Chapter 5

TIME PROFILE OF THE EXTRACTED BEAM

In Chapter 4 the time needed for a particle to be extracted after becoming unstable has been evaluated. The transit time formulae can be used to estimate the temporal structure of the extracted beam, that is the number of particles reaching the electrostatic septum as a function of time, when the beam is slowly brought into the resonance. This will be expressed as the density in time \( P(t) \), defined such that \( P(t) \, dt \) is the number of particles that reach the ES between \( t \) and \( t+dt \). This function depends on the extraction method and can thus be used to compare different extraction techniques.

5.1 "Strip" profile

During the extraction process, the stable region is shrunk slowly, such that a narrow strip of particles becomes unstable and is thus extracted. This situation is represented in Figure 5.1.

The first particle will reach the ES at \( t = T_{c,d} \) and the last one at \( t = T_{c,d}(1=\Lambda \Delta) \). Note that the maximum time does not correspond to \( \Delta = 1 \). This is due to the fact that the particles that start very near to the stable point \( P_2 \) will be overtaken by the inward movement of separatrix B. These particles will be extracted along the following separatrix. This is

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* The name "strip" will be reserved for the narrow region of monoenergetic particles on the side of a stable triangle. The name "band" will be introduced later for the series of strips in a beam with a momentum spread.
equivalent to considering them as starting near $P_3$ (they are in fact replaced by particles from the separatrix C that are overtaken by the movement of separatrix A). The delay with which they appear near $P_3$ will not be considered, for the moment. The value of $\Lambda_F$ will be given later in this Chapter.

As noticed in Chapter 4, all the particles starting with $0<\Lambda<0.1$, reach the electrostatic septum at virtually the same moment, so the spill will start with a spike at $t = T_{c,d}$. If $\rho(\Lambda)$ is the linear probability density of particles in the strip, and $N_{\text{strip}}$ is the total number of particles contained in the strip, the spike will contain $N_{\text{spike}}$ particles, where:

$$N_{\text{spike}} = N_{\text{strip}} \int_0^{\frac{1}{\sqrt{G_2 F}}} \rho(\Lambda) d\Lambda + N_{\text{strip}} \int_{\frac{1}{\sqrt{G_2 F}}}^{\frac{1}{\sqrt{G_5 F}}} \rho(\Lambda) d\Lambda$$  \hspace{1cm} (5.1)

where $\int_0^{\frac{1}{\sqrt{G_2 F}}} \rho(\Lambda) d\Lambda = 1$. Thus the initial spike can be approximately described by:

$$N_{\text{strip}} P_{\text{spike}}(t) dt = N_{\text{spike}} 6(t-T_{c,d}) dt.$$  \hspace{1cm} (5.2)

After the initial spike, the spill shape can be evaluated by noticing that the particles coming out between $T(\Lambda)$ and $T(\Lambda)+d\Lambda$ are the ones which started between $\Lambda$ and $\Lambda+d\Lambda$. This means:

$$N_{\text{strip}} P_{\text{tail}}(t) dt = N_{\text{strip}} P_{\text{tail}}(T(\Lambda)) dt - N_{\text{strip}} \rho(\Lambda) d\Lambda = N_{\text{strip}} \frac{d\Lambda}{dt} dt$$  \hspace{1cm} (5.3)

for $t_0 < t < t_F$, which corresponds to $0.1 < \Lambda < \Lambda_F$. The time profile for the elementary strip is then given by the sum of $P_{\text{spike}}$ and $P_{\text{tail}}$.

$$P_{\text{strip}}(t) dt = P_{\text{spike}}(t) dt + P_{\text{tail}}(t) dt.$$  \hspace{1cm} (5.4)

Thus to evaluate the time profile of the elementary strip $P_{\text{strip}}$, $\Lambda_F$, $\rho(\Lambda)$ and $\Lambda(t)$ have to be determined. The shape of this elementary spill is the key to calculating the spill profile for the whole beam and for including the influence of ripple (an explanation of ripple is given later in Section 5.8).

### 5.2 Linear probability density

Assuming that the initial beam is smoothly distributed in phase space and that the resonance is applied adiabatically, then the density probability $\rho(\Lambda)$ in the strip will be proportional to $1/\nu(\Lambda)$, where $\nu(\Lambda) = dX/dt$ is the velocity in normalised phase space of the particles. With the Kobayashi Hamiltonian translated to $P_3$, $\nu(\Lambda)$ is given by (4.2) \((b)\)

$$\frac{dX}{dt} = -\frac{\partial H}{\partial X} = -\frac{S}{4} \left( 12hX + 6\sqrt{3}hX^2 + 3X^2 - 3X^2 \right).$$  \hspace{1cm} (4.2) \((b)\)
Restricting (4.2) \((b)\) to trajectories close to separatrix A, substituting the definitions of \(\lambda\) and \(\Lambda\) neglecting the term in \(\lambda^2\), yields

\[
\frac{dX'}{dt} = v(\Lambda) = -3Sh^2 \left( -\lambda - 3\Lambda + 3\lambda^2 \right)
\]

which yields

\[
\rho(\Lambda) = k \frac{1}{v(\Lambda)} = k \frac{1}{-3Sh^2 \left( -\lambda - 3\Lambda + 3\lambda^2 \right)}
\]

where \(k\) is a normalisation constant defined by

\[
\int_0^1 \rho(\Lambda)d\Lambda = 1.
\]

Thus

\[
k = \frac{-9Sh^2}{2\ln \frac{\lambda}{3}}.
\]

Finally, the line density is given by

\[
\rho(\Lambda) = \frac{3}{2(3\lambda^2 - 3\Lambda - \lambda)\ln \frac{\lambda}{3}}
\]

(5.5)

### 5.3 Inverting the Transit Time

The second element that is necessary for the evaluation of \(P_{\text{tail}}\), is \(d\Lambda/dt\). The time needed for a particle to reach the electrostatic septum starting from \(X = -\lambda h\) and \(X' = -2\sqrt{3}\lambda h\), near separatrix A, is:

\[
T_{s,d} = \frac{1}{\sqrt{3}E} \ln \left| \frac{n + 3}{A} \frac{2\sqrt{3} - A}{A} \frac{\Lambda}{1 - \Lambda} \frac{3}{\lambda_{F,d} - \frac{1}{\sqrt{3} E\varepsilon}} \right|.
\]

(4.19)

Re-arrangement of (4.19) to solve for \(\Lambda\), and the use of \(t\) instead of \(T_{s,d}\), to stress that it is now the independent variable, yields

\[
e^{\sqrt{3}t} = \frac{n}{n + 3} \frac{2\sqrt{3} - A}{A} \frac{\Lambda}{1 - \Lambda} \left( \frac{3}{\lambda_{F,d} - \frac{1}{\sqrt{3} E\varepsilon}} \right)
\]
and, using (4.18),

\[ \frac{n + 3}{3n} e^{3\alpha} = \frac{2\sqrt{3} - A}{A} \frac{\Lambda}{1 - \Lambda} \left( \frac{1}{A(2\sqrt{3} - A)} \right) - \frac{1}{\sqrt{3}e} \frac{\dot{\epsilon}}{\epsilon} \left( 1 + \ln \left[ \frac{2\sqrt{3} - A}{A} \frac{\Lambda}{1 - \Lambda} \right] \right). \]  

(5.6)

As \( \lambda \) is of the order of \( \frac{\dot{\epsilon}}{\epsilon} \), and \( \epsilon << 1 \), the first term in the denominator of the last fraction in (5.6) can be neglected with respect to the second. Equation (5.6) can then be rewritten as

\[ R = K + K \ln R \]  

(5.7)

where

\[ R = \frac{2\sqrt{3} - A}{A} \frac{\Lambda}{1 - \Lambda}; \quad K = \frac{\dot{\epsilon}}{\sqrt{3}e} \frac{n + 3}{3n} e^{\dot{\epsilon}t}. \]  

(5.8)

In order to invert \( T(\Lambda) \), it is necessary to solve (5.7) with respect to \( R \). This equation has real solutions only when \( K \geq 1 \), which corresponds to \( t \geq T_{c,d} \). This is to be expected since no particle is extracted prior to \( T_{c,d} \). Let \( R = R(K) \) be the solution of equation (5.7), then:

\[ \Lambda = \frac{g(K)}{1 + g(K)} \]  

(5.9)

where \( g(K) = \frac{A}{2\sqrt{3} - A} R(K) \). Referring back to equation (5.3), the aim is to evaluate

\[ \rho(\Lambda) \frac{d\Lambda}{dt} = \frac{3}{2(3\lambda^2 - 3\lambda - \lambda) \ln \frac{\lambda}{3}} \frac{d\Lambda}{dt} \approx \frac{3}{2(3\lambda^2 - 3\lambda) \ln \frac{\lambda}{3}} \frac{d\Lambda}{dt}. \]  

(5.10)

Substitution of expression (5.9), yields:

\[ \rho(\Lambda) \frac{d\Lambda}{dt} = -\left( 1 + g(K) \right)^2 \frac{\dot{g}(K)}{2g(K) \ln \frac{\lambda}{3}} \left( 1 + g(K) \right)^2 = -\frac{1}{2 \ln \frac{\lambda}{3}} \frac{\dot{g}(K)}{g(K)} = -\frac{1}{2 \ln \frac{\lambda}{3}} \frac{\dot{R}(K)}{R(K)}. \]  

(5.11)

which no longer depends on \( A \). Note that neglecting \( \lambda \) with respect to \( (A^2 - A) \) in equation (5.10), means that \( \Lambda \gg \lambda \), which is always true for \( \Lambda > 0.1 \), and \( (1 - \Lambda) \gg \lambda \). This will be shown later to be also true for \( \Lambda < \Lambda_F \).

Derivation of (5.7) with respect to time, gives
\[
\dot{R} = \dot{K} + K \ln(R) + K \frac{\dot{R}}{R}
\]

\[
\dot{R} = \frac{1 + \ln(R)}{1 - \frac{K}{R}} \dot{K}
\]

which, substituted into (5.11), gives

\[
\rho(\Lambda) \frac{d\Lambda}{dt} = -\frac{1}{2 \ln \frac{\lambda}{3}} \frac{1 + \ln(R(K))}{R(K) - K}  \dot{K} = \frac{1}{\ln \frac{\lambda}{3}} \frac{1 + \ln(R(K))}{R(K) - K} \frac{n + 3}{n} \frac{\pi \dot{Q}}{e} e^{\sqrt{\epsilon} t}.
\] (5.12)

Note that neglecting the derivatives of \(n\) and \(e\) was already included in the model when it was assumed that the stable region does not change during the extraction time. A plot of expression (5.12) is shown in Figure 5.2 compared with a numerical simulation.

![Figure 5.2. Strip Profile (blue line) shown with simulation results (points).](image)

When \(K \gg 1\), then \(R \gg K\) and \(K\) can be neglected in the denominator. From (5.7), \(\frac{1 + \ln(R)}{R} = \frac{1}{K}\). Thus, for \(K \gg 1\), the asymptotic value of the spill density is

\[
\rho(\Lambda) \frac{d\Lambda}{dt} \bigg|_{t \rightarrow \infty} \approx -\frac{1}{2 \ln \frac{\lambda}{3}} \frac{\dot{K}}{K} = -\frac{\sqrt{3} \epsilon}{2 \ln \frac{\lambda}{3}}
\] (5.13)

### 5.4 Spill length

The end of the spill (and thus the spill length) can be evaluated by noticing that separatrix B in its movement overtakes some particles with \(\Lambda\) sufficiently close to unity that their velocity is slower than the velocity of the separatrix itself. Those particles will then be extracted along the following separatrix. The effect is that some particle near \(P_2\) will
disappear to reappear near $P_3$. The spill therefore ends with the particle whose velocity is equal to the velocity of the separatrix. The $X'$ coordinate of this particle is given by

$$\frac{dX'}{dt} = -\frac{S}{4}(12h + 6\sqrt{3}hX' + 3X'^2 - 3X^2) = -\frac{8\pi}{\sqrt{3S}} \hat{Q}. \quad (5.14)$$

Neglecting the term in $X$ and $X^2$ with respect to the right-hand side, and substituting $X' = -2\sqrt{3}Ah$, yields

$$9Sh^2\Lambda^2 - 9Sh^2\Lambda - \frac{8\pi}{\sqrt{3S}} \hat{Q} = 0$$

whose solutions are

$$\Lambda_{1,2} = \frac{1 \mp \left(1 + \frac{16\pi\hat{Q}}{9\sqrt{3S^2h^2}}\right)}{2}.$$  

The solution of interest is the one close to the stable point $P_2$,

$$\Lambda_F = 1 + \frac{8\pi\hat{Q}}{9\sqrt{3S^2h^2}} = 1 + \frac{2\pi\hat{Q}}{\sqrt{3e^2}} = 1 + \frac{\dot{\varepsilon}}{3\sqrt{3e^2}} \quad (5.15)$$

Note that $1 - \Lambda_F >> \lambda$. This justifies the omission of $\lambda$ in equation (5.10) in the previous section. The spill will thus end at

$$T_{s,d}(\Lambda_F) \approx \frac{1}{\sqrt{3e}} \ln \left| \frac{n}{n+3} \frac{2\sqrt{3} - A}{A} \frac{\Lambda_F}{1 - \Lambda_F} \left(\frac{\dot{\varepsilon}}{\sqrt{3e^2}}\right)^\frac{3}{1 + \left[\frac{2\sqrt{3} - A}{A} \frac{\Lambda_F}{1 - \Lambda_F}\right]} \right| \approx \frac{1}{\sqrt{3e}} \ln \left| \frac{n}{n+3} \frac{2\sqrt{3} - A}{A} \left(\frac{3\sqrt{3e^2}}{\dot{\varepsilon}}\right)^\frac{3}{1 + \left[\frac{2\sqrt{3} - A}{A} \frac{3\sqrt{3e^2}}{\dot{\varepsilon}}\right]} \right|$$

where the term in $\lambda$ in the last fraction has been neglected as in (5.6). Expressing this time in units of $T_{c,d}$, yields

$$\frac{T_{s,d}(\Lambda_F)}{T_{c,d}} \approx \frac{1}{\sqrt{3e}} \ln \left| \frac{n}{n+3} \frac{2\sqrt{3} - A}{A} \left(-\frac{3\sqrt{3e^2}}{\dot{\varepsilon}}\right)^{-1} \right| \approx \frac{1}{\sqrt{3e}} \ln \left| \frac{n}{n+3} \left(-\frac{3\sqrt{3e^2}}{\dot{\varepsilon}}\right)^{-1} \right|.$$
\[
\ln\left(\frac{n}{n+3}\right) + \ln\left(\frac{2\sqrt{3} - A}{A}\right) + 2\ln\left(-\frac{3\sqrt{3}e^2}{\dot{e}}\right) - \ln\left(1 + \ln\left(\frac{2\sqrt{3} - A}{A} - \frac{3\sqrt{3}e^2}{\dot{e}}\right)\right) \approx 2.
\]

Where all the addenda have been considered negligible with respect to \(\ln|e^2/\dot{e}|\). Hence this complicated derivation converges to the simple result,

\[
T_{s,d}(\Lambda_F) = 2T_{c,d}.
\]

Thus the first particle reaches the electrostatic septum at \(T_{c,d}\), the last one at \(2T_{c,d}\), and the spill length is \(T_{c,d}\).

### 5.5 Width of the initial spike

It is now possible to evaluate the time needed for the last particle to be overtaken by the separatrix. Consider the velocity of the particle along the side of the stable triangle and, similar to Chapter 4, add the velocity of the separatrix:

\[
\frac{dX}{dt} = -3S^2\left(3\lambda^2 - 3\lambda - \lambda\right) + \frac{8\pi}{\sqrt{3S}}\dot{Q}.
\]

This yields

\[
\lambda = \frac{e}{\sqrt{3}}\left(3\lambda^2 - 3\lambda - \lambda\right) - \frac{2\pi}{\dot{e}}\dot{Q}.
\]

Finally, the time for a particle starting at \(\Lambda_i\) to be overtaken by separatrix B is

\[
T = \int_{\Lambda_i}^{\Lambda_F} \frac{1}{\lambda} - \frac{1}{\sqrt{3}\varepsilon\lambda^2 - \sqrt{3}\varepsilon\lambda - \frac{e}{\sqrt{3}}\lambda - \frac{2\pi}{\dot{e}}\dot{Q}} \, d\Lambda \approx \frac{1}{\sqrt{3}\varepsilon} \int \left| \frac{\Lambda_i - \frac{2\dot{e}}{\sqrt{3}\varepsilon}}{2\sqrt{3}\varepsilon(\Lambda_i - 1) + \frac{2\dot{e}}{3\varepsilon}} \right| \, d\Lambda.
\]

For \(\Lambda_i \rightarrow \Lambda_F\), \(T\) goes to infinity, as expected since this is the time needed for the separatrix to reach a particle which moves with the same velocity. So it is necessary to consider \(\Lambda_i = \Lambda_F + 2\lambda/\sqrt{3}\), that is one step of the separatrix away from \(\Lambda_F\). Then

\[
T = \frac{1}{\sqrt{3}\varepsilon} \ln\left| \frac{1}{6e} \right| \]

which may be a large fraction of \(T_{c,d}\) for small \(e\). However, for the sake of simplicity, and considering that most of the particles are near the stable points and thus will be overcome in a few turns, this delay will be neglected and all the particles starting near \(P_2\) will be considered as if they were starting near \(P_3\).
5.6 Population of spike and tail

It is interesting to evaluate the fraction of the beam contained in $P_{spike}$ and $P_{tail}$. This can be done in a straightforward way by integrating the density $\rho(\lambda)$. It is easier to integrate $P_{tail}$ between $\Lambda = 0.1$ and $\Lambda = \Lambda_F$, as in this range $\lambda$ can be neglected in the density. It results,

$$\int_{0.1}^{\Lambda_F} \rho(\lambda) d\lambda = \ln \frac{\lambda}{2} \ln \frac{27 \sqrt{3} e}{3} \approx \frac{1}{2}$$

(5.18)

neglecting $\ln(27 \sqrt{3} e)$ and $\ln(3)$ with respect to $\ln(\lambda)$ yields the result that 1/2 of the beam is in $P_{tail}$ and 1/2 in the initial spike. Thus the two components of the strip profile are equally important. This result is of consequence for the response to ripple and the efficiency of feedback systems. For perturbations up to frequencies corresponding to the width of the initial spike (of the order of 100 kHz for revolution periods of the order of 1 μs), half of the beam behaves coherently with a definite delay while the other half is extended over a period that will cause overlap with ripple frequencies as low as 1 kHz.

5.7 Time profile without ripple

Up to now, only the microscopic behaviour has been considered. When performing an extraction, the whole beam is present and the total spill is the important feature. In the following the aim is to derive the global behaviour from the elementary strip profile.

In the Steinbach diagram, which shows the stable and unstable regions in the amplitude vs tune (or momentum) space, the strip corresponds to a fixed amplitude and tune (momentum), as shown in Figure 5.3.

![Image of Steinbach diagram]

Figure 5.3. A strip in the Steinbach diagram corresponds to a certain amplitude and momentum.

In general the beam may have any shape in the amplitude-momentum space, and the beam distribution shown in the figure is meant to represent a generic distribution. The particular cases of wide and narrow beam, with respect to the resonance, will be analysed later.
The total time profile of the beam is obtained by summing the contributions of the different strips. In a schematic way this is shown in Figure 5.4:

![Figure 5.4. Strip profiles for different strips. The total spill is the sum of all the strips.](image)

where \( P_{\text{strip,1}}, P_{\text{strip,2}}, P_{\text{strip,3}} \) represent the profiles in time for strips that become unstable at different times or with different amplitudes and so on. In principle they are all different and the extracted current is given by:

\[
I(t) = \sum_i P_{\text{strip,i}}(t).
\]

The evaluation of \( I(t) \), can be greatly simplified when the total extraction time is much longer than the duration of a strip. In this case, indeed, the relative variation of \( \varepsilon \), and of the other parameters, during the strip extraction is small and thus all the terms \( P_{\text{strip,i}}(t) \), relative to the same \( \varepsilon \), effectively contributing to the total extracted current are equal. Then, the sum of all these terms just gives the integral over one of them. Since the same argument applies to all the amplitudes, or modified tune distances \( \varepsilon \), involved then the extracted current equals the number of particles that become unstable as a function of time \( N_T(t) \). This means that:

- if the extraction is performed in amplitude the time profile is the amplitude distribution,
- if the extraction is performed in momentum the time profile is the momentum distribution,
- combinations of amplitude and momentum yield the combined distribution.

### 5.8 Ripple

As it has just been shown, under ideal extraction conditions, apart from an initial and a final transient, the intuitive result is found, and the effort of this complicated analysis seems worthless. Unfortunately, ideal conditions are never met in reality and the fields in the magnets, and thus all the parameters that depend on them, are not perfectly stable, but oscillate around their value. This effect is called ripple and in general many ripple...
frequencies with different amplitudes are present at once. In the following only one ripple
frequency will be considered.

The effect of a ripple in tune is to move the “V” of the resonance back and forth while a
ripple in the resonant sextupole strength causes the slope of the “V” to increase and
decrease, as shown in Figure 5.5.

In the very simple model in which a particle is extracted as soon as it enters the resonance,
the profile in time \( \frac{dN}{dt} \) of the extracted beam is given by the distribution of particles that
become unstable per unit tune variation \( \chi = \frac{dN}{dQ} \) and by the relative motion between
the resonance and the stack:

\[
\frac{dN}{dt} = \frac{dN}{dQ} \frac{dQ}{dt} = \chi \cdot (\dot{Q}_0 + \dot{Q}_r) = \chi \cdot (\dot{Q}_0 + \omega \delta Q_R \cos(\omega t))
\]  

(5.19)

where a sinusoidal ripple has been assumed. If the product of the ripple frequency \( \omega \) and
ripple amplitude \( \delta Q_R \) equals or exceeds the stack velocity \( \dot{Q}_0 \), the extracted beam is
completely or over modulated, respectively, and there are instants during which no beam is
extracted because the resonance is “escaping” faster than the beam.

This approximation holds until the transit time of the particles is short compared to the
period of the ripple. As soon as the transit time is no longer negligible, a more accurate
analysis is needed. As it has already been shown, the transit time is not the same for all the
particles, thus a mixed concept considering both the transit time and the number of
particles having a certain fixed delay has to be developed. In practice, this means
evaluating the time profile for the set of particles that become unstable in one step of the
ejection process, taking into account all the possible amplitudes.

The “width” of this profile is the quantity that has to be compared to the ripple period to
have an indication of the overlapping of these elementary profiles. If these overlap, the
continuity of the extracted beam will not be broken even if the product \( \omega \delta Q_R \) exceeds \( \dot{Q}_0 \).
5.9 Comparison of extraction methods

To extract the beam, either the particles have to be progressively moved into the resonance stopband or the stopband must be moved onto the beam. This means that some physical quantities have to be changed. The varying parameter, the way in which it is changed and the initial geometry in the amplitude momentum space distinguish the extraction method.

Of the many possibilities illustrated in Section 3.11, the three cases in which the average extracted momentum does not vary during extraction are considered:

- **Amplitude selection by moving the resonance**: the beam is narrow in momentum and the tune of the particles is changed, by varying the focusing of the machine, see Figure 5.6 (a).

- **Amplitude-momentum selection by moving the beam** [13]: the beam is wide in momentum and the tune changes, due to chromaticity, when accelerating the particles, see Figure 5.6 (b).

- **RF knock-out** [20]: the chromaticity is zero (or quasi-zero), so that all particles have the same tune. The particles are made unstable by blowing up the beam with transverse RF excitation. Thus the resonance is reached in amplitude, see Figure 5.6 (c).

![Figure 5.6. Comparison of the main extraction methods: (a) moving the resonance, (b) moving the beam, (c) increasing the particle amplitude.](image-url)

In the amplitude selection case, the movement of the resonance maroons the large amplitude particles first in the unstable region. Since the momentum spread is small, the basic element of spill marooned in the elementary extraction process (one elementary movement of the resonance) is nothing more than the strip profile just described. So the “building block” of this particular spill starts with a very narrow peak containing half of the particles involved. Moreover in the initial phase the transit time is very short and sensitivity to the ripples is high. At the end of the spill the transit times are longer and the particles in the tail are distributed over a longer period.

In the RF-knockout method the blow-up velocity is fixed by the spill length. When looked at in tune, the Steinbach diagram is similar to the amplitude selection case one. The beam
in this case has null width and the resonance is reached in amplitude. A ripple in tune would move the beam right and left, which causes the resonance line to move up and down in the “Δp/p diagram”. As the particles enter the resonance at high amplitudes, and thus large tune distances, the transit time is short and the sensitivity to ripples is high.

In the amplitude-momentum selection case, particles of all amplitudes become unstable at the same moment. This leads to an enlarged leading peak, which can fill the time interval during which no beam enters the resonance in a much more efficient way. A more detailed study is presented to estimate the smoothing of the ripples with this method in the following sections.

5.10 Elementary "band" for a wide momentum spread

Consider the amplitude-momentum selection case. The beam is wide in tune and a stationary situation develops in which all betatron amplitudes are extracted simultaneously. In Figure 5.7 an amplitude-momentum selection extraction is shown and the elementary band of beam that becomes unstable in one step is highlighted.

A hollow beam is considered to set a minimum ε which can be considered large with respect to its variation in the transit time. If the number of particles that would normally be in the hole is small, then a good representation for the beam with no hole is found. This approximation is not so restricting since firstly the radial particle density across a beam rises from zero at the centre and secondly particles with sufficiently small amplitude will cross the resonance without being extracted. Let the band that is extracted contain particles with tune shifts between δQ_{min} and δQ_{max}. The time profile of this band, is given by considering the time profile (5.4) for each ε value between ε_{F} = 6πδQ_{min}, corresponding to T_{F} = T_{c,d}(ε_{F}), and ε_{0} = 6πδQ_{max}, corresponding to T_{0} = T_{c,d}(ε_{0}), and summing at each instant all these contributions, as shown in Figure 5.8.

Figure 5.7. Amplitude-momentum selection extraction. A complete band of particles becomes unstable in one step.

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* Here the word “band” is used to indicate the series of strips corresponding to different momentum and emittance that become unstable simultaneously.
The resultant spill can be written as,

\[ N_B P_{\text{band}}(t) = \int_{\varepsilon_0}^{\varepsilon} N_{\text{strip}}(\varepsilon) P_{\text{strip}}(t, \varepsilon) d\varepsilon = \int_{\varepsilon_0}^{\varepsilon} N_{\text{strip}}(\varepsilon) \left( P_{\text{spike}}(t, \varepsilon) + P_{\text{tail}}(t, \varepsilon) \right) d\varepsilon \] (5.20)

where \( N_B \) is the total number of particles in the band and \( N_{\text{strip}}(\varepsilon) d\varepsilon \) is the number of particles marooned with modified tune distance between \( \varepsilon \) and \( \varepsilon + d\varepsilon \).

Figure 5.8. The total flux of particles is obtained summing the “strip profiles” for all the amplitudes present.

5.11 Simplified model

The exact integral is far too complicated to be evaluated analytically. So, to facilitate the task, each \( P_{\text{strip}}(t) \) is approximated by one delta function plus a rectangle representing \( P_{\text{spike}} \) and \( P_{\text{tail}} \) respectively. The width of the rectangles, as mentioned in section 5.4, is equal to the time needed for the first particle to reach the ES, that is \( T_{c,d} \), and hence the height is \( 1/T_{c,d} \). Not all the \( P_{\text{strip}}(t, \varepsilon) \) give their contribution at any given instant. Most of them are zero, either because they have not yet “started” or because they have already “finished”. At time \( t \), only the \( \varepsilon \) for which

\[ t/2 < T_{c,d}(\varepsilon) < t \]

give a non zero contribution. Thus the part of the integral arising from \( P_{\text{tail}} \), becomes

\[ N_B P_{\text{band,tail}}(t) = \int_{\varepsilon(t/2)}^{\varepsilon(t)} N_T(\varepsilon) \frac{1}{2 T_{c,d}(\varepsilon)} d\varepsilon. \]

Only half of the particles are in the rectangle

To be correct the integration limits should be \( \max(\varepsilon(t), \varepsilon_0) \), which corresponds to \( \min(t, T_F) \), and \( \min(\varepsilon(t/2), \varepsilon_0) \) which corresponds to \( \max(t/2, T_0) \).
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After a change of the integration variable, 
\[ \int_{\min(t(t/2),e_0)}^{\max(t(t'/2),e_0)} \frac{N_T(e)}{2} \frac{1}{T_{c,d}(e)} d\epsilon = \int_{\min(t(T_p))}^{\max(t(T_p))} \frac{N_T(e(T))}{2} \frac{1}{T} \frac{d\epsilon}{dT} dT. \] (5.21)

It is necessary to find \( e(T) \), that is to invert

\[ T_{c,d} = \frac{1}{\sqrt{3e}} \ln \left( \frac{n}{n+3} \left( \frac{3}{\lambda - \frac{1}{\sqrt{3e}} \frac{\dot{e}}{e}} \right) \right). \]

To simplify the calculation, it is assumed that the ES is far away so that \( n >> 1 \), thus \( n/(n+3) \approx 1 \). Neglecting \( \lambda \) as usual, gives

\[ T_{c,d} = \frac{1}{\sqrt{3e}} \ln \left( 3 \left( \frac{\dot{e}}{\frac{\sqrt{3e}}{e}} \right) \right) = \frac{1}{\sqrt{3e}} \ln \left( \frac{3\sqrt{3e}}{\dot{e}} \right). \]

which can be written

\[ \frac{1}{2} \sqrt{\frac{\dot{e}}{\sqrt{3}}} T_{c,d} = \frac{1}{\sqrt{3}} \ln \left( -\frac{3\sqrt{3}}{\dot{e}} \right). \]

It is necessary to invert the equation

\[ Y = \ln(X)/X. \]

A simple fit to the inversion, valid to within 10% in the range \( 4 < X < 30000 \), is

\[ X = -1.42 \ln(Y)/Y. \]

Figure 5.9. Fit to the inversion (Logarithmic scale).
Note that $X = 10$ corresponds to $\frac{\dot{e}}{e^2} \approx 5 \cdot 10^{-2}$ and values of $X$ smaller than this should be avoided to stay within the limits of the transit time formulae ($\frac{\dot{e}}{e^2}$ measures the relative variation of the stable region during the transit time). Thus

$$
\varepsilon = -1.42 \sqrt{\frac{\dot{e}}{3\sqrt{3}}} \left( \ln \left( \frac{1}{2} \sqrt{\frac{\dot{e}}{\sqrt{3}}} \right) \right) = -1.42 \cdot \frac{2}{\sqrt{3}} \ln \left( \frac{1}{2} \sqrt{\frac{\dot{e}}{\sqrt{3}}} \right) T
$$

(5.22)

from which

$$
\frac{d\varepsilon}{dT} = -1.42 \cdot \frac{2}{\sqrt{3}} \cdot \frac{1}{T^2} \left( 1 - \ln \left( \frac{1}{2} \sqrt{\frac{\dot{e}}{\sqrt{3}}} \right) \right).
$$

(5.23)

It is now necessary to estimate $N_T(\varepsilon)d\varepsilon$. This is the number of particles that are in the border of thickness $dH$ of a triangle of surface $3\sqrt{3}H^2 = 3\sqrt{3} \left( \frac{2 \varepsilon}{3S} \right)^2$. This border corresponds to the annulus of width $dR$ around a circle of the same surface $\pi R^2$ in the initial beam. Thus,

$$
N_T(\varepsilon)d\varepsilon = N_B \rho(R(\varepsilon))d\left(3\sqrt{3}H^2\right) = N_B \rho\left( \sqrt{\frac{3\sqrt{3}}{\pi} \frac{2 \varepsilon}{3S^2}} \right) \cdot \left( \frac{\sqrt{3}}{3S^2} \right) \cdot \frac{\sqrt{3}}{S} \cdot d\varepsilon
$$

(5.24)

where $N_B$ is the total number of particles in the band.

### 5.12 Uniform distribution in phase space

Assuming a uniform phase space distribution in the initial beam, of value $\rho_0 = 1/\text{Triangle Area} = \frac{3S^2}{4\sqrt{3}e^2_0}$, yields for the integration of the tails from all the differential strips in the band,

$$
P_{\text{band, tail}}(t) = \frac{1}{N_B} \int_{\text{max}(t,T_r)}^{\text{min}(t,T_r)} \frac{N_T(T(\varepsilon))}{2} \frac{d\varepsilon}{T} dT =
$$

$$
= \rho_0 \frac{8\sqrt{3}}{3S^2} (-1.42)^2 \cdot \frac{4}{3} \int_{\text{min}(t,T_r)}^{\text{max}(t/2,T_r)} \frac{\ln \left( \frac{1}{2} \sqrt{\frac{\dot{e}}{\sqrt{3}}} \right) - \left( \ln \left( \frac{1}{2} \sqrt{\frac{\dot{e}}{\sqrt{3}}} \right) \right)^2}{T^4} dT =
$$

$$
= \frac{4(-1.42)^2}{3e_0^2} \int_{\text{min}(t,T_r)}^{\text{max}(t/2,T_r)} \frac{\ln \left( \frac{1}{2} \sqrt{\frac{\dot{e}}{\sqrt{3}}} \right) - \left( \ln \left( \frac{1}{2} \sqrt{\frac{\dot{e}}{\sqrt{3}}} \right) \right)^2}{T^4} dT.
$$
Using the standard formulae

\[ \int \frac{\ln(ax)}{x^p} \, dx = -\frac{\ln(ax)}{(p-1)x^{p-1}} - \frac{1}{(p-1)^2 x^{p-1}} \]

and

\[ \int \frac{(\ln(x))^2}{x^p} \, dx = -\frac{(\ln(x))^2}{(p-1)x^{p-1}} - \frac{2\ln(x)}{(p-1)^2 x^{p-1}} - \frac{2}{(p-1)^3 x^{p-1}}, \]

the integral can be evaluated. It results that

\[ P_{\text{band,tail}}(t) = \frac{4(-1.42)^2}{3e_0^2} \times \]

\[ \left[ \frac{\ln \left( \frac{1}{2} \sqrt{\frac{-\dot{e}}{\sqrt{3}}} \right)}{3T^3} - \frac{1}{9T^3} + \frac{\left( \frac{1}{2} \sqrt{\frac{-\dot{e}}{\sqrt{3}}} \right)^2}{3T^3} - 2\ln \left( \frac{1}{2} \sqrt{\frac{-\dot{e}}{\sqrt{3}}} \right) - \frac{1}{9T^3} + \frac{2}{27T^3} \right] \]

\[ = \frac{4(-1.42)^2}{3e_0^2} \left[ \frac{9 \ln \left( \frac{1}{2} \sqrt{\frac{-\dot{e}}{\sqrt{3}}} \right)^2}{27T^3} - 3\ln \left( \frac{1}{2} \sqrt{\frac{-\dot{e}}{\sqrt{3}}} \right) - 1 \right] \]

and finally, for times \( 2T_0 < t < T_F \), that is apart from the initial and final parts

\[ P_{\text{band,tail}}(t) = \frac{4(-1.42)^2}{3e_0^2} \times \]

\[ \left[ \frac{63 \ln \left( \frac{1}{2} \sqrt{\frac{-\dot{e}}{\sqrt{3}}} \right)^2}{27T^3} - (144\ln(2) + 21)\ln \left( \frac{1}{2} \sqrt{\frac{-\dot{e}}{\sqrt{3}}} \right) + \frac{72\ln(2)^2 + 241\ln(2) - 7}{27T^3} \right] \] (5.25)

To correctly evaluate the initial and the final part of the beam, the correct integration limits have to be considered. In fact for times \( t < T_0 \), no particle has reached the septum and \( P_{\text{band,tail}} = 0 \). For times \( T_0 < T < 2T_0 \), the integral has to be performed between \( T \) and \( T_0 \) (there are no particles with \( \varepsilon = \varepsilon(T/2) \)). For the same reason for times \( T_F < T < 2T_F \), the integral has to be performed between \( T_F \) and \( T/2 \). For times greater than \( 2T_F \), \( P_{\text{band,tail}} = 0 \) again.
The contribution from the initial spikes of the elementary strips in the band, gives

$$P_{\text{band, spikes}}(t) = \int_{t_0}^{t} N_T(\varepsilon(T)) \frac{1}{2} \delta(t - T) \frac{d\varepsilon}{dT} dT = \frac{4(-1.42)^2}{3\varepsilon_0^2} \ln \left( \frac{1}{2} \sqrt{\frac{\varepsilon}{\sqrt{3}}} \right) - \ln \left( \frac{1}{2} \sqrt{\frac{\varepsilon}{\sqrt{3}}} \right)^2 \cdot \frac{1}{t^3}.$$ 

The graph of the spill shape for a particular case is drawn in Figure 5.10.

![Figure 5.10. Band profile for $\delta Q_{\text{max}} = 3.782 \times 10^{-3}$, $\delta Q_{\text{min}} = 3.782 \times 10^{-4}$, $Q = 5.673 \times 10^{-8}$, $S = 27.7897 \, \text{m}^{-1/2}$. The two contributions and the sum are shown.](image)

5.13 Gaussian distribution in phase space

Beams are often assumed to have a gaussian distribution in phase space

$$\rho(R) = \frac{1}{2\pi E_x} e^{-\frac{R^2}{E_x E_{x}}} \quad (5.26)$$

where $E_x$ is the horizontal one-$\sigma$ emittance. Assuming that the area of the biggest stable triangle corresponds to an emittance of $n$-$\sigma$, the one-$\sigma$ emittance is

$$E_x = \frac{\text{Area}}{\pi n^2} = \frac{4\sqrt{3}\varepsilon_0^2}{3S^2 \pi n^2} \quad (5.27)$$

Thus

$$\rho(\varepsilon) = \frac{3S^2 n^2}{8\sqrt{3}\varepsilon_0^2} e^{-\frac{\varepsilon^2}{2\varepsilon_0^2}} \quad (5.28)$$

*The definition of emittance used is phase space area divided by $\pi$*
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\[
N_T(\varepsilon) d\varepsilon = N_T n^2 e^{-\frac{\varepsilon^2}{2\sigma^2}} d\varepsilon 
\]  
(5.29)

and

\[
P_{\text{band, tail}}(t) = \int_{\text{max}(t, T_T)}^{\text{min}(t, T_T)} N_T(T(\varepsilon)) \frac{1}{2} \frac{d\varepsilon}{dT} dT = \frac{2n^2(-1.42)^2}{3e_0^2} \times \left\{ \ln \left( \frac{1}{2} \frac{\delta}{\sqrt{3}} \right) - \left( \ln \left( \frac{1}{2} \frac{\delta}{\sqrt{3}} \right) \right)^2 \right\}^2 \right\} dT. 
\]  
(5.30)

The integral can be evaluated numerically and plotted, as shown in Figure 5.11.

![Figure 5.11. Band profile for a gaussian beam for $Q_{\text{max}} = 3.782 \times 10^{-3}$, $Q_{\text{min}} = 3.782 \times 10^{-4}$, $S = 5.673 \times 10^{-8}$, $S = 27.7897 \, \text{m}^{1/2}$, $E_{\text{tot}} = 9 \times E_{\text{rms}}$. The two contributions and the sum are shown.](image)

If the gaussian and the uniform cases are plotted on the same graph, the comparison in Figure 5.12 is obtained.

![Figure 5.12. Band profile for gaussian and uniform beam. A $3\sigma$ emittance of the gaussian beam has been considered as total emittance.](image)
From the plots above, it appears that the width of the “band profile” is of the order of $T_0$. Thus it can be expected that when the ripple period is of the order of $T_0$, the width of the profile partially fills the time interval during which no beam enters the resonance and smoothes the modulation of the extracted beam. The band profiles have been worked out in this Chapter considering a constant $\dot{Q}$. Thus considering them as “filling the gap” in the presence of ripple is an approximation which neglects the instantaneous variation and uses an average $\dot{Q}$ ($\dot{Q} = \dot{Q}_0 + \omega \delta Q R \cos(\omega t) \approx \dot{Q}_0$). This can be accepted by considering that $\dot{Q}$ appears in the expression for the transit time inside a logarithm and visually by comparing the profiles in time for a variation in $\dot{Q}$ of 2 orders of magnitude, as shown in figure 5.13:

![Figure 5.13. Time profiles for gaussian and uniform beams for different values of $\dot{Q}$](image)

Even a large variation in $\dot{Q}$ does not vary the orders of magnitude and, to a first approximation, the band profile can be considered when analysing the spill with ripple. The enlarged leading peak quoted in Section 5.9 can be thus estimated in hundreds of turns to be compared with the tens of turns of the strip profile, demonstrating the superior ripple tolerance of the amplitude-momentum selection scheme.

### 5.14 Simulations of extraction with ripple

All the theory and approximations made so far, call for a check to verify the correctness of the conclusions drawn. With this in mind, three numerical simulations have been made:

1) amplitude selection extraction with a mono-energetic beam;
2) amplitude-momentum selection with a uniform distribution in phase space;
3) amplitude-momentum selection with a gaussian distribution in phase space.

In order to be consistent with the numerical examples shown so far, the following parameters have been used for the simulations:

$$T_{\text{rev}} = 0.734 \ \mu\text{s}$$

$$\dot{Q} = -5.673 \times 10^{-8} \Rightarrow \dot{Q} = -0.0257 \ \text{s}^{-1}.$$  

It should be noted that these simulation parameters correspond to the extraction of protons at 60 MeV from the PIMMS ring and are therefore representative of a practical situation.
From inspection of Figure 5.12, it results that the width of the “band profile” should be able to fill gaps of the order of $\Delta T = 100$. To demonstrate this effect a round number for the tune ripple frequency has been chosen as

$$f_{\text{ripple}} = 4000 \text{ Hz} \Rightarrow \Delta T = 113.$$ 

The corresponding amplitude for 100% modulation of the extracted beam in the “instantaneous transfer” model is

$$A_{\text{ripple}} = 10^{-6}.$$ 

The tune ripple is excited by a quadrupole gradient ripple on a single quadrupole family. The corresponding normalised quadrupole gradient ripple amplitude is easily obtained with a lattice program. It should be noticed that the relative ripple amplitude, in this case, is just $1.3 \times 10^{-6}$, which is a very tough constraint on the quality of the power supplies! Since the binning in the analysis of the results cannot be infinitely large, a slightly larger (20% larger) amplitude has been used in the simulations.

The program used to simulate the extraction is MAD [21]. Note that the particle tracking does not follow the approximated hamiltonian used in the theoretical derivation, but is an exact tracking through the PIMMS synchrotron.

In order to limit the computing time and to have a sufficiently large number of extracted particles, only a small fraction of a beam is generated which occupies the region of interest. This is schematically shown in Figure 5.14.

The resonant sextupole strength is raised linearly from zero to its final value in 4000 turns. The beams are generated such that a small part of them is unstable once the sextupole is fully raised. In order to extract those particles that are already unstable, after the sextupole has reached its full strength, the program lets the particles circulate for 6000 turns before extraction starts. As an example the main steps in the preparation of the beam are shown in Figures 5.15 to 5.17 where the initial distribution, the distribution after 4000 turns (sextupole reaches its final value) and 10000 turns (just before extraction starts) are shown for the uniform, band-like distribution. The time profile is analysed using bins 30 turns wide, which correspond to an integration time of $22 \mu$s for an hypothetical extracted current measurement.
Figure 5.15. Initial beam distribution for the uniform band case.

Figure 5.16. Beam distribution after raising the sextupole.

Figure 5.17. Beam distribution just before extraction starts.
5.14.1 Amplitude selection
As is evident from inspection of Figure 5.18, in this case the extracted beam is 100% modulated. This is to be expected since the basic element for building the time profile is the strip profile, whose width is small with respect to the 4 kHz period.

Figure 5.18. Time profile without (upper) and with (lower) ripple.
5.14.2 Amplitude-momentum selection (uniform distribution)

For the “uniform phase-space distribution” – “band profile”, the basic element for the time profile construction is the band profile shown in Figure 5.10. It is clear that 113 time units after the band profile start, there is still some beam in the resonance that fills the “gap”. The simulation, shown in Figure 5.19, confirms this intuition, showing that the continuity in the extracted flux is not broken, although the modulation is visible.

Figure 5.19. Time profile without (upper) and with (lower) ripple.
5.14.3 Amplitude-momentum selection (gaussian distribution)

For the “gaussian phase-space distribution” – “band profile”, the basic element for the time profile construction is the band profile shown in Figure 5.11. By direct inspection, it is clear that 113 time units after the band profile start, the particle flux is still close to maximum and, consequently, a very good smoothing of the ripple is expected. The simulation of Figure 5.20 shows that in this case the extracted spill is almost indistinguishable with or without ripple.

Figure 5.20. Time profile without (upper) and with (lower) ripple.
5.14.4 *Amplitude-momentum selection (gaussian distribution) II*

In order to have a second check for the “gaussian phase-space distribution” – “band profile”, the same simulation has been run with a 10 times larger ripple amplitude. The simulation of Figure 5.21 shows that also in this case the extracted spill is very well behaved.

![Figure 5.21. Time profile without (upper) and with (lower) ripple.](image-url)
5.15 Concluding remarks

At kHz frequencies and above the amplitude-momentum selection extraction has been shown to offer a very useful smoothing effect on the spill. This allows a relaxation of the tolerances on the power supplies in a region where the constraints are very tough.

It should be noticed that the time profile is expressed in number of turns and the revolution period varies with machine size and extraction energy. Full suppression of ripple effects starts for ripple periods of the order of $T_0 = T_{c,d}(dQ_{\text{max}})$, that is the transit time for the largest amplitude particles. For PIMMS, a typical proton-carbon ion medical synchrotron, this is of the order of 4 kHz for the protons at the minimum extraction energy. The smoothing diminishes to near zero at around 1 kHz. At higher extraction energies, the revolution period diminishes and the full smoothing threshold moves up in frequency, which reduces the efficiency at a given frequency. However, at higher extraction energies, the magnet power supplies work at a higher current and the relative ripple is lower. Fortuitously, this gain outweighs the loss in “band smoothing” due to the higher revolution frequency.

The suppression of ripple in the spill is of paramount importance for raster scanning. Other smoothing techniques are available:

- front-end acceleration by RF noise or empty bucket channelling (efficient up to 2 kHz)
- compensation via scanning speed (efficient up to 1 kHz)
- feedback (efficient up to 1 kHz)
- geometric smoothing from beam size (efficient at 10 kHz and above)
- eddy currents smoothing (efficiency increases from 1 kHz upwards)

However the intrinsic smoothing from the band profile is embedded in the extraction design and is, as such, totally reliable not requiring any active components. Moreover it covers the difficult region of the few kHz.