Lattice Gauge Fixing for Parameter Dependent Covariant Gauges

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Abstract

We propose a non-perturbative procedure to fix generic covariant gauges on the lattice. Varying the gauge parameter, this gauge fixing provides a concrete method to check numerically the gauge dependence of correlators measured on the lattice. The new algorithm turns out to converge with a good efficiency. As a preliminary physical result, we find a sensitive dependence of the gluon propagator on the gauge parameter.
Lattice gauge fixing is unavoidable to compute non perturbatively the propagators of the fundamental fields appearing in the QCD Lagrangian [1]. It is also necessary in non-perturbative renormalization schemes [2, 3] which use gauge dependent matrix elements to renormalize composite operators, and it becomes a fundamental technical ingredient in the so called non gauge invariant quantizations of chiral theories [4].

Up to now, the Landau gauge is the only covariant gauge for which an efficient numerical algorithm is available.

A few years ago, an unconventional method to implement a generic covariant gauge on the lattice was proposed [5]. In principle this procedure takes into account the contributions from each gauge orbit with appropriate weight. With this method a gauge dependence of the gluon propagator, in particular at zero momentum, had been found [6]. On the other hand the implementation of this method for physical lattices can be numerically demanding.

Here we follow a more conservative procedure to fix a generic covariant gauge on the lattice that stems from ref. [7]. It is based, as in the Landau case, on the minimization of a functional $H_A[G]$ chosen in such a way that its absolute minima correspond to a gauge transformation $G$ satisfying the appropriate gauge condition (see section 2). In the continuum it is easy to show [8] that this procedure is equivalent to the Faddeev-Popov quantization for covariant gauges, i.e. the two procedures lead to the same matrix elements for a generic gauge dependent operator. As usual the Faddeev-Popov factor can be written as a Gaussian integral of local Grassman variables, the resulting effective action is invariant under the BRST transformations and the correlation functions of the operators satisfy the appropriate Slavnov-Taylor identities. Moreover in the perturbative region, the renormalized correlation functions can be compared with the same quantities computed in the standard perturbation theory.

The numerical implementation of this procedure on the lattice uses a straight-forward generalization of the standard Landau algorithm, i.e. a steepest descendent iterative algorithm minimizes a discretization of the new functional $H_A[G]$. In order to have an efficient and feasible gauge fixing procedure, the discretized functional $H_U[G]$ must be chosen carefully. In fact a na"ive discretization of this functional, due to its complicated structure, would lead to an algorithm which either does not converge or takes too much computer time [9].

The main purpose of this paper is to show that a simple discretization $H_U[G]$ exists and leads to a minimization algorithm which converges with a good efficiency. The application of this algorithm to the non-perturbative evaluation of gauge dependent Green functions like the quark and gluon propagator will clarify the gauge dependence of the fitted parameters. Non trivial Slavnov-Taylor identities provide a consistency check of the lattice technique in computing gauge dependent quantities. In order to show the feasibility of our method, we have computed the gluon propagator at different values of the gauge parameter at small volumes, finding a sensitive gauge dependence.

The plan of the paper is as follows. In section 2 we review the method proposed to fix a generic covariant gauge on the lattice. In section 3 we describe a very simple and efficient discretization of the gauge fixing functional. In section 4 we give the details of our numerical simulations and report our main numerical results.
2 Covariant Gauge Fixing

In this section we set the notation and briefly formulate the covariant gauge fixing method in the continuum. In the following we will neglect the problem of Gribov copies, assuming that a gauge section in the space of gauge fields intersects all gauge orbits once and only once. In the Landau gauge the expectation value of a gauge dependent operator is given by

$$\langle \mathcal{O} \rangle = \int \delta A_\mu \delta \eta \delta \bar{\eta} \mathcal{O} e^{-S(A) - S_{\text{ghost}}(\eta, \bar{\eta}, A)} \delta(\partial_\mu A_\mu) ,$$

and the gauge fixing condition can be enforced non-perturbatively by minimizing Gribov’s functional [10]

$$F_A[G] \equiv ||A^G||^2 = \int \text{Tr} \left( A^G_\mu A^G_\mu \right) d^4 x .$$

The Landau gauge is readily extended to a general covariant gauge-fixing condition of the form

$$\partial_\mu A^G_\mu(x) = \Lambda(x) ,$$

where $\Lambda(x) = \lambda^a(x) \frac{\mathbb{T}^a}{2}$ belongs to the Lie algebra of the group and $\text{Tr}(T^a T^b) = 2 \delta^{ab}$. Since gauge-invariant quantities are not sensitive to changes of gauge condition, it is possible to average over $\Lambda(x)$ with a Gaussian weight

$$\langle \mathcal{O} \rangle = \int \delta \Lambda e^{-\frac{1}{\alpha} \int d^4 x \text{Tr}[\Lambda^2]} \int \delta A_\mu \delta \eta \delta \bar{\eta} \mathcal{O} e^{-S(A) - S_{\text{ghost}}(\eta, \bar{\eta}, A)} \delta(\partial_\mu A_\mu - \Lambda) ,$$

obtaining the standard formula

$$\langle \mathcal{O} \rangle = \int \delta A_\mu \delta \eta \delta \bar{\eta} \mathcal{O} e^{-S(A) - S_{\text{ghost}}(\eta, \bar{\eta}, A)} e^{-\frac{1}{\alpha} \int d^4 x \text{Tr}[(\partial_\mu A_\mu)^2]} .$$

This formula is adopted as a definition for the expectation value of gauge dependent operators. The invariance under BRST transformations of the effective action in eq. (5) leads to the Slavnov-Taylor identities among different correlation functions. In particular the longitudinal part of the gluon propagator has to be equal to the free one, i.e. [8]

$$\frac{2}{N^2 - 1} \text{Tr} \langle \partial_\mu A_\mu(x) \partial_\nu A_\nu(y) \rangle \propto \alpha \delta(x - y) ,$$

where $N$ is the number of colors. Following the usual technique, the gauge-fixing condition (3) is obtained non perturbatively by minimizing the new functional [7]

$$H_A[G] \equiv \int d^4 x \text{Tr} \left[ (\partial_\mu A^G_\mu - \Lambda) (\partial_\nu A^G_\nu - \Lambda) \right] ,$$

which reaches its absolute minima when eq. (3) is satisfied. Therefore following the standard notation, each absolute minima corresponds to a Gribov copy of the equation (3). Due to the complexity of the functional (7), it may also have relative minima which do not satisfy the gauge condition in eq. (3) (spurious solutions) [7] because the stationary points of $H_A[G]$ correspond to the following gauge condition

$$D_\nu \partial_\nu (\partial_\mu A^G_\mu - \Lambda) = 0 .$$
On the lattice, the expectation value of a gauge dependent operator $O$ in a generic covariant gauge is

$$\langle O \rangle = \frac{1}{Z} \int d\Lambda e^{-\frac{1}{\alpha} \sum_x Tr[\Lambda^2]} \int dU O(U^{G_\alpha}) e^{-\beta S(U)},$$

which is the straight-forward discretization of Eq. (4), where $\Lambda$ is dimensionless on the lattice. $S(U)$ is the Wilson lattice gauge invariant action and $G_\alpha$ is the gauge transformation that minimizes the discretized version of the functional (7). On the lattice, the correct adjustment to the measure is included in eq. (9) by evaluating the operator over the gauge rotated links. Therefore it is not necessary to introduce ghost fields but it is mandatory to fulfill the gauge fixing condition numerically in order to get $G_\alpha$.

3 The Driven Discretization

In the Wilson discretization of gauge theories, the fundamental fields are the links $U_\mu$ which act as parallel transporters of the theory. Hence the lattice fields $A_\mu$ are derived quantities which tend to the continuum gluon field as the lattice spacing vanishes. As a consequence on the lattice it is possible to choose different definitions of $A_\mu$ formally equal up to $O(a)$ terms. In quantum field theory this ambiguity is well understood because any pair of operators, differing from each other by irrelevant terms, will tend to the same continuum operator. This feature, checked in perturbation theory, has been verified numerically at the non-perturbative level in ref. [11], where it has been shown that different definitions of the gluon field give rise to Green’s functions proportional to each other, guaranteeing the uniqueness of the continuum gluon field.

The freedom to choose the lattice definition of $A_\mu$ can be used to build discretized functionals which lead to efficient gauge-fixing algorithms. In the standard Landau gauge fixing, for example, the discretization of the functional (2) is given by

$$F_U[G] = -\frac{1}{VTa^2g^2} Tr \sum_{x,\mu} \left[ U^{G_\mu}_\mu(x) + U^{G_\mu}_\mu(x) - 2I \right],$$

where $V$ is the 3-dimensional volume and $T$ the time size of the lattice. This formula corresponds, only up to $O(a)$ terms, to the naïve discretization of eq. (2) that is obtained from the standard lattice definition of the gluon field

$$A_\mu(x) \equiv \left[ \frac{U_\mu(x) - U_\mu^\dagger(x)}{2i\alpha g} \right]_{\text{Traceless}}.$$ (11)

On the other hand $F_U[G]$ has the important property that it depends only linearly on $G(\bar{x})$, when the iterative algorithm visits the lattice point $\bar{x}$. This feature would be spoiled if $F_U[G]$ were defined assuming literally the naïve discretization (11) in eq. (2).

In order to study the convergence of the algorithm, two quantities are usually monitored as a function of the number of iteration steps: $F_U[G]$ itself and

$$\theta_F = \frac{1}{VT} \sum_x Tr[\Delta_{\Delta F}^\dagger],$$

where
where
\[ \Delta_F(x) = \left[X_F(x) - X_F^\dagger(x)\right]_{\text{Traceless}} \propto \frac{\delta F_U[G]}{\delta \epsilon} , \] (13)

being \( G = e^{i a \cdot \tau} \) and
\[ X_F(x) = \sum_\mu \left(U_\mu(x) + U_\mu^\dagger(x - \mu)\right) . \] (14)

\( \Delta_F(x) \) is proportional to the first derivative of \( F_U[G] \) and reaches zero as the functional is extremized. Therefore the quality of the gauge fixing is determined by the parameter \( \theta_F \) which corresponds to the continuum quantity \( \int d^4 x \text{Tr}(\partial_\mu A_\mu)^2 \).

A naïve discretization of \( H_A[G] \) will generate a quadratic dependence on \( G \), which could prevent the convergence of the algorithm. This obstacle has been overcome by taking advantage of the freedom to choose the gluon field definition. In fact, as in the Landau case, it has been possible to find a discretization of \( H_A[G] \) ("driven discretization") that depends linearly on \( G(\bar{x}) \) and corresponds, up to \( O(a) \) terms, to the continuum limit, eq. (7). This aim can be reached by choosing each different term of \( H_U[G] \) in order to guarantee the local linear dependence on \( G(\bar{x}) \), instead of being the algebraic consequence of a particular \( A_\mu \) definition. We propose the following compact form of \( H_U[G] \):
\[ H_U[G] = \frac{1}{VT} a^4 g^2 T \text{Tr} \sum_x J^G(x)J^{G\dagger}(x) , \] (15)

where
\[ J(x) = N(x) - i g \Lambda(x) , \]
\[ N(x) = -8I + \sum_\nu \left(U_\nu^\dagger(x - \nu) + U_\nu(x)\right) . \] (16)

It is easy to see that locally \( H_U[G] \) transforms linearly in \( G(\bar{x}) \) and its continuum limit is the functional (7). \( H_U[G] \) is positive semidefinite and, unlike the Landau case, it is not invariant under global gauge transformations.

The functional \( H_U[G] \) can be minimized using the same numerical technique adopted in the Landau case. In order to study the convergence of the algorithm, two quantities can be monitored as a function of the number of iteration steps: \( H_U[G] \) itself and
\[ \theta_H = \frac{1}{VT} \sum_x \text{Tr}[\Delta_H \Delta_H^\dagger] , \] (17)

where
\[ \Delta_H(x) = \left[X_H(x) - X_H^\dagger(x)\right]_{\text{Traceless}} \propto \frac{\delta H_U[G]}{\delta \epsilon} \] (18)

and
\[ X_H(x) = \sum_\mu \left(U_\mu(x)J(x + \mu) + U_\mu^\dagger(x - \mu)J(x - \mu)\right) \]
\[ - 8J(x) - 72I + igN(x)\Lambda(x) . \] (19)
\( \Delta_H \) is the driven discretization of the eq. (8); it is proportional to the first derivative of \( H_U[G] \) and, analogously to the continuum, it is invariant under the transformations \( \Lambda(x) \to \Lambda(x) + C \), where \( C \) is a constant matrix belonging to the \( SU(3) \) algebra. During the minimization process \( \theta_H \) decreases to zero and \( H_U[G] \) becomes constant. The quality of the convergence is measured by the final value of \( \theta_H \).

4 Numerical Simulations and Results

We have generated 50 \( SU(3) \) thermalized link configurations using the Wilson action with periodic boundary conditions at \( \beta = 6.0 \) for \( 8^4 \) and \( 8^3 \times 16 \) volumes. Following the prescription contained in eq. (9), for each Monte Carlo configuration and for each lattice site we have extracted a matrix \( \Lambda(x) \) according to a Gaussian distribution at a fixed \( \alpha \) value. The gauge-fixing code implements an iterative overrelaxed minimization algorithm for \( F_U[G] \) and \( H_U[G] \). We have monitored the quantities \( F_U \) and \( \theta_F \) for the standard Landau gauge-fixing algorithm and \( H_U[G] \) and \( \theta_H \) for the new one after every lattice sweep. We have found that the value of the overrelaxing parameter \( \omega \) adopted for the Landau algorithm \( \omega = 1.72 \) is a good choice also for the new gauge fixing.

In Fig. 1 we report the values of \( \theta_H \) for different \( \alpha \) values as a function of the gauge fixing sweeps for a typical thermalized configuration at \( \beta = 6.0 \) and \( V \cdot T = 8^4 \). Each sweep of

![Graph](image.png)

Figure 1: Typical behaviour of \( \theta_H \) vs gauge fixing sweeps for different choices of \( \alpha \), for \( \beta = 6.0 \) and \( V \cdot T = 8^4 \).

the new algorithm takes \( \sim 15\% \) more computer time with respect to the Landau case. For \( \alpha = 0 \), which in this new procedure corresponds to the Landau gauge and is obtained by taking \( \Lambda = 0 \), the number of sweeps necessary to fix the gauge is \( \simeq 1.2 \) times that of the
standard algorithm. In the case of $\alpha \neq 0$, the number of sweeps to minimize $H_U[G]$ with a given precision increases when $\alpha$ decreases.

Once the configuration have been rotated in a given gauge, we have tested our procedure computing the following two point correlation functions

\[
\langle A_0 A_0 \rangle(t) \equiv \frac{1}{V^2} \sum_{x,y} \text{Tr}(A_0(x,t)A_0(y,0)),
\]

\[
\langle A_i A_i \rangle(t) \equiv \frac{1}{3V^2} \sum_i \sum_{x,y} \text{Tr}(A_i(x,t)A_i(y,0)),
\]

\[
\langle \partial A \partial A \rangle(t) \equiv \frac{1}{4V} \sum_{\mu,\nu} \sum_x \text{Tr}(\partial_\mu A_\mu(x,t)\partial_\nu A_\nu(x,0)),
\]

where $\mu$ and $\nu$ run from 1 to 4, $i$ from 1 to 3 and the trace is over the color indices. In eqs. (20)-(22) the definition (11) is adopted for the gluon field on the lattice and $\partial_\mu$ indicates the usual backward derivative. The correlators in eqs. (20) and (21) are relevant to the investigation of the QCD gluon sector, while we use the correlation in eq. (22) to check the Slavnov-Taylor identity for the longitudinal component of the gluon propagator. The statistical errors for the correlation functions have been estimated by the jacknife method.

In Fig. 2, the Green functions computed with the standard Landau gauge fixing $\langle A_i A_i \rangle_F$ and $\langle A_i A_i \rangle_H$ evaluated with the new algorithm with $\alpha = 0$, are reported. The remarkable agreement shows that the new algorithm reproduces very well the results of the standard
one within an overall scale factor close to 1. Moreover, the possible spurious solutions of the new gauge condition apparently do not affect this operator at least at $\alpha = 0$.

Although in this paper we did not perform a systematic study of the Gribov copies for this gauge fixing procedure, a rough preliminary search shows, for $\alpha = 0$, the same pattern of copies of the Landau case.

In Fig. 3 the data for the correlation (20) are reported. For $\alpha = 0$ they show the well known flat behaviour forced by the Landau gauge. For $\alpha \neq 0$ the data show an almost flat curve with an enhancement at $t = 1$ due to the effects of contact terms. In this case the flatness stems from the fact that $\int d^3x \partial_0 A_0 \simeq 0$ because the average value of $\Lambda$ on each time slice is negligible.

In Fig. 4 our results for the correlations $\langle A_i A_i \rangle_H$ for different $\alpha$ values as a function of the time slice $t$ are reported for a set of 50 SU(3) configurations at $\beta = 6.0$ and $V \cdot T = 8^4$.

Although in the case of small volume and a small number of configurations, the gauge dependence of the gluon propagator is clearly shown. For increasing $\alpha$ values, the time dependence of the gluon propagator becomes flatter since it approaches a limit where the gauge fixing effects disappear. It is also interesting to note that the $\alpha$ dependence of the gluon propagator shown in Fig. 4 does not seem to be re-absorbed by an overall scaling factor. This plot shows the feasibility of this procedure to study the gauge dependence of physically interesting correlators.

For a finite volume $V \cdot T$ with periodic boundary conditions, the Slavnov-Taylor identity
Figure 4: Behaviour of $\langle A_i A_i \rangle_H$ (see eq. (21)) obtained with the new functional $H_U[G]$ for different $\alpha$ values as function of time $t$ for a set of 50 thermalized configurations at $\beta = 6.0$ with a volume $V \cdot T = 8^4$. The errors are jacknife and the data have been slightly displaced in $t$ for clarity.

A thorough study of this problem will be presented in a forthcoming paper [12]. Eq. (23) shows that the shape of the correlation function (22) is proportional to a $\delta$-function up to the square of the $\partial_\mu A_\mu(x)$ renormalization constant, together with volume and $O(a)$ effects. In order to verify eq. (23), in Fig. 5 we plot the correlation functions (22) versus the time $t$. The agreement between these data and eq. (23) is an indication of the absence of possible spurious solutions. The small signal at $t = 2, 8$ is likely due to the residual correlation induced by the enhancement in the point $t = 1$.

5 Conclusions

In this paper we have described an efficient method to fix non-perturbatively a generic covariant gauge on the lattice. This procedure is equivalent to the Faddeev-Popov quantization for covariant gauges. Therefore it can be used to compare the lattice renormalized correlation functions with the same quantities computed in continuum perturbation theory. We have tested the algorithm on different SU(3) lattices for different values of the gauge parameter $\alpha$. Our numerical data show that the statistical fluctuations of measured quantities is comparable to the standard Landau gauge fixing algorithm.
We have computed the correlation functions relevant for the investigations of the gluon propagator at a few $\alpha$ values and we have found a sensitive dependence on the gauge parameter. In collaboration with the Boston University Center for Computational Science, we are applying this method to the study of the gluon propagator on physical lattices.

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References

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