2R_{33} + R \propto \frac{\Sigma''}{\Sigma} + \frac{a'\Sigma'}{a\Sigma} - \frac{4}{r_0^2 u} - \frac{r_0^2 u}{4a} f'^2 - \frac{r_0^2 u}{8a^2} (f^2 - 1)^2 = 0$$

$$R_{32} - \frac{1}{2} R \propto 24 \frac{\Sigma' u'}{\Sigma u} - 24 \frac{u'^2}{u^2} + 16 a ' \frac{\Sigma'}{a \Sigma} + 4 \frac{a'^2}{2a^2} - \frac{16}{a} + 4 \frac{r_0^2 u^4}{\Sigma^2} v'^2 - 2 \frac{r_0^2 u}{a} f'^2 - 8 \frac{r_0^2 u^4}{a \Sigma^2} f'^2 v'^2 + \frac{r_0^2 u}{a^2} (f^2 - 1)^2 - \frac{48}{a^2} = 0$$

$$R_{33} = R_{44} \propto \frac{a''}{a} + \frac{a' \Sigma'}{a \Sigma} - \frac{2}{a} + \frac{r_0^2 u}{4a} f'^2 - \frac{r_0^2 u^4}{a \Sigma^2} f'^2 v'^2 + \frac{r_0^2 u}{4a^2} (f^2 - 1)^2 = 0$$

$$R_{44} \propto \frac{u''}{u} + \frac{u' \Sigma'}{u \Sigma} - \frac{u'^2}{u^2} + \frac{u'a'}{ua} - \frac{4}{r_0^2 u} + \frac{r_0^2 u^4}{3 \Sigma^2} v'^2 - \frac{r_0^2 u}{6a} f'^2 + 2 \frac{r_0^2 u^4}{3 a \Sigma^2} f'^2 v'^2 - \frac{r_0^2 u}{12a^2} (f^2 - 1)^2 = 0$$

$$R_{15} \propto v'' + \frac{v'}{v} \left( \frac{\Sigma'}{\Sigma} + \frac{a'}{a} + \frac{u'}{u} \right) \frac{2}{a} v f'^2 = 0,$$

$$R_{35} \propto f'' + f' \left( \frac{\Sigma'}{\Sigma} + \frac{u'}{u} \right) + 4 u^3 \frac{u}{2} f v^2 - \frac{f}{a} (f^2 - 1) = 0$$

Note that the equations (25) and (26) are the “Yang-Mills” equations. This system of ordinary differential equations is extremely difficult for the analytical investigation. Therefore

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8The deduction of these gravitational equations for any group $G$ is given in the Appendix.

9although there exists a closed-form solution for the simplest case when $a(r) = const$ (flux tube solution [5]).
we will search for a numerical solution of this system.

III. NUMERICAL INVESTIGATION

Now we must write the initial conditions. At the origin \( r = 0 \) we can expand all functions by this manner

\[
a(x) = 1 + \frac{a_2}{2} x^2 + \cdots, \tag{27}
\]

\[
\Sigma(x) = \Sigma_0 + \frac{\Sigma_2}{2} x^2 + \cdots, \tag{28}
\]

\[
u(x) = u_0 + \frac{u_2}{2} x^2 + \cdots, \tag{29}
\]

\[
v(x) = v_1 x + \frac{v_3}{6} x^3 + \cdots, \tag{30}
\]

\[
f(x) = f_0 + \frac{f_2}{2} x^2 + \cdots, \tag{31}
\]

here we introduce the dimensionless coordinate \( x = r/\sqrt{a_0} \) \((a_0 = a(0))\) and redefine \( a(x)/a_0 \to a(x), \sqrt{a_0} v(x) \to v(x), r_0^2/a_0 \to r_0^2 \). Then we can rescale time and the constant \( r_0 \) so that \( \Sigma_0 = u_0 = 1 \). Thus we have only the following initial conditions for the numerical calculations

\[
a_0 = 1, \quad u_0 = 1, \quad \Sigma_0 = 1, \quad v_0 = 0, \quad f_0 = f_0, \tag{32}
\]

\[
a'_0 = 0, \quad u'_0 = 0, \quad \Sigma'_0 = 0, \quad v'_0 = v_1, \quad f'_0 = 0, \tag{33}
\]

We see that our system depends on two parameters only: \( f_0 \) and \( v_1 \). The constrained equation (22) for the initial data give us

\[
r_0^2 = \frac{1 + \sqrt{1 + 3 \left[v_1^2 + \frac{1}{4} (f_0^2 - 1)^2\right]}}{v_1^2 + \frac{1}{4} (f_0^2 - 1)^2} \tag{34}
\]

this equation is written in dimensionless variables. The numerical calculations of Eq.’s (21)-(26) are presented in the Figs. 1-5. The initial data for these calculations are the following:

\[
f_0 = 0.2, \quad v_1 = 0.3, 0.5, 0.6, 0.61, 0.615, 1.0, 2.0.
\]

\[\text{of course the 7D metric depends on the three parameters: } a_0, f_0 \text{ and } v_1.\]
FIG. 1. Function $a(x)$. Initial data: $f_0 = 0.2$, for curve 1: $v_1 = 0.3$, for curve 2: $v_1 = 0.5$, for curve 3: $v_1 = 0.6$, for curve 4: $v_1 = 0.61$, for curve 5: $v_1 = 0.615$, for curve 6: $v_1 = 1.0$, and for curve 7: $v_1 = 2.0$. 
FIG. 2. Function $u(x)$. Initial data as for Fig. 1.
FIG. 3. Function $\Sigma(x)$. Initial data as for Fig. 1.

FIG. 4. Function $f(x)$. Initial data as for Fig. 1.
FIG. 5. Function $v(x)$. Initial data as for Fig. 1.

IV. PHYSICAL DISCUSSION

Figs. 1-5 show us that in accordance with some relation between $f_0$ and $v_1$ there are two types of solutions:

1. There exists some value of the radial coordinate $r_1$ that $a(\pm r_1) = 0$, $u(\pm r_1) = s(\pm r_1) = \infty$. Probably this means that at $r = \pm r_1$ points we have a singularity. This 4D part of the MD metric we can name as gravitational flux tube. As we have some flux of color electric/magnetic field between the points $r = -r_1$ and $r = +r_1$ where the color electric/magnetic charges are located.

2. There exists some value of the radial coordinate $r_2$ that $a(\pm r_2), s(\pm r_2) < \infty, u(\pm r_2) = 0$. The value $u(\pm r_2) = 0$ means that the interval $ds^2 = 0$ on the hypersurface $r = \pm r_2, t, \theta, \phi = const$. Since the value of $a(\pm r_2)$ is finite we can name this type of the solutions as the WH-like.

This type of the solutions is close to the 5D case investigated in [1]. In this Ref. it was shown that the 5D vacuum Kaluza-Klein theory for the metric ($r_0$ is radius of the U(1)
gauge group, $Q = nr_0$ is the magnetic charge, $\chi$ is the $5^{th}$ coordinate)

$$ds^2 = e^{2\nu(r)}dt^2 - r_0^2e^{2\psi(r)-2\nu(r)} [d\chi + \omega(r)dt + n \cos \theta d\phi]^2$$

$$- dr^2 - a(r)(d\theta^2 + \sin^2 \theta d\phi^2),$$

(35)

has the following solutions

1. $E > H$. WH-like solution located between two $ds^2 = 0$ hypersurfaces.

2. $E = H$. Infinite flux tube with constant electric and magnetic fields.

3. $E < H$. Finite flux tube located between two singularities at points $r = \pm r_1$ where are the electric and magnetic charges.

In the first 5D case the solution exists for $|r| > r_0$ where $e^{2\nu} < 0$ and the metric is asymptotically flat. The whole construction can be interpreted as the Euclidean WH with the Lorentzian throat $^{12}$ [6]. The question is: whether this situation can be kept in the 7D case for the second type of the solutions? In Ref’s [7], [8] a similar idea had been investigated about the changing of the signature of the 4D metric in some MD metric on the regular $T$-hole horizon.

Also in our case the numerical calculations show that on the $(f_0, v_1)$ plane there is a curve that separates regions with the different solution type. Evidently, in the first rough approximation the equation for this curve is

$$\frac{a''}{a_0} = 2 \frac{a''}{a_0} - r_0^2u_0 \left(f_0^2 - 1\right)^2 = 0$$

(36)

In Fig. 6 these regions with the different type of solutions are shown.

$^{11}$E = q/a and $H = Q/a$ are the electric and magnetic charges.

$^{12}$The words Euclidean and Lorentzian can be exchanged.
FIG. 6. The curves C₁ and C₂ separate the regions with different types of the solutions. In the 2-regions with $a''_0 < 0$ we have the flux tube solutions and for the 1-regions $a''_0 > 0$ - the wormhole-like solutions.

In the $(f_0, v_1)$ plane we can single out a few cases allowing a more detailed analysis.

A. $f_0 = \pm 1, v_1 = 0$

Immediately from the “Yang-Mills” equations (25), (26) we see that

$$v(r) = 1, \quad f(r) = \pm 1$$

(37)

All terms with gauge fields in (21)-(24) vanish, and at the origin $(r = 0)$ the equation for the initial data give us

$$\frac{1}{a_0} + \frac{3}{a_0 r_0} = 0$$

(38)

As $a(r), u(r) > 0$ this relation cannot be satisfied. This allows us to say that in the absence of the gauge fields (off-diagonal components of the MD metric) the gravitational flux tubes and WH-like solutions do not exist.
B. $f = 0$

In this case (25) equation is easy to integrate

$$v' = \frac{q \Sigma}{r_0^2 au^4}$$

(39)

And from (17), (18) we see that there are only the radial color component of the electric and magnetic fields.

It is very interesting to note that there is some very simple analytical solution in the case $a(r) = \text{const}$ [5]. The solution is

$$a(r) = \frac{2q^2}{7} = \frac{r_0^2}{8} = \text{const}$$

(40)

$$\Sigma(r) = \cosh \left( \frac{7r}{2\sqrt{2}q} \right)$$

(41)

$$v(r) = \frac{\sqrt{2}}{r_0} \sinh \left( \frac{7r}{2\sqrt{2}q} \right)$$

(42)

$$u(r) = 1$$

(43)

$$q = \sqrt{\frac{7a}{2}}$$

(44)

We can apply the following gauge transformation to the potentials of Eqs. (12 - 14) with $f(r) = 0$

$$A'_\mu = S^{-1} A_\mu S - i(\partial_\mu S^{-1})S$$

(45)

where

$$S = \begin{pmatrix} \cos \frac{\theta}{2} & -e^{-i\phi} \sin \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

(46)

With this gauge transformation we find that the gauge potentials become

$$A'_\theta = (0; 0; 0)$$

(47)

$$A'_\phi = (\cos \theta - 1)(0; 0; 1)$$

(48)

$$A'_t = v(r)(0; 0; 1)$$

(49)

In fact this means that the potential (47)-(49) is an Abelian one. This solution corresponds to the points $x_1$ and $x_2$ in Fig. 6 and we can call it an infinite gravitational flux tube. It is very close to the Levi-Civita-Robinson-Bertotti solution [9], [10], [11] in 4D electrogravity.
This remark poses a very interesting problem: how are the other solutions on the curve $C_1$, $C_2$ (Fig. 6)? Are these solutions infinite flux tubes or something else\textsuperscript{13}?

For the question which other Lie groups can lead to similar results we can say that for any $N > 2$ there is an inclusion $\text{SU}(2) \subset \text{SU}(N)$. This means that for any such $\text{SU}(N)$ we can use the ansatz (12)-(14). As one knows, also all other compact semi-simple Lie groups (besides $\text{U}(1)$, of course), have $\text{SU}(2)$ as one of their subgroups. Therefore, for all these Lie groups similar statements are also valid. For the other Lie groups it is complicated to find out the spherically symmetric ansatz for the off-diagonal components of the MD metric.

We would like to point out the following dimensional reduction expression for the Ricci scalar $R(E)$ on the total space of the principal bundle

\[
\int d^4 x d^4 y \sqrt{|\det G_{AB}|} R(E) = V_G \int d^4 x \sqrt{|g|} \varphi^{d/2} \left[ R(M) + R(G) - \frac{1}{4} \varphi F^a_{\mu \nu} F^a_{\mu \nu} + \frac{1}{4} d(d - 1) \partial_{\mu} \varphi \partial^{\mu} \varphi \right]
\]

here $R(M), R(G)$ are the Ricci scalars of the base and structural group of the principal bundle respectively; $V_G$ is the volume of the group $G$, cf. [3], eq. (8.12). Immediately we see that for an arbitrary Lie group $G$ we face the problem of how to get the appropriate ansatz for the gauge field $A^a_{\mu}$.

In this context it should be noted that also in the models discussed here, a generalized Birkhoff theorem (cf. [12]) is valid. This means that a spherically symmetric solution possesses a further isometry without additional assumptions. Of course, this result rests on the symmetries chosen for our model.

**V. THE POSSIBLE PHYSICAL APPLICATIONS**

Probably, the most interesting above-mentioned solution in 7D gravity on the principal bundle with the $\text{SU}(2)$ structural group is the WH-like solution. We remember that if the $G_{55}, G_{66}, G_{77}$ components of the MD metric are not the dynamical variables then this

\textsuperscript{13}Unfortunately, here the numerical investigation is more difficult because of a sensitive dependence on initial data. This fact becomes also clear by the following consideration: These lines are the borderlines between two different domains of attraction, and there this sensitive behaviour is quite typical.
MD gravity is equivalent to the pure 4D gravity + SU(2) Yang-Mills theory. It can be supposed that in the Universe there exist regions where the $G_{55}, G_{66}, G_{77}$ metric components are nondynamical variables and that there exist other regions where these components are dynamical variables. The composite WH can be of such a kind. It means that we have our above-mentioned WH-like solution as a throat of composite WH and two 4D Yang-Mills black holes attached to this throat (see Fig. 7).

![Diagram of the composite wormhole](image)

**FIG. 7.** The composite wormhole with the 1-throat as MD wormhole-like solution and two black 2-holes attached to its 3-ends.

Such construction can polarize a space-time foam by the following way [13]. Without presence of the 7D (5D) throat the handles of the space-time foam are located disordered in the space-time (Fig. 8a). But after the appearance of the throat of composite WH the location of these handles is ordered (Fig. 8b). Such model can be a geometrical model of the renormalization of the color/electric charge (the case for the electric charge is described in [13]). This composite WH with the polarization of the space-time foam is a continuation of Wheeler’s idea about “charge without charge” and “mass without mass” in the vacuum gravity.
FIG. 8. A polarization of space-time foam in the presence of MD insertion. Fig. 8a presents the unpolarized space-time foam. Fig. 8b presents the polarized space-time foam. 1 are the virtual wormholes, 2 is the multidimensional insertion.

In this connection we can recall Wheeler’s quote in Ref. [14]: “We are therefore led to consider the view that the electron is nothing but a collective state of excitation of the foam-like medium . . . In other words the electron is not a natural starting point for the description of nature, according to the present reinterpretation of the views of Lorentz. Instead it is a first order correction to vacuum physics. That vacuum, that zero order state of affairs, with its enormous concentrations of electromagnetic energy and multiply-connected topologies, has to be described properly before one has the starting point for a proper perturbation theoretic development.” May be all these words can be used to a neutrino with mass, i.e. neutrino is a wormhole in the polarized space-time foam. In these cases there remains to solve the problem of the geometric description of spin. A similar 6-dimensional model, called $M$-fluxbranes, has been discussed recently in [15].

It is interesting to note that the above-mentioned gravitational flux tube solutions cannot be the throat of the composite WH as they have a singularity on the place of the $ds^2 = 0$ hypersurface for the WH-like solution. It imposes some restriction on the possible relation
between color electric and magnetic fields in the composite WH.

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APPENDIX A: GRAVITATIONAL EQUATIONS FOR THE GRAVITY ON THE PRINCIPAL BUNDLE

We consider the MD gravity on the principal bundle with the structural group $G$ [16]. In this case the extra dimensions are the group $G$, and the 4D physical spacetime is the base of this bundle.

According to above-mentioned theorem the MD metric on the total space is

$$ds^2 = \Sigma_{\tilde{A}}\Sigma^{\tilde{A}}$$ (A1)

where

$$\Sigma^{\tilde{A}} = h_{\tilde{B}}^{\tilde{A}}dx^{\tilde{B}},$$ (A2)

$$\Sigma^{\tilde{a}} = \varphi(x^\alpha)\sigma^{\tilde{a}} + h_{\tilde{\mu}}^{\tilde{a}}(x^\alpha)dx^{\tilde{\mu}},$$ (A3)

$$\sigma^{\tilde{a}} = h_{\tilde{b}}^{\tilde{a}}dx^{\tilde{b}},$$ (A4)

$$\Sigma^{\tilde{\mu}} = h_{\tilde{\nu}}^{\tilde{\mu}}(x^\alpha)dx^{\tilde{\nu}}$$ (A5)

Here $x^{\tilde{B}}$ are the coordinates on the total space ($\tilde{B} = 0, 1, 2, 3, 5, \ldots, N$, is the MD index $\dim(G) = N$), $x^a$ is the coordinates on the group $G$ ($a = 5, \ldots, N$), $x^{\mu} = 0, 1, 2, 3$ are the coordinates on the base of the bundle, $\tilde{A}$ is the $N$-bein index, $h_{\tilde{B}}^{\tilde{A}}$ is the $N$-bein, $\sigma^{\tilde{a}}$ are the 1-forms on the group $G$ satisfying $d\sigma^{\tilde{a}} = f_{bc}^{\tilde{a}}\sigma^{\tilde{b}}\sigma^{\tilde{c}}$, $(f_{bc}^{\tilde{a}})$ are the structural constants for the group $G$. We must note that the functions $\varphi, h_{\tilde{\mu}}^{\tilde{a}}, h_{\tilde{\mu}}^{\tilde{\nu}}$ can depend only on the $x^{\mu}$ points on the base as the fibres of our bundle are locally homogeneous spaces. The matrix $h_{\tilde{B}}^{\tilde{A}}$ has the following form

$$h_{\tilde{B}}^{\tilde{A}} = \begin{pmatrix} \varphi h_{\tilde{b}}^{\tilde{a}} & h_{\tilde{\mu}}^{\tilde{a}} \\ 0 & h_{\tilde{\nu}}^{\tilde{\mu}} \end{pmatrix}$$ (A6)
The inverse matrix \( h^{-B}_{A} \) is

\[
h^{-B}_{A} = \begin{pmatrix} \varphi^{-1} h^{-b}_{a} h^{b}_{\mu} \\ 0 \end{pmatrix}
\]

(A7)

here \( h^{-b}_{\mu} = -\varphi^{-1} h^{b}_{a} h^{a}_{\mu} h^{\nu}_{A} \). Also we see that we have only the following degrees of freedom: \( \varphi(x^\alpha), h^{b}_{\mu}(x^a) \) and \( h^{b}_{\nu}(x^\alpha), h^{a}_{\mu} \) is given and not varying. Varying with respect to \( h^{A}_{\mu} = (h^{a}_{\mu}, h^{\nu}_{\mu}) \) leads to the equations

\[
R^{\bar{a}}_{A} - \frac{1}{2} h^{\bar{a}}_{A} R = 0 \tag{A8}
\]

here \( \bar{A} = \bar{a}, \bar{\nu} \), \( R^{\bar{A}}_{\bar{B}} \) is the MD Ricci tensor. Let \( x^{a} \) be the coordinates on the group \( G \) then

\[
\varphi^{\sigma^a} = \varphi h^{b}_{\sigma} dx^{b} \tag{A9}
\]

Varying with respect to \( \varphi(x^\mu) \) leads to the following result

\[
\frac{\delta}{\delta \varphi} (hR) = \frac{\delta (h^{b}_{a}/\varphi)}{\delta \varphi} \frac{\delta (hR)}{\delta (h^{b}_{a}/\varphi)} = -\frac{1}{\varphi^2} h^{b}_{a} \left( R^{\bar{a}}_{\bar{b}} - \frac{1}{2} h^{\bar{a}}_{\bar{b}} R \right) = 0 \tag{A10}
\]

here \( h = \det h^{A}_{\bar{B}}, R \) is the MD Ricci scalar for the metric on the total space. As \( h^{\nu}_{a} = 0 \) we can write

\[
h^{b}_{a} \left( R^{\bar{a}}_{\bar{b}} - \frac{1}{2} h^{\bar{a}}_{\bar{b}} R \right) + h^{\nu}_{\bar{a}} \left( R^{\bar{a}}_{\bar{\nu}} - \frac{1}{2} h^{\bar{a}}_{\bar{\nu}} R \right) = h^{A}_{\bar{a}} \left( R^{\bar{A}}_{A} - \frac{1}{2} h^{\bar{A}}_{A} R \right) = 0 \tag{A11}
\]

From (A8) and (A11) we see that

\[
h^{A}_{\bar{a}} \left( R^{\bar{A}}_{A} - \frac{1}{2} h^{\bar{A}}_{A} R \right) + h^{\mu}_{\bar{a}} \left( R^{\bar{\mu}}_{\bar{A}} - \frac{1}{2} h^{\bar{\mu}}_{\bar{A}} R \right) = h^{\bar{A}}_{\bar{a}} \left( R^{\bar{B}}_{\bar{A}} - \frac{1}{2} h^{\bar{B}}_{\bar{A}} R \right) = 0. \tag{A12}
\]

This means that

\[
R = 0. \tag{A13}
\]

Hence from (A11) we can write

\[
h^{A}_{\bar{a}} R^{\bar{A}}_{\bar{a}} = R^{\bar{\mu}}_{\bar{a}} \tag{A14}
\]

Finally we have the following equation system for the MD gravity on the principal bundle

\[
R^{\bar{a}}_{A} = 0 \tag{A15}
\]

\[
R^{\bar{a}}_{\bar{a}} = R^{\bar{\mu}}_{\bar{a}} + \cdots + R^{\bar{N}}_{\bar{a}} = 0. \tag{A16}
\]
We note that the (A16) Eq. is an analog for the Brans-Dicke scalar gravity. In addition we see that (A15) can be wrote as

\[ R_{\bar{\mu}A} = 0 \quad \text{or} \quad h_A^B R_{\bar{\mu}B} = R_{\bar{\mu}A} = 0 \quad (A17) \]

Figure captions

Fig. 1. Function \(a(x)\). Initial data: \(f_0 = 0.2\), for curve 1: \(v_1 = 0.3\), for curve 2: \(v_1 = 0.5\), for curve 3: \(v_1 = 0.6\), for curve 4: \(v_1 = 0.61\), for curve 5: \(v_1 = 0.615\), for curve 6: \(v_1 = 1.0\), and for curve 7: \(v_1 = 2.0\).

Fig. 2. Function \(u(x)\). Initial data as for Fig. 1.

Fig. 3. Function \(\Sigma(x)\). Initial data as for Fig. 1.

Fig. 4. Function \(f(x)\). Initial data as for Fig. 1.

Fig. 5. Function \(v(x)\). Initial data as for Fig. 1.

Fig. 6. The curves \(C_1\) and \(C_2\) separate the regions with different types of the solutions. In the 2-regions with \(a''_0 < 0\) we have the flux tube solutions and for the 1-regions \(a''_0 > 0\) - the wormhole-like solutions.

Fig. 7. The composite wormhole with the 1-throat as MD wormhole-like solution and two black 2-holes attached to its 3-ends.

Fig. 8. A polarization of space-time foam in the presence of MD insertion. Fig. 8a presents the unpolarized space-time foam. Fig. 8b presents the polarized space-time foam. 1 are the virtual wormholes, 2 is the multidimensional insertion.