Casimir energy of a non-uniform string

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Abstract

The Casimir energy of a non-uniform string built up from two pieces with different speed of sound is calculated. A standard procedure of subtracting the energy of an infinite uniform string is applied, the subtraction being interpreted as the renormalization of the string tension. It is shown that in the case of a homogeneous string this method is completely equivalent to the zeta renormalization.

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I. INTRODUCTION

In the standard setting of the Casimir effect problem, a bounded configuration space of the quantum field system is considered [1,2]. However unbounded configuration space can be nonhomogeneous, i.e., it can consist from separate regions endowed with different physical characteristics. In the case of electromagnetic field such characteristics are permittivity $\varepsilon$ and permittivity $\mu$ of the media. It is obvious that the vacuum energy should depend on the configuration of these nonhomogenities. Hence the problem on finding out this dependence, i.e. on calculation of the Casimir energy, can be set.

When calculating the vacuum energy it proves to be essential whether the velocity of light (in the general case the velocity of relevant quanta) is continuous when crossing the interface between regions with different characteristics. From a mathematical stand point the constant velocity of light implies that the coefficients of the higher (usually the second) derivatives in the corresponding dynamical equation are continuous functions in space. If it is the case then the calculation of the Casimir energy (or the Casimir forces) turns out to be practically the same as for an empty space with a perfectly conducting shell having the shape of the interface between media [3–8].

The discontinuity of the speed of light on the boundaries between different regions results in considerable mathematical difficulties. Certain results have been obtained here only recently. The Casimir energy was calculated for a compact dilute dielectric ball proceeding from the Green’s function formalism and employing the “naive” zeta function technique for removal of the divergences without detailed procedure of analytic continuation [9–12]. It was assumed that the difference of the light velocities inside ($c_1$) and outside ($c_2$) is small and only the term proportional to $(c_1 - c_2)^2$ in the Casimir energy has been kept. In Ref. [13] it was shown that the heat kernel coefficient $a_2$, responsible for the pole contribution to the Casimir energy, vanishes in this approximation (it proves to be proportional to $(c_1 - c_2)^3$). This implies that the complete zeta regularization should provide a finite answer in this problem.

In calculations accomplished in Refs. [9,10,12] a key role was played by the so called contact terms which cannot be incorporated in the standard zeta function technique at least in a straightforward way. Their physical origin is still unclear.

Thus it is worth investigating these problems in the framework of a more simple model. Such a model, preserving all essential features of field theory, is a string built up from two pieces which have different velocities of sound (the velocities of vibration propagation). In papers by I. Brevik and co-authorths [14–16] a piecewise uniform string, with the same velocity of sound at each of its constituents, has been studied. Such string model is a one-dimensional analog of the electromagnetic field in media obeying the condition $\varepsilon\mu = c^{-2}$, where $c$ is the constant velocity of light. In this model the Casimir energy has been investigated in detail both at zero and finite temperature.

The present paper seeks to calculate the Casimir energy of a nonuniform open string of total length $R$, which is built up of the pieces of length $r$ and $R - r$ having the sound velocities $v_1$ and $v_2$ respectively. At the joint point ($x = r$) the continuity of the string shape and its tangents is imposed. The string ends $x = 0$ and $x = R$ are subjected to the Dirichlet or Neumann boundary conditions. The equations defining the eigenfrequencies of the string are derived. This material is presented in Sect. II. Proceeding from this in Sect. III the
Casimir energy of the string is calculated by making use of the mode-by-mode summation method [7,5], which relies on the contour integration. The renormalization of the string tension by subtracting the energy of an infinite uniform string enables one to remove the divergences in a unique way. It is shown that in the limit $R \to \infty$ the Casimir energy of the string does not depend on the boundary conditions imposed at $x = R$ and is expressed in terms of the polylogarithm function. In the Conclusion (Sect. IV) the results obtained are discussed briefly. In the Appendix it is shown that the subtraction procedure employed in the paper is equivalent to the zeta renormalization when considering a usual uniform string.

II. CLASSICAL DYNAMICS OF A NON-UNIFORM STRING

The model we are considering is a string built up from two pieces of length $r$ and $R - r$ and endowed with the sound velocities $v_1$ and $v_2$, respectively. The string is stretched along the $X^1$ axis (which we shall denote by $x$ in what follows) and can vibrate in $D - 2$ transverse dimensions $X^2 \ldots X^D$. The both pieces of the string satisfy the wave equation

$$\frac{1}{v_j^2} \frac{\partial^2 X_i}{\partial t^2} - \frac{\partial^2 X_i}{\partial x^2} = 0, \quad j = 1, 2, \quad i = 2, 3, \ldots, D. \tag{2.1}$$

At the ends the string is subjected to the boundary conditions, which we shall take to be of either Dirichlet or Neumann type. Consequently, we shall consider four cases:

$$X^i(t, 0) = 0, \quad X^i(t, R) = 0, \quad (DD),$$
$$X^i(t, 0) = 0, \quad \left. \frac{\partial X^i}{\partial x} \right|_{x=R} = 0, \quad (DN),$$
$$\left. \frac{\partial X^i}{\partial x} \right|_{x=0} = 0, \quad X^i(t, R) = 0, \quad (ND),$$
$$\left. \frac{\partial X^i}{\partial x} \right|_{x=0} = 0, \quad \left. \frac{\partial X^i}{\partial x} \right|_{x=R} = 0 \quad (NN), \quad i = 2, \ldots, D. \tag{2.2}$$

At the joining point $x = r$ we impose the continuity conditions

$$\lim_{x \to r^-} X^i(t, x) = \lim_{x \to r^+} X^i(t, x),$$
$$\lim_{x \to r^-} \left( \frac{\partial X^i}{\partial x} \right) = \lim_{x \to r^+} \left( \frac{\partial X^i}{\partial x} \right), \quad i = 2, \ldots, D. \tag{2.3}$$

The wave equations (2.1) together with the boundary conditions (2.2) and the continuity conditions (2.3) lead to the equations which determine the eigenfrequencies of the string $\omega$

$$f_1(\omega) = \sin(\alpha \omega) + \delta v \sin(\beta \omega) = 0, \quad (DD),$$
$$f_2(\omega) = \cos(\alpha \omega) - \delta v \cos(\beta \omega) = 0, \quad (DN),$$
$$f_3(\omega) = \cos(\alpha \omega) + \delta v \cos(\beta \omega) = 0, \quad (ND),$$
$$f_4(\omega) = \sin(\alpha \omega) - \delta v \sin(\beta \omega) = 0, \quad (NN), \tag{2.4}$$
where
\[ \delta v = \frac{v_1 - v_2}{v_1 + v_2}, \quad \alpha = \frac{r}{v_1} + \frac{R - r}{v_2}, \quad \text{and} \quad \beta = \frac{r}{v_1} - \frac{R - r}{v_2}. \]
Let us note, that \( \alpha > 0 \) and \( \alpha > \beta \).

III. GROUND STATE ENERGY OF THE STRING AND ITS RENORMALIZATION

When calculating the Casimir energy, the main problem encountered here is the removal of the divergences, which arise inevitably in such calculations. In quantum field theory [17] this procedure is accomplished in the course of the renormalization of the physical parameters, which specify the field theory model under investigation (the masses of the quanta, coupling constants and so on). In order for this procedure to be correct from mathematical standpoint, divergent expressions should be regularized, the regularization being removed after renormalization.

We define the renormalized Casimir energy in a standard way [1]
\[ E_C = \frac{1}{2} \sum_n (\omega_n - \overline{\omega}_n), \quad (3.1) \]
where the \( \omega \)'s are the eigenfrequencies of the string, i.e., the roots of the frequency equations (2.2), and the \( \overline{\omega} \)'s are the roots of the same equations but in the limit when the total string length \( R \) and \( r \) tend to infinity, and the difference of the velocities \( \delta v \) tends to zero. Hence in this limit the initial string becomes a uniform infinite string. In quantum field theory this implies the removal of the Minkowski space contribution [1]. In the Appendix it is shown that the method of calculation of the vacuum energy, outlined above, is completely equivalent to the zeta regularization in the case of usual uniform string.

It is convenient to represent the sum (3.1) in terms of the contour integral [18]
\[ E_C = \frac{D - 2}{4\pi i} \oint_C dz \frac{d}{dz} \ln \frac{f(z)}{\overline{f}(z)}, \quad (3.2) \]
where the function \( f(z) \) is any function \( f_l(z), \quad l = 1, \ldots, 4 \) defining the frequency equations in (2.4), and \( \overline{f} \) is obtained from \( f \) when passing to the limit described above. The contour \( C \) encloses all the positive roots of the equation \( f(\omega) = 0 \). This contour can be deformed into the semicircle \( C_\Lambda \) of radius \( \Lambda \) in the right half-plane and the interval of the imaginary axes \( (-i\Lambda, i\Lambda) \). When the radius \( \Lambda \) is fixed, the contour \( C \) encloses a finite number of the roots of the equation \( f(z) = 0 \). The sum of these roots is obviously finite. Hence the radius \( \Lambda \) is a regularization parameter, and taking the limit \( \Lambda \to \infty \) means the removal of the regularization.

The subtraction in Eq. (3.2) accomplished with the function \( \overline{f}(z) \) can be interpreted as the renormalization of the parameters which specify the classical energy of the string. From the general consideration one can assume that this energy is proportional to the total length of the string \( R \), i.e., \( E^{cl} = T \cdot R \), where \( T \) is a dimensional parameter \([T] = L^{-2}\). It is determined by a concrete string model that should certainly involve nonlinearities. For
example, in the relativistic string model [19] this parameter is the string tension. Due to the quantum corrections the string tension should be renormalized in the following way

\[ T_{\text{ren}} = T + \frac{D-2}{R} \frac{1}{4\pi i} \oint_C dz \frac{d}{dz} \ln f(z). \]  

(3.3)

After removing the regularization \((\Lambda \to \infty)\) the integral in (3.3) obviously diverges. However it is assumed that this divergence is canceled by the respective infinity of the “bare” string tension \(T\). As a result the renormalized (physical) string tension \(T_{\text{ren}}\) proves to be finite. Here we follow the standard procedure of removing the divergences in quantum field theory [17]. In view of the oscillating behavior of the functions \(f_l(z), \ l = 1, \ldots, 4\) on the semicircle \(C_\Lambda\) their asymptotics for \(R \to \infty\) are simply these functions themselves, i.e., \(f_l(z) = f_l(z), \ z \in C_\Lambda\). Therefore the integration along this part of the contour \(C\) does not give contribution into the Casimir energy (3.2). On the imaginary axes the asymptotics of the functions \(f_l(z)\) when \(R, r \to \infty\) can be find easy

\[ f_l(z) \to \frac{e^{a y}}{2}, \ \alpha = \frac{r}{v_1} + \frac{R-r}{v_2}. \]  

(3.4)

As a result we obtain the final expression for the Casimir energy

\[ E_C = \frac{D-2}{2\pi} \int_0^\infty dy \ln [h_l(y)], \ l = 1, \ldots, 4 \]  

(3.5)

with the functions \(h_l(y)\) given by

\[ h_1(y) = 1 - e^{-2a y} + 2\delta v e^{-a y} \sinh(\beta y), \ (\text{DD}), \]
\[ h_2(y) = 1 + e^{-2a y} - 2\delta v e^{-a y} \cosh(\beta y), \ (\text{DN}), \]
\[ h_3(y) = 1 + e^{-2a y} + 2\delta v e^{-a y} \cosh(\beta y), \ (\text{ND}), \]
\[ h_4(y) = 1 - e^{-2a y} - 2\delta v e^{-a y} \sinh(\beta y), \ (\text{NN}). \]  

(3.6)

As a simple consistency check, we apply the formulas (3.5) and (3.6) to the uniform string \((v_1 = v_2 \equiv v)\) of length \(R\) with the expected result [20]

\[ E_C = \frac{D-2}{2\pi} \int_0^\infty dy \ln \left(1 - e^{-2\frac{\pi}{r} y}\right) = -\frac{\pi(D-2)}{24R} v \]

in the DD and NN cases and

\[ E_C = \frac{D-2}{2\pi} \int_0^\infty dy \ln \left(1 + e^{-2\frac{\pi}{r} y}\right) = \frac{\pi(D-2)}{48R} v \]

in the DN and ND cases.

Let us consider the string configuration, when the second piece of the string becomes infinitely long (i.e. \(R\) tends to infinity, while \(r\) remains fixed). Then we have

\[ h_1^\infty(y) = h_2^\infty(y) = 1 - \delta v e^{-2\frac{\pi}{r} y} \equiv h_-(y), \ (\text{DD}), \ (\text{DN}), \]
\[ h_3^\infty(y) = h_4^\infty(y) = 1 + \delta v e^{-2\frac{\pi}{r} y} \equiv h_+(y), \ (\text{ND}), \ (\text{NN}). \]  

(3.7)
These formulas imply that when $R \to \infty$ the Casimir energy becomes independent of the type of the boundary condition imposed at $x = R$. Certainly, this is an appealing property of the vacuum energy of the string under consideration. For $R \to \infty$ it is also possible to obtain the explicit form of this energy

$$E_\pm = \frac{D - 2}{2\pi} \int_0^\infty dy \ln [h_\pm(y)]$$

$$= \frac{D - 2}{2\pi} \int_0^\infty dy \ln \left(1 \pm \delta v \ e^{-2\pi y/\Omega_1}\right)$$

$$= -\frac{D - 2}{4\pi} \frac{v_1}{r} \text{Li}_2(\pm \delta v), \quad (3.8)$$

where

$$\text{Li}_\nu(z) = \sum_{k=1}^\infty \frac{x^k}{k^\nu}, \quad |z| < 1,$$

$$= \frac{z}{\Gamma(\nu)} \int_0^\infty \frac{t^{\nu-1}dt}{e^t - z}, \quad \Re \nu > 0, \quad |\text{arg}(1 - z)| < \pi$$

is the polylogarithm function [21].

Our consideration of non-uniform string can be generalized in a straightforward way to the finite temperature. It can be done by the substitution [22]

$$dy \to 2\pi \theta \sum_{n=0}^\infty \delta(y - \Omega_n) \ dy, \quad (3.9)$$

where $\theta$ is the temperature, $\Omega_n = 2\pi n \theta$ are the Matsubara frequencies, and the prime means that the term $n = 0$ is taken with half weight and

**IV. CONCLUSION**

The non-uniform string with finite $r$ and $R \to \infty$ is, in some sense, an analogue of the radial part of the problem concerning the calculation of the vacuum energy of a pure dielectric compact ball placed in a homogeneous unbounded medium or cavity in such a medium [6,9]. The main lesson of our consideration is the following. When defining the counter term in Eq. (3.1), in order to renormalize the Casimir energy, we take not only the limit $R, r \to \infty$ but also put $\delta v \to 0$. It means that all the physical parameters, specifying the problem under study, should be involved in determination of the counter term.

It is worth noting that we do not use here the contact terms in order to get a finite result for the Casimir energy.

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APPENDIX: RIEMANN ZETA FUNCTION IN THE UNIFORM STRING MODEL

The representation of the spectral zeta functions in terms of contour integrals, with the integrands being the relevant frequency equations, is of wide use [13,23]. This representation is a direct application of the principle of argument theorem from the complex analysis [18,24]. However, this theorem alone does not afford the required analytical continuation of the zeta function. The details of this technique can be clearly demonstrated by considering the zeta function for the usual uniform string. In this case we are dealing with the Riemann zeta function the analytic continuation of which is well known [18,24].

The eigenfrequencies of the uniform string of the length \( R \) with the Dirichlet boundary conditions

\[ \omega_n = \frac{n\pi}{R}, \quad n = 1, 2, \ldots \]  

(A1)

are the positive roots of the equation

\[ \sin(\omega R) = 0. \]  

(A2)

For simplicity the velocity \( v \) is taken to be 1. Let us define the zeta function in the problem at hand by the standard formula

\[ \zeta(s) = \sum_{n=1}^{\infty} \omega_n^{-s}, \quad \Re s > 1. \]  

(A3)

When \( R = \pi \) we obviously have the Riemann zeta function.

Now we represent the sum (A3) in terms of the contour integral

\[ \zeta(s) = \frac{1}{2\pi i} \oint_C z^{-s} \frac{d}{dz} \ln[\sin(zR)] dz, \] 

(A4)

where the contour \( C \) encloses all the positive roots of Eq. (A2). The point \( z = 0 \) is the branching point of the function \( z^{-s} = \exp(-s \ln z) \) in Eq. (A4). Obviously this point should be left outside the closed contour \( C \) in order to integrate one valued function. For the logarithm we take as usual the branch acquiring real values on the positive half-axes \( 0 < \Re z < \infty \). Let us deform (still formally) the contour \( C \) in a semicircle of infinitely large radius laying in the right half-plane of the complex variable \( z \) and close it by imaginary axes \( -\infty < \Im z < \infty \). For \( \Re s > 1 \) the integration along the semicircle can be dropped. As a result one gets

\[ \zeta(s) = \frac{1}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_{0}^{\infty} dy y^{-s} \frac{d}{dy} \ln[\sinh(yR)] \]  

(A5)

\[ = \frac{R}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_{0}^{\infty} dy y^{-s} \coth(yR). \]  

(A6)

The integral in Eq. (A6) converges at the upper limit only for \( \Re s > 1 \) and at the lower limit for \( \Re s < 0 \). The latter is due to the singular behaviour \( \sim \varepsilon^{-s} \) of the integral along
the semicircle of infinitely small radius $\varepsilon$ at the origin. This contribution vanishes only for $\Re s < 0$. Thus, in the general case the contour $C$ in Eq. (A4) cannot pass through the point $z = 0$.

However this contour can be retained if we take, instead of Eq. (A2), a new frequency equation

$$\frac{\sin(\omega R)}{\omega R} = 0,$$

which has the same positive roots as Eq. (A2), but the point $z = 0$ does not satisfy it. As a result we obtain, instead of Eq. (A6),

$$\zeta(s) = \frac{R}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_0^\infty dy \, y^{-s} \left( \coth(yR) - \frac{1}{yR} \right).$$

(A8)

This integral representation of the zeta function is defined for $1 < \Re s < 2$ and gives in fact the same sum (A3) but with explicit substitution of the roots (A1). Really, let us present the function $\coth(yR)$ in Eq. (A8) as a series (see, for example, Eq. 1.4.21.4 in [25])

$$\coth z = \frac{1}{z} + 2\pi \sum_{n=1}^\infty \frac{1}{z^2 + \pi^2 n^2}.$$  

(A9)

After that integration in (A8) can be done

$$\zeta(s) = \frac{R}{\pi} \sin \left( \frac{\pi s}{2} \right) 2R \sum_{n=1}^\infty \int_0^\infty \frac{y^{1-s} dy}{y^2 R^2 + \pi^2 n^2}$$

$$= \frac{R}{\pi} \sin \left( \frac{2\pi s}{2} \right) \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{1-s}{2} \right) \frac{1}{R} \sum_{n=1}^\infty \left( \frac{n\pi}{R} \right)^{-s}$$

$$= \sum_{n=1}^\infty \left( \frac{n\pi}{R} \right)^{-s}. \quad (A10)$$

Thus the contour integration of the frequency equation (A2) or (A7) gives us nothing new as compared with the sum (A3) with explicit frequencies (A1).

As known [18,24] analytic continuation of the series (A3) with $R = \pi$ to the region $\Re s < 1$ is provided by the formula

$$\zeta_R(s) = \frac{i\Gamma(1-s)}{2\pi} \int_\infty^{(0+)} \frac{(-z)^{s-1}}{e^z - 1} \, dz.$$  

(A11)

Here $\zeta_R(z)$ is the Riemann zeta function, and the path of integration encircles the real positive axes $0 \leq \Re z \leq \infty$. When $\Re s > 1$ the integration contour around the origin can be drawn to the point, and the remaining integral gives

$$\zeta_R(s) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx.$$  

(A12)

Expanding the function $1/(e^x - 1)$ in powers of $e^x$ and integrating each term we arrive at the sum
\[ \zeta_R = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re s > 1. \quad (A13) \]

If \( \Re s < 1 \), the integral around the origin in Eq. (A11) does not vanish, and it should be taken into account when reducing the contour integral representation (A11) to the ordinary integral (see, for example, [24]).

The Riemann zeta function defined in the entire plane \( s \), safe for the point \( s = 1 \), by the contour integral (A11) obeys the Riemann reflection formula [18,24]

\[ \zeta_R(s) = \frac{(2\pi)^s}{\pi} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta_R(1 - s). \quad (A14) \]

This formula relates the values of the zeta function left and right from the line \( \Re s = 1 \). At the point \( s = 1 \) the function \( \zeta_R(s) \) has a simple pole

\[ \zeta_R(s) \big|_{s \to 1} \approx \frac{1}{s - 1} + \gamma, \quad (A15) \]

where \( \gamma \) is the Euler constant. It is this pole that prevents the use of the definition (A13) for \( \Re s < 1 \).

Thus we see that even for analytic continuation of the simple sum (A13) a special choice of the path and the integrand is needed. Direct summation of the frequencies by contour integration does not afford analytic continuation.

However if we define the zeta function for a uniform string by subtracting the vacuum energy of an infinite string from Eq. (A13), as it is usually done when calculating the Casimir energy (see Eq. (3.1)), then the resulting zeta function proves to be the analytic function for \( \Re s < 0 \) and exactly that function which follows from Eq. (A11) or from the reflection formula (A14), that is the same.

So let us define the zeta function in the following way

\[ \zeta(s) = \sum_n \left( \omega_n^{-s} - \bar{\omega}_n^{-s} \right), \quad (A16) \]

where the frequencies \( \bar{\omega} \) are defined by Eq. (A2) when \( R \to \infty \). The corresponding limiting form of this equation can be found unambiguously upon rotating the integration path to the imaginary axes

\[ \lim_{R \to \infty} \sinh(yR) = \frac{e^{yR}}{2}. \quad (A17) \]

Now we apply Eq. (A5) to the both terms in definition (A16)

\[ \zeta(s) = \frac{s}{\pi} \sin \left( \frac{\pi s}{2} \right) \int_0^\infty y^{-s-1} \ln \left( 1 - e^{-2yR} \right) dy. \quad (A18) \]

Here the integration by parts has been done, that is legitimated for \( \Re s < 0 \). Expanding the logarithm in (A18) in power series and integrating each term we obtain for \( \Re s < 0 \)
\[ \zeta(s) = \frac{(2\pi)^s}{\pi} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \]
\[ = \frac{(2\pi)^s}{\pi} \sin \left( \frac{\pi s}{2} \right) \Gamma(1 - s) \zeta(1 - s). \] (A19)

It is exactly the same result that follows from the Riemann reflection formula (A14).

Thus the analytic continuation of the Riemann zeta function (A13) into the region \( \Re s < 1 \), which is given by Eq. (A11), implies in essence that for \( \Re s < 1 \) one considers the initial series (A13) with subtraction (A16). Certainly for \( \Re s > 1 \) the new definition of the zeta function (A16) cannot be reduced to the initial series (A13), hence it cannot play the role of Eq. (A11), i.e., it cannot afford, in a rigorous mathematical sense, analytic continuation of the series (A13) to the region \( \Re s < 1 \). The new definition (A16) simply “guesses” the result of analytic continuation of the Riemann zeta function (A13) to this region.

To our opinion, this consideration shows, in a simple and clear way, the relationship between the analytic continuation, providing the mathematical basis of the zeta renormalization technique, and the method of subtraction or counter terms, widely used by physicists.

Summarizing we arrive at the conclusion that the calculation of the Casimir energy of a non-uniform string in the present paper is completely equivalent to the zeta regularization in the case of a usual uniform string.
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