Four-Neutrino Mixing and Long-Baseline Experiments

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Abstract

We consider the two four-neutrino schemes that are compatible with all neutrino oscillation data. We present the range of the corresponding mixing parameters allowed by the results of neutrino oscillation experiments. We discuss the implications for long-baseline experiments and we suggest the possibility to reveal the presence of CP-violating phases through a comparison of the oscillation probability measured in long-baseline experiments with the corresponding average oscillation probability measured in short-baseline experiments.

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I. INTRODUCTION

The three indications in favor of neutrino oscillations found in solar neutrino experiments [1], in atmospheric neutrino experiments [2] and in the accelerator LSND experiment [3] require the existence of three independent scales of neutrino mass-squared differences (see [4]): $\Delta m_{\text{sun}}^2 \sim 10^{-6} - 10^{-4} \text{eV}^2$ (MSW) or $\Delta m_{\text{sun}}^2 \sim 10^{-11} - 10^{-10} \text{eV}^2$ (VO) [5], $\Delta m_{\text{atm}}^2 \sim 10^{-3} - 10^{-2} \text{eV}^2$ [6,7], $\Delta m_{\text{SBL}}^2 \sim 1 \text{eV}^2$ [3]. Hence, at least four massive neutrinos must exist in nature.

Considering the minimal possibility of four massive neutrinos [8–15], we have the mixing relation

$$\nu_{\alpha L} = \sum_{k=1}^{4} U_{\alpha k} \nu_{kL} \quad (\alpha = e, \mu, \tau, s), \quad (1.1)$$

that connects the flavor neutrino fields $\nu_{\alpha L}$ ($\alpha = e, \mu, \tau, s$) with the fields $\nu_{kL}$ ($k = 1, \ldots, 4$) of neutrinos with masses $m_k$. $U$ is the $4 \times 4$ unitary mixing matrix. The field $\nu_{sL}$ describes sterile neutrinos, that do not participate to weak interactions and do not contribute to the invisible decay width of the $Z$-boson, whose measurement have shown that the number of light active neutrino flavors is three (see [16]), corresponding to $\nu_e$, $\nu_\mu$ and $\nu_\tau$.

The purpose of this paper is to present the implications of the results of neutrino oscillation experiments for the mixing of four light massive neutrinos. We will discuss also some implications for long-baseline neutrino oscillation experiments [17].

The plan of the paper is as follows. In Section II we introduce the two four-neutrino mixing schemes that are allowed by the data and we review some of their properties. In Section III we derive the constraints on four-neutrino mixing obtained from the results of neutrino oscillation experiments. In Sections IV and V we discuss some implications for long-baseline disappearance and appearance experiments, respectively. In Section VI we summarize our achievements.

In the discussion of long-baseline experiments we will neglect matter effects, that may be relevant for long-baseline accelerator experiments whose beams propagate for hundreds of kilometers in the crust of the Earth, but are rather complicated to study in the framework of four-neutrino mixing (see [15]) and will be treated in detail in another article [18].

II. THE MIXING SCHEMES

It has been proved in [10–12] that among all the possible four-neutrino mixing schemes only two can accommodate the results of all neutrino oscillation experiments:

$$\text{(A)} \quad \begin{array}{c} \text{atm} \\ \text{sun} \end{array} \begin{array}{c} m_1 < m_2 < m_3 < m_4 \end{array}, \quad \text{(B)} \quad \begin{array}{c} \text{sun} \\ \text{atm} \end{array} \begin{array}{c} m_1 < m_2 < m_3 < m_4 \end{array}. \quad (2.1)$$
These two spectra are characterized by the presence of two pairs of close masses separated by a gap of about 1 eV which provides the mass-squared difference \( \Delta m^2_{SBL} = \Delta m^2_{41} \) responsible for the short-baseline (SBL) oscillations observed in the LSND experiment. In the scheme A, we have \( \Delta m^2_{\text{atm}} = \Delta m^2_{21} \) and \( \Delta m^2_{\text{sun}} = \Delta m^2_{43} \) whereas in scheme B, \( \Delta m^2_{\text{atm}} = \Delta m^2_{43} \) and \( \Delta m^2_{\text{sun}} = \Delta m^2_{21} \). In order to treat simultaneously the two schemes A and B, let us rename the mass eigenstates indices as

\[
\begin{align*}
  c &= 1, \ d = 2, \ a = 3, \ b = 4, \text{ in scheme A,} \\
  a &= 1, \ b = 2, \ c = 3, \ d = 4, \text{ in scheme B.}
\end{align*}
\] (2.2)

(2.3)

With this convention, in both schemes \( \nu_a \) and \( \nu_b \) are the two massive neutrinos whose mass-squared difference is responsible for solar neutrino oscillations and \( \nu_c \) and \( \nu_d \) are the two massive neutrinos whose mass-squared difference generates atmospheric neutrino oscillations:

\[
\Delta m^2_{\text{sun}} = \Delta m^2_{ba}, \quad \Delta m^2_{\text{atm}} = \Delta m^2_{dc}.
\] (2.4)

For \( \Delta m^2_{SBL} \) in the following we will use the range

\[
0.20 \text{eV}^2 \lesssim \Delta m^2_{SBL} \lesssim 2.0 \text{eV}^2,
\] (2.5)

that is allowed by the 99% CL maximum likelihood analysis of the LSND data in terms of two-neutrino oscillations [3] and by the 90% CL exclusion curves of the Bugey reactor \( \bar{\nu}_e \) disappearance experiment [19] and of the BNL E776 [20] and KARMEN [21] experiments (see also [22])².

The mixing in both schemes A and B is constrained by the results of neutrino oscillation experiments [10–12,4]. In order to quantify these constraints, let us define the quantities \( \Sigma_\alpha \), with \( \alpha = e, \mu, \tau, s \), as

\[
\Sigma_\alpha \equiv \sum_{k=\text{c,d}} |U_{\alpha k}|^2.
\] (2.6)

Physically \( \Sigma_\alpha \) quantify the mixing of the flavor neutrino \( \nu_\alpha \) with the two massive neutrinos whose mass-squared difference is relevant for the oscillations of atmospheric neutrinos (\( \nu_1, \nu_2 \) in scheme A and \( \nu_3, \nu_4 \) in scheme B).

Let us write down the expressions, that we will need in the following, for the oscillation probabilities in short baseline experiments and in atmospheric and long-baseline experiments. The neutrino oscillation probabilities in short-baseline (SBL) experiments are given by [10]

\[\text{In the following we will denote the mass-squared differences } m^2_k - m^2_j \text{ with } \Delta m^2_{kj}.\]

\[\text{In order to obtain reliable results, in this paper we use 99% CL limits when possible. Limits with 90% CL are used only when they are the only available information from the corresponding experiment.}\]
\[ P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{SBL} = \frac{1}{2} A_{\alpha\beta} \left( 1 - \cos \frac{\Delta m_{SBL}^2 L}{2E} \right) \quad (\beta \neq \alpha), \quad (2.7) \]
\[ P_{\bar{\nu}_{\alpha} \rightarrow \bar{\nu}_{\beta}}^{SBL} = 1 - \frac{1}{2} B_{\alpha\alpha} \left( 1 - \cos \frac{\Delta m_{SBL}^2 L}{2E} \right), \quad (2.8) \]

where \( L \) is the source-detector distance, \( E \) is the neutrino energy and the oscillation amplitudes are given by

\[ A_{\alpha\beta} = 4 \left| \sum_{k=a,b \text{ or } k=c,d} U_{\beta k} U_{\alpha k}^* \right|^2, \quad B_{\alpha\alpha} = 4 \Sigma_{\alpha} (1 - \Sigma_{\alpha}). \quad (2.9) \]

The two expressions for \( A_{\alpha\beta} \) obtained by summing over \( k = a, b \) or \( k = c, d \) are equivalent because of the unitarity of the mixing matrix. The oscillation probabilities of neutrinos and antineutrinos in short-baseline experiments are equal. Since the expressions (2.7) and (2.8) have the same form as the usual probabilities in the case of two-neutrino mixing, the oscillation amplitudes \( A_{\alpha\beta} \) and \( B_{\alpha\alpha} \) are equivalent to the usual parameter \( \sin^2 \vartheta \) used in the analysis of the experimental results (\( \vartheta \) is the two-neutrino mixing angle). The average probabilities measured by short-baseline experiments that are sensitive to values of \( \Delta m^2 \) much smaller than the actual value of \( \Delta m_{SBL}^2 \) are given by

\[ \langle P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{SBL} \rangle = \frac{1}{2} A_{\alpha\beta} \quad (\alpha \neq \beta), \quad \langle P_{\bar{\nu}_{\alpha} \rightarrow \bar{\nu}_{\beta}}^{SBL} \rangle = 1 - \frac{1}{2} B_{\alpha\alpha}. \quad (2.10) \]

The probability of \( \nu_{\alpha} \rightarrow \nu_{\beta} \) transitions in atmospheric and long-baseline experiments is given by\(^3\) [10]

\[ P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{LBL} = \left| \sum_{k=a,b} U_{\beta k} U_{\alpha k}^* \right|^2 + \left| U_{\alpha c}^* U_{\beta c} + U_{\alpha d}^* U_{\beta d} \exp \left( -i \frac{\Delta m_{atm}^2 L}{2E} \right) \right|^2. \quad (2.11) \]

The average oscillation probability measured in experiments sensitive to values of \( \Delta m^2 \) smaller than the actual value of \( \Delta m_{atm}^2 \) is given by

\[ \langle P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{LBL} \rangle = \left| \sum_{k=a,b} U_{\alpha k}^* U_{\beta k} \right|^2 + \left( \sum_{k=c,d} |U_{\alpha k}|^2 |U_{\beta k}|^2 \right), \quad (2.12) \]

for both neutrinos and antineutrinos.

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\( ^3 \)When in the following we will need to write the explicit expressions for oscillation probabilities that are different for neutrinos and antineutrinos, we will consider, for simplicity, only neutrinos. The corresponding expressions for antineutrinos are obtained by taking the complex conjugate of the elements of the mixing matrix.
The negative results of short-baseline $\bar{\nu}_e$ and $\nu_\mu$ disappearance experiments and the results of solar neutrino experiments imply that $\Sigma_e$ and $\Sigma_\mu$ are constrained by [10,12]

$$\Sigma_e \leq a_0^e(\Delta m^2_{SBL}) \quad \text{and} \quad \Sigma_\mu \leq a_0^\mu(\Delta m^2_{SBL}) \quad \text{or} \quad 1 - \Sigma_\mu \leq a_0^\mu(\Delta m^2_{SBL}). \quad (3.1)$$

The quantities $a_0^e$ and $a_0^\mu$, that depend on $\Delta m^2_{SBL}$, are given by

$$a_0^\alpha = \frac{1}{2} \left( 1 - \sqrt{1 - B_0^\alpha} \right) \quad (\alpha = e, \mu), \quad (3.2)$$

where $B_0^\alpha$ is the experimental upper bound, that depends on $\Delta m^2_{SBL}$, for the amplitude $B_0^\alpha$, obtained from the exclusion plots of short-baseline $\bar{\nu}_\alpha$ disappearance experiments (see [23]). From the 90% CL exclusion curves of the Bugey reactor $\bar{\nu}_e$ disappearance experiment [19] and of the CDHS [24] accelerator $\nu_\mu$ disappearance experiment it follows that $a_0^e \lesssim 3 \times 10^{-2}$ for $\Delta m^2_{SBL}$ in the LSND range (2.5) and $a_0^\mu \lesssim 0.2$ for $\Delta m^2_{SBL} \gtrsim 0.4$ eV$^2$ (see [23]).

Furthermore, it is clear that the observed disappearance of atmospheric $\nu_\mu$'s implies that $\Sigma_\mu$, that represents the mixing of $\nu_\mu$ with the two massive neutrinos responsible for atmospheric neutrino oscillations, cannot be small. Indeed, it has been shown in [12] that the up-down asymmetry of high-energy $\mu$-like events generated by atmospheric neutrinos measured in the Super-Kamiokande experiment ($A_\mu = 0.311 \pm 0.043 \pm 0.01$ [25]) implies the bound

$$1 - \Sigma_\mu \lesssim 0.55. \quad (3.3)$$

Hence, $\Sigma_\mu$ is large also for $\Delta m^2_{SBL} \lesssim 0.3$ eV$^2$, that is below the sensitivity of the short-baseline $\nu_\mu$ disappearance experiments performed so far.

The constraints in Eqs. (3.1) and (3.3) show that $\Sigma_e$ and $1 - \Sigma_\mu$ are small (in both schemes A and B). The smallness of $\Sigma_e$ implies that the oscillations of $\nu_e$'s and $\bar{\nu}_e$'s in atmospheric and long-baseline experiments are suppressed. Indeed, rather strong bounds on the transition probability $1 - P^\text{LBL}_{\bar{\nu}_e \to \nu_\mu}$ of $\nu_e$'s in long-baseline experiments (LBL) have been derived in [13].

Using the fact that the oscillations of electron neutrinos are negligible in atmospheric experiments and the fact that the Super-Kamiokande up-down asymmetry has been measured through an average over the neutrino energy spectrum and propagation distance, we can derive a lower bound for $\Sigma_\mu$ stronger than the one in Eq. (3.3). We will use the updated [6] value of the Super-Kamiokande up-down asymmetry,

$$A_\mu \equiv \frac{D - U}{D + U} = 0.32 \pm 0.04 \pm 0.01, \quad (3.4)$$

where $D$ and $U$ are the numbers of down-going and up-going $\mu$-like events, respectively. Summing the statistic and systematic errors in quadrature, we obtain

$$A_\mu \geq 0.224 \quad (99\% \text{ CL}). \quad (3.5)$$
Since the asymmetry is due to the disappearance of $\nu_\mu$’s into $\nu_\tau$’s or $\nu_s$’s, we have\(^4\)

\[
A_\mu = \frac{\langle P_{\nu_\mu \rightarrow \nu_\mu}^{SBL} \rangle - \langle P_{\nu_\mu \rightarrow \nu_\mu}^{LBL} \rangle}{\langle P_{\nu_\mu \rightarrow \nu_\mu}^{SBL} \rangle + \langle P_{\nu_\mu \rightarrow \nu_\mu}^{LBL} \rangle},
\]

(3.6)

where $\langle P_{\nu_\mu \rightarrow \nu_\mu}^{SBL} \rangle$ is the average survival probability of $\nu_\mu$’s in short-baseline experiments, that is the same as the average survival probability of downward-going atmospheric $\nu_\mu$’s, and $\langle P_{\nu_\mu \rightarrow \nu_\mu}^{LBL} \rangle$ is the average survival probability of $\nu_\mu$’s in long-baseline experiments, that is the same as the average survival probability of upward-going atmospheric $\nu_\mu$’s. From Eqs. (2.10), (2.12) and from the definition (2.6), we have

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu}^{SBL} \rangle = 1 - 2\Sigma_\mu(1 - \Sigma_\mu),
\]

(3.7)

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu}^{LBL} \rangle = \langle P_{\nu_\mu \rightarrow \nu_\mu}^{SBL} \rangle - 2|U_{\mu d}|^2(\Sigma_\mu - |U_{\mu d}|^2).
\]

(3.8)

Taking into account the obvious constraint $0 \leq |U_{\mu d}|^2 \leq \Sigma_\mu$, the long-baseline average survival probability of $\nu_\mu$’s is bounded by

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu}^{SBL} \rangle - \frac{\Sigma_\mu^2}{2} \leq \langle P_{\nu_\mu \rightarrow \nu_\mu}^{LBL} \rangle \leq \langle P_{\nu_\mu \rightarrow \nu_\mu}^{SBL} \rangle.
\]

(3.9)

Using the lower bound in Eq. (3.9), for the asymmetry (3.6) we have the upper bound

\[
A_\mu \leq \frac{\Sigma_\mu^2}{7\Sigma_\mu^2 - 8\Sigma_\mu + 4}.
\]

(3.10)

Inverting this inequality, we get

\[
\Sigma_\mu \geq \frac{4A_\mu - 2\sqrt{A_\mu(1 - 3A_\mu)}}{7A_\mu - 1}.
\]

(3.11)

This is the lower bound for $\Sigma_\mu$ given by the asymmetry (3.6). Thus, in the framework of the four-neutrino schemes under consideration the asymmetry $A_\mu$ cannot be larger than $1/3$, a condition that is satisfied by the measured asymmetry (3.4). From Eqs. (3.5) and (3.11) we obtain

\[
1 - \Sigma_\mu \lesssim 0.38 \equiv a_\mu^{SK}.
\]

(3.12)

This is the new upper bound for $1 - \Sigma_\mu$, that is more stringent than the one in Eq. (3.3).

\(^4\)Here we neglect corrections due to matter effects, whose calculation would require a complicated fit of the data, beyond our possibilities. However, the matter corrections are expected to be small because of the average over a wide range of neutrino energies.
Furthermore, using the expression (3.8) for \( \langle P_{\bar{\nu}_\mu}^{\text{LBL}} \rangle \), we can derive the range of \( |U_{\mu d}|^2 \) that is allowed by the lower bound (3.5) for \( \mathcal{A}_\mu \) (if \( |U_{\mu d}|^2 \) were allowed to vary in the whole range \([0, \Sigma]\), the asymmetry could vanish). After a straightforward calculation we obtain that \( |U_{\mu d}|^2 \) is bounded in the range

\[
\lambda^-_\mu \leq |U_{\mu d}|^2 \leq \lambda^+\mu ,
\]

with

\[
\lambda^\pm_\mu = \frac{\Sigma_\mu}{2} \pm \frac{1}{2} \sqrt{\frac{(1 - 7 A_\mu) \Sigma_\mu^2 - 4 A_\mu (1 - 2 \Sigma_\mu)}{1 + A_\mu}} .
\]

The allowed range for \( |U_{\mu d}|^2 \) as a function of \( \Sigma_\mu \) obtained from the lower bound (3.5) for the asymmetry is shown by the shadowed area in Fig. 1. Since \( \Sigma_\mu \) can be close to one, we have the allowed range

\[
0.21 \lesssim |U_{\mu d}|^2 \lesssim 0.76 .
\]

The same allowed range obviously applies also to \( |U_{\mu c}|^2 \), with the constraint (2.6) (for example, from Fig. 1 one can see that \( |U_{\mu d}|^2 \simeq 0.21 \) is possible only if \( \Sigma_\mu \simeq 0.8 \) and the corresponding value of \( |U_{\mu c}|^2 \) is 0.59, well within the range (3.15)).

Up to now we have derived the upper bounds for \( \Sigma_\mu \) and \( 1 - \Sigma_\mu \) implied by the results of neutrino oscillation experiments. However, \( \Sigma_\mu \) and \( 1 - \Sigma_\mu \), albeit small, cannot vanish, because non-zero values of both \( \Sigma_\mu \) and \( 1 - \Sigma_\mu \) are required in order to generate the \( \bar{\nu}_e \to \bar{\nu}_\mu \) and \( \nu_e \to \nu_\mu \) transitions observed in the LSND experiment [3]. Indeed, we are going to derive lower bounds for \( \Sigma_\mu \) and \( 1 - \Sigma_\mu \) from the results of the LSND experiment.

Let us consider the amplitude \( A_{\mu e} \) of \( \bar{\nu}_e \to \bar{\nu}_\mu \) and \( \nu_e \to \nu_\mu \) oscillations in the LSND experiment. From Eq. (2.9), using the Cauchy–Schwarz inequality, we obtain two inequalities:

\[
A_{\mu e} \leq 4 \Sigma_e \Sigma_\mu , \tag{3.16}
\]

\[
A_{\mu e} \leq 4 (1 - \Sigma_e) (1 - \Sigma_\mu) . \tag{3.17}
\]

It is clear that both \( \Sigma_e \) and \( 1 - \Sigma_\mu \) must be bigger than zero and smaller than one in order to have a non-vanishing amplitude \( A_{\mu e} \) in the LSND experiment.

From the inequalities (3.16) and (3.17) we obtain the upper bound

\[
1 - \Sigma_\mu \leq 1 - \frac{A_{\min e}}{4 \sigma_{\mu}^e} \equiv \sigma_{\mu}^{\text{LSND}} , \tag{3.18}
\]

already derived in [12], and the lower bounds

\[
\Sigma_e \geq \Lambda^-_e \quad \text{and} \quad 1 - \Sigma_\mu \geq \Lambda^-_\mu , \tag{3.19}
\]

with

\[
\Lambda^-_e = \Lambda^-_\mu = \frac{1}{2} \left( 1 - \sqrt{1 - A_{\min e}} \right) \simeq \frac{A_{\min e}}{4} , \tag{3.20}
\]
where $A_{\mu e}^{\text{min}}$ is the minimum value of the amplitude $A_{\mu e}$ allowed by the results of the LSND experiment and the approximation is valid for small $A_{\mu e}^{\text{min}}$, as follows from the LSND data. Obviously, the minimum (3.20) depends on $\Delta m_{SBL}^2$ through the dependence from $\Delta m_{SBL}^2$ of $A_{\mu e}^{\text{min}}$.

Summarizing, from the results of all neutrino oscillation experiments we have the bounds

$$
\Lambda_\mu^- (\Delta m_{SBL}^2) \leq \Sigma_\mu \leq 1 - \Lambda_\mu^+ (\Delta m_{SBL}^2),
$$

(3.21)

with $\Lambda_\mu^- = \Lambda_\mu^+$ given by Eq.(3.20) and

$$
\Lambda_\mu^+ = a_\mu^0, \quad \Lambda_\mu^+ = \min \left[ a_\mu^0, a_{\mu}^{SK}, a_{\mu}^{LSND} \right].
$$

(3.22)

In Eq. (3.21) we have emphasized that the bounds for $\Sigma_\mu$ and $\Sigma_\mu$ depend on the value of $\Delta m_{SBL}^2$, that must lie in the LSND range (2.5).

The dashed line in Fig. 2 represents the lower bound (3.19) for $\Sigma_\mu$ obtained from the 99% CL maximum likelihood allowed region of the LSND experiment [3]. The solid line in Fig. 2 is the upper bound (3.1) for $\Sigma_\mu$ obtained from the 90% CL exclusion curve of the Bugey reactor $\bar{\nu}_e$ disappearance experiment [19].

The dash-dotted line in Fig. 2 shows the upper bound for $\Sigma_\mu$ obtained from the negative result of the CHOOZ long-baseline reactor $\bar{\nu}_e$ disappearance experiment [26]. This bound has been obtained taking into account that the survival probability of $\bar{\nu}_e$’s in long-baseline experiments is bounded by [13]

$$
P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}^{\text{LRL}} \leq \Sigma_\mu^2 + (1 - \Sigma_\mu)^2.
$$

(3.23)

The CHOOZ experiment has measured a survival probability averaged over the energy spectrum [26]

$$
P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}^{\text{CHOOZ}} = 1.010 \pm 0.028 \pm 0.027 = 1.010 \pm 0.039,
$$

(3.24)

where we added in quadrature the independent statistical and systematic errors. Assuming a Gaussian error distribution and taking into account that the survival probability is physically limited in the interval $[0, 1]$, we obtain the 90% CL lower bound\(^5\)

$$
P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}^{\text{CHOOZ}} \geq 0.942 \equiv P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}^{\text{CHOOZ, min}}.
$$

(3.25)

Taking into account the fact that $\Sigma_\mu$ is small, from Eqs. (3.23) and (3.25) we obtain

$$
\Sigma_\mu \leq \frac{1}{2} \left( 1 - \sqrt{1 - 2 \left( 1 - P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}^{\text{CHOOZ, min}} \right)} \right) = 3.0 \times 10^{-2}.
$$

(3.26)

\(^5\)The lower bound (3.25) has been obtained using a Gaussian distribution normalized in the interval $[0, 1]$, according to the Bayesian prescription (see [27,16,28]). The lower bound obtained with the Unified Approach [29] is $P_{\bar{\nu}_e \rightarrow \bar{\nu}_e}^{\text{CHOOZ}} \geq 0.945$, in reasonable agreement with (3.25).
This is the upper bound represented by the dash-dotted line in Fig. 2. One can see that the CHOOZ bound is compatible with the Bugey bound (solid line) and a more sensitive experiment it is necessary in order to probe the range of $\Sigma_e$ allowed by the results of the LSND experiment (shadowed area).

The bounds (3.1), (3.12), (3.18) and (3.19) for $1 - \Sigma_\mu$ are illustrated in Fig. 3. The shadowed area enclosed by the solid line is excluded by the 90% CL exclusion curve of the CDHS [24] short-baseline $\nu_\mu$ disappearance experiment using Eq. (3.1) ($1 - \Sigma_\mu \leq a^0_\mu$ or $\Sigma_\mu \leq a^0_\mu$ [10,23]). The vertically hatched area limited by the dash-dotted line is excluded by the inequality (3.12) obtained the up-down asymmetry of high-energy $\mu$-like events generated by atmospheric neutrinos measured in the Super-Kamiokande experiment. The diagonally hatched region is excluded by the results of the LSND experiment, taking into account also the Bugey 90% CL exclusion curve. The region above the dashed line is excluded by the upper bound (3.18) and the region below the dotted line is excluded by the lower bound (3.19). Taking into account all these constraints, the allowed region for $1 - \Sigma_\mu$ is given by the white area in Fig. 3.

IV. LONG-BASELINE DISAPPEARANCE EXPERIMENTS

From the expression (2.11) with $\alpha = \beta$ (and the corresponding one for antineutrinos) it follows that the probability of $^{(\gamma)}\nu_\alpha$ transitions in other states in long-baseline experiments is bounded by [13]

$$2\Sigma_\alpha (1 - \Sigma_\alpha) \leq 1 - P_{^{(\gamma)}_\alpha \rightarrow^{(\gamma)}_\alpha}^{\text{LBL}} \leq 2\Sigma_\alpha (1 - \frac{1}{2} \Sigma_\alpha).$$

(4.1)

It is interesting to notice that the lower bound coincides with the average probability of $\nu_\alpha$ transitions in other states in short-baseline experiments. Indeed, from Eqs. (2.8) and (2.9) we have

$$1 - \langle P_{^{(\gamma)}_\alpha \rightarrow^{(\gamma)}_\alpha}^{\text{SBL}} \rangle = \frac{1}{2} B_{\alpha\alpha} = 2\Sigma_\alpha (1 - \Sigma_\alpha).$$

(4.2)

This is obviously due to the fact that long-baseline $\nu_\alpha$ disappearance experiments are sensitive also to short-baseline oscillations due to $\Delta m^2_{\text{SBL}}$, but since these oscillations are very fast, only their average is observed in long-baseline experiments. The surprising fact is that the interference terms in the long-baseline disappearance probability are such that the disappearance in long-baseline experiments cannot be smaller than the average disappearance in short-baseline experiments. As we will see later, the analogous propriety for $\nu_\alpha \rightarrow \nu_\beta$ transitions is not true.

If we consider $\alpha = e$ in Eq. (4.1), taking into account the bound (3.21) for $\Sigma_e$, we have

$$\frac{A_{\mu e}^{\text{min}}}{2} \simeq 2\Lambda^- (1 - \Lambda^-) \leq 1 - P_{^{(\gamma)}_{\mu \rightarrow^{(\gamma)}_e}}^{\text{LBL}} \simeq 2\Lambda^+ (1 - \frac{1}{2} \Lambda^+) \simeq 2a^0_e,$$

(4.3)

with the approximations applying, respectively, for small $A_{\mu e}^{\text{min}}$ and small $a^0_e$, as realized in practice. Figure 4 shows the values of the bounds (4.3) obtained from the existing data.
The solid line in Fig. 4 shows the upper bound in (4.3) obtained from the exclusion curve of the Bugey $\bar{\nu}_e$ short-baseline disappearance experiment and the dotted line shows the lower bound in (4.3) obtained from the lower edge of the region in the $A_{\mu e}-\Delta m_{SBL}^2$ plane allowed at 99% CL by the results of the LSND experiment. The shadowed region in Fig. 4 is allowed by all data and one can see that it extends below the CHOOZ bound (dash-dotted line obtained from Eq. (3.25)). The lower bound for $1-P^{LBL}_{\nu_\alpha \rightarrow \nu_\alpha}$ lies above $7 \times 10^{-4}$, a limit that will be very difficult to reach. However, if $\Delta m^2_{SBL}$ lies in the lower part of the LSND-allowed region, the lower bound $1-P^{LBL}_{\nu_\alpha \rightarrow \nu_\alpha}$ is substantially higher and may be reachable in the near future (see [30]).

Considering the disappearance of $\nu_\mu$’s in long-baseline experiments, it is clear that, since $\Sigma_\mu$ is large, the corresponding probability can vary almost from zero to one. Indeed, using the limits (3.21) for $\Sigma_\mu$ in Eq.(4.1) we obtain

$$A_{\mu e}^{\text{min}} \simeq 2\Lambda^- (1-\Lambda^-) \leq 1 - P^{LBL}_{\nu_\alpha \rightarrow \nu_\alpha} \leq \left( 1 - \Lambda^- \right) \left( 1 + \frac{1}{2} \Lambda^- \right) \simeq 1 - \frac{A_{\mu e}^{\text{min}}}{8}. \quad (4.4)$$

The expression (4.1) gives the bounds for the transition probability $1-P^{LBL}_{\nu_\alpha \rightarrow \nu_\alpha}$, that depends on the neutrino energy. However, if a long baseline experiment is sensitive to values of $\Delta m^2$ smaller than the actual value of $\Delta m^2_{\text{atm}}$, its transition probability is given by the average (2.12). For $\alpha = \beta$ in Eq. (2.12), taking into account the definition of $\Sigma_\alpha$ in Eq. (2.6) and the obvious constraints $|U_{\alpha c}|^2, |U_{\alpha d}|^2 \leq \Sigma_\alpha$, one can derive the bounds

$$2\Sigma_\alpha (1-\Sigma_\alpha) \leq 1 - \langle P^{LBL}_{\nu_\alpha \rightarrow \nu_\alpha} \rangle \leq 2\Sigma_\alpha \left( 1 - \frac{3}{4} \Sigma_\alpha \right). \quad (4.5)$$

Therefore, only the upper bound for the average transition probability is different from the corresponding one for the transition probability, but the difference in practice is negligible for $\alpha = e$, because $\Sigma_e$ is very small. Hence, the bounds in Fig. 4 apply also to $1-\langle P^{LBL}_{\nu_\alpha \rightarrow \nu_\alpha} \rangle$.

For the average transition probability of $\nu_\mu$’s into other states, a lower bound that is stronger than the lower bound implied by Eq. (4.5) (that coincides with the one in Eq. (4.4)) is given by the asymmetry (3.6), from which we get

$$\langle P^{LBL}_{\nu_\mu \rightarrow \nu_\mu} \rangle = \frac{1 - A_\mu}{1 + A_\mu} \langle P^{SBL}_{\nu_\mu \rightarrow \nu_\mu} \rangle. \quad (4.6)$$

Taking into account the limit (3.5) for $A_\mu$ and approximating the upper bound $\langle P^{\text{SBL}}_{\nu_\mu \rightarrow \nu_\mu} \rangle \leq 1 - 2\Lambda^- (1-\Lambda^-)$ with one, we obtain

$$1 - \langle P^{LBL}_{\nu_\mu \rightarrow \nu_\mu} \rangle \gtrsim 0.37. \quad (4.7)$$

This is the lower bound for the average transition probability of $\nu_\mu$’s into other states in order to accommodate the up-down asymmetry observed in the Super-Kamiokande experiment. We have also an upper bound for this transition probability, that is given by Eq. (4.5):
\[ 1 - \langle P_{\nu_\mu \to \nu_\mu}^{\text{LBL}} \rangle \leq \begin{cases} 
\frac{(1 - \Lambda_\mu^+)(1 + 3\Lambda_\mu^+)}{2} & \text{for } \Lambda_\mu^+ < 1/3, \\
2/3 & \text{for } \Lambda_\mu^+ \geq 1/3, 
\end{cases} \tag{4.8} \]

where we have taken into account the fact that, from Eqs. (3.12) and (3.22), \( \Lambda_\mu^+ \) can be as large as about 0.38. It is interesting to notice that there is the possibility that \( 1 - \langle P_{\nu_\mu \to \nu_\mu}^{\text{LBL}} \rangle \) is larger than \( 1/2 \). Since in the case of two-neutrino mixing the average transition probability of any flavor neutrino into other states is smaller than \( 1/2 \) (equal in the case of maximal mixing), the measurement in long-baseline experiments of an average transition probability of \( \nu_\mu \)'s into other states larger than \( 1/2 \) would be an evidence of the existence of a neutrino mass-squared difference larger than the atmospheric one and more than two-neutrino mixing. Equation (4.8) shows that this is possible in the framework of four-neutrino mixing, given the existing experimental data.

V. LONG-BASELINE APPEARANCE EXPERIMENTS

Taking into account the definition (2.9) for \( A_{\alpha\beta} \), from Eq. (2.11) it follows that the probability of \( (\nu_\alpha \to \nu_\beta) \) transitions with \( \alpha \neq \beta \) in long-baseline experiments is bounded by [13]

\[ \frac{1}{4} A_{\alpha\beta} \leq p_{\nu_\alpha \to \nu_\beta}^{\text{LBL}} \leq \frac{1}{4} A_{\alpha\beta} + \Sigma_\alpha \Sigma_\beta, \tag{5.1} \]

and by the unitarity upper limits

\[ p_{\nu_\alpha \to \nu_\beta}^{\text{LBL}} \leq 1 - p_{\nu_\alpha \to \nu_\beta}^{\text{SBL}}, \quad p_{\nu_\alpha \to \nu_\beta}^{\text{LBL}} \leq 1 - p_{\nu_\beta \to \nu_\alpha}^{\text{SBL}}. \tag{5.2} \]

The corresponding allowed range for \( p_{\nu_\alpha \to \nu_\beta}^{\text{LBL}} \) obtained from the experimental data has been presented in Ref. [13]. The other constrained channels are \( \nu_\alpha \to \nu_\beta \) with \( \alpha = e, \mu \) and \( \beta = \tau, s \), for which we have only the upper limits

\[ p_{\nu_\alpha \to \nu_\beta}^{\text{LBL}} \leq 1 - p_{\nu_\alpha \to \nu_\alpha}^{\text{SBL}}, \quad (\alpha = e, \mu \quad \beta = \tau, s), \tag{5.3} \]

due to the conservation of probability, with the upper limits for \( 1 - p_{\nu_\alpha \to \nu_\alpha}^{\text{SBL}} \) and \( 1 - p_{\nu_\mu \to \nu_\mu}^{\text{SBL}} \) given in Eqs. (4.3) and (4.4), respectively.

Confronting the interval (5.1) with the expression for the average probability of \( (\nu_\alpha \to \nu_\beta) \) transitions in short-baseline experiments given in Eq. (2.10), one can see that, as anticipated in the discussion after Eq. (4.2), the probability of long-baseline \( (\nu_\alpha \to \nu_\beta) \) transitions can be smaller than the average probability of short-baseline \( (\nu_\alpha \to \nu_\beta) \) transitions. This is due to interference effects produced by \( \Delta m^2_{\text{atm}} \). Hence we have

\[ p_{\nu_\alpha \to \nu_\beta}^{\text{LBL}} \leq \langle p_{\nu_\alpha \to \nu_\beta}^{\text{SBL}} \rangle \quad \text{(possible)}. \tag{5.4} \]

For a fixed source-detector distance in a long-baseline experiment, the realization of the possible inequality (5.4) depends on the value of the neutrino energy. Therefore, using
a detector with sufficient energy resolution (or with a sufficiently narrow-band beam and varying the neutrino energy) it is possible to check experimentally if the inequality (5.4) can be realized. The inequality (5.4) is interesting because it can be realized only if there are at least two mass-squared differences, $\Delta m^2_{\text{SBL}}$ and $\Delta m^2_{\text{atm}}$. Indeed, it is impossible in the case of two-neutrino mixing that implies, with an obvious notation,

$$P_{\nu^c \nu} = \langle P_{\nu^c \nu} \rangle \quad \text{only } \Delta m^2_{\text{SBL}},$$

$$P_{\nu^c \nu} \geq \langle P_{\nu^c \nu} \rangle = 0 \quad \text{only } \Delta m^2_{\text{atm}}.$$  \hfill (5.5)

However, the possibility of the inequality (5.4) is not unique to the four-neutrino schemes, and it can be realized, for example, in three-neutrino schemes with $\Delta m^2_{31} = \Delta m^2_{\text{SBL}}$ and $\Delta m^2_{21} = \Delta m^2_{\text{atm}}$.

The experimental check of the inequality (5.4) could provide some information on the relative phases of the elements of the mixing matrix. Indeed, confronting Eqs. (2.11) and (2.10), taking into account of the definition (2.9) for $A_{\alpha\beta}$, one can see that the inequality (5.4) for neutrinos is realized if

$$\text{Re} \left[ U_{\alpha c} U^{\ast}_{\beta c} U^{\ast}_{\alpha d} U_{\beta d} \exp \left( \frac{-i \Delta m^2_{\text{atm}} L}{2E} \right) \right] < \text{Re} \left[ U_{\alpha c} U^{\ast}_{\beta c} U^{\ast}_{\alpha d} U_{\beta d} \right].$$

The product $U_{\alpha c} U^{\ast}_{\beta c} U^{\ast}_{\alpha d} U_{\beta d}$ is invariant under rephasing of the mixing matrix, i.e. the transformations $U_{ak} \to e^{i\gamma_k} U_{ak}$ and the transformations $U_{ak} \to e^{-i\gamma_k} U_{ak}$. Using these transformations we can consider, without loss of generality, $U_{ac} = |U_{ac}|$, $U_{\beta c} = |U_{\beta c}|$, $U_{ad} = |U_{ad}|$, $U_{\beta d} = |U_{\beta d}| e^{i\xi}$ and write the inequality (5.7) as

$$\cos(\xi - \varphi) < \cos \xi,$$

where $\varphi \equiv \Delta m^2_{\text{atm}} L/2E$. If we confine $\xi$ in the interval $[0, 2\pi)$, the inequality (5.8) is not realized for any value of $\varphi$ only if

$$\xi = \pi,$$

and it is realized for all values of $\varphi \neq 0$ only if

$$\xi = 0.$$  \hfill (5.10)

Since the two extreme cases (5.9) and (5.10) correspond to the absence of CP-violating phases in the $\alpha, \beta$-1, 2 sector of the mixing matrix, it is clear that the observation of the inequality (5.4) for some, but not all values of $\varphi$ would be an evidence in favor of CP violation in the lepton sector due to neutrino mixing. This fact can also be seen by noting that if CP is conserved all the elements of the mixing matrix are real and there are two possibilities:

1) $\frac{1}{4} A_{\alpha \beta} = |U_{\alpha c} U_{\beta c} + U_{ad} U_{\beta d}|^2 = (|U_{\alpha c} U_{\beta c}| + |U_{ad} U_{\beta d}|)^2$ if $U_{\alpha c} U_{\beta c} U_{ad} U_{\beta d} > 0$, \hfill (5.11)

2) $\frac{1}{4} A_{\alpha \beta} = |U_{\alpha c} U_{\beta c} + U_{ad} U_{\beta d}|^2 = (|U_{\alpha c} U_{\beta c}| - |U_{ad} U_{\beta d}|)^2$ if $U_{\alpha c} U_{\beta c} U_{ad} U_{\beta d} < 0$. \hfill (5.12)
In the first case we have
\[ |U_{\alpha\beta}U_{\beta\epsilon} + U_{\alpha\delta}U_{\delta\epsilon}e^{-i\varphi}|^2 \leq \frac{1}{4} A_{\alpha\beta}, \quad (5.13) \]
and
\[ P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{\text{LBL}} (\varphi = \pi) - P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{\text{LBL}} (\varphi = 0) = -4|U_{\alpha\epsilon}U_{\beta\epsilon}U_{\alpha\delta}U_{\delta\epsilon}| \cos \xi, \quad (5.17) \]
\[ P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{\text{LBL}} (\varphi = \frac{3}{2} \pi) - P_{\nu_{\alpha} \rightarrow \nu_{\beta}}^{\text{LBL}} (\varphi = \frac{1}{2} \pi) = -4|U_{\alpha\epsilon}U_{\beta\epsilon}U_{\alpha\delta}U_{\delta\epsilon}| \sin \xi. \quad (5.18) \]

The ratio of Eqs. (5.18) and (5.17) gives \( \tan \xi \) independently from the value of \( |U_{\alpha\epsilon}U_{\beta\epsilon}U_{\alpha\delta}U_{\delta\epsilon}| \).

Let us finally consider the average transition probabilities that are measured in long-baseline experiments that are sensitive to values of \( \Delta m^2 \) much smaller than the actual value of \( \Delta m^2_{\text{atm}} \). Considering \( \bar{\nu}_{\mu} \rightarrow \bar{\nu}_{\epsilon} \) transitions, from Eq. (2.12) we have
\[ \langle P_{\nu_{\mu} \rightarrow \nu_{\epsilon}}^{\text{LBL}} \rangle = \frac{1}{4} A_{\mu\epsilon} + |U_{\mu\epsilon}|^2 |U_{ee}|^2 + |U_{\mu\delta}|^2 |U_{ed}|^2. \quad (5.19) \]
Taking into account the bounds (3.21) for \( \Sigma_{\epsilon} \) and \( \Sigma_{\mu} \), the allowed range (3.13) for \( |U_{\mu\delta}|^2 = \Sigma_{\mu} - |U_{\mu\epsilon}|^2 \) and the constraint \( 0 \leq |U_{ed}|^2 \leq \Sigma_{e} \) for \( |U_{ed}|^2 = \Sigma_{e} - |U_{ee}|^2 \), one can find that the average transition probability (5.19) is bounded by
\[ \frac{1}{4} A_{\mu\epsilon} + \lambda_{\mu}^{-}(\min) \Lambda_{\epsilon}^{-} \leq \langle P_{\nu_{\mu} \rightarrow \nu_{\epsilon}}^{\text{LBL}} \rangle \leq \frac{1}{4} A_{\mu\epsilon} + \lambda_{\mu}^{+}(\max) \Lambda_{\epsilon}^{+}. \quad (5.20) \]

The lower bound is reached for \( |U_{ed}|^2 = 0 \) and \( |U_{\mu\delta}|^2 = \Sigma_{\mu} - \lambda_{\mu}^{-}(\min) \) or \( |U_{ed}|^2 = \Sigma_{e} \) and \( |U_{\mu\delta}|^2 = \lambda_{\mu}^{-}(\min) \). The quantity \( \lambda_{\mu}^{-}(\min) \) is the minimum value of \( \lambda_{\mu}^{-} \) in the allowed range for \( \Sigma_{\mu} \). For the values of \( \Delta m^2_{\text{SBL}} \) in which \( \Sigma_{\mu} \) is limited by the Super-Kamiokande asymmetry constraint (3.12), the minimum of \( \lambda_{\mu}^{-} \), shown in Fig. 1, occurs for
\[\Sigma_\mu = \frac{8 A_\mu - \sqrt{2(1 - 3 A_\mu)(1 + A_\mu)}}{2(7 A_\mu - 1)} \]  
(5.21)

The upper bound in Eq.(5.20) is reached for \( |U_{ed}|^2 = 0 \) and \( |U_{\mu d}|^2 = \Sigma_\mu - \lambda_\mu(\text{max}) \) or \( |U_{ed}|^2 = \Sigma_e \) and \( |U_{\mu d}|^2 = \lambda_\mu(\text{max}) \). The quantity \( \lambda_\mu(\text{max}) \) is the maximum value of \( \lambda_\mu \) in the allowed range for \( \Sigma_\mu \), that, as shown in Fig. 1, occurs always for the maximum allowed value of \( \Sigma_\mu, \Sigma_\mu^{\text{max}} = 1 - \Lambda_\mu^- \).

Figure 5 shows the allowed range (5.20) as a function of \( \Delta m_{23}^2 \) (shadowed region). The dashed line in Fig. 5 represents the unitarity upper bound \( \langle P_{LBL}(\nu_{\mu}^{-} \rightarrow \nu_{\mu}^{-}) \rangle \leq 1 - \langle P_{LBL}(\nu_{e}^{-} \rightarrow \nu_{e}^{-}) \rangle \), with the upper bound for \( 1 - \langle P_{LBL}(\nu_{\mu}^{-} \rightarrow \nu_{e}^{-}) \rangle \) given by Eq. (4.5). Since \( \Lambda_\mu^+ = a_\mu^0 \) is small, the dashed line in Fig. 5 practically coincides with the solid line in Fig. 4.

The other constrained channels are \( \nu_\alpha \rightarrow \nu_\beta \) with \( \alpha = e, \mu \) and \( \beta = \tau, s \), for which \( \langle P_{LBL}(\nu_{\alpha}^{-} \rightarrow \nu_{\beta}^{-}) \rangle \) has only the unitarity upper limits given by the average of Eq. (5.3).

**VI. CONCLUSIONS**

We have presented the constraints on the neutrino mixing parameters obtained from the results of neutrino oscillation experiments in the framework of the two four-neutrino schemes (2.1) that are compatible with all neutrino oscillation data. We have shown that the parameters \( \Sigma_e \) and \( 1 - \Sigma_\mu \) defined in Eq. (2.6) are small, but do not vanish. Their allowed ranges are presented in Figs. 2 and 3. We have also derived the bounds for the mixing of \( \nu_\mu \) with the two massive neutrinos whose mass-squared difference is responsible for atmospheric neutrino oscillations (Eq. (3.13)). In Sections IV and V we presented the bounds for the oscillation probabilities in long-baseline experiments (some of which have been derived already in Ref. [13]). We have also shown that a comparison of the probability measured in long-baseline appearance experiments with the corresponding average probability measured in short-baseline experiments may give information on the relative phases of the relevant elements of the mixing matrix and, in particular, may provide an evidence for the existence of CP violation in the lepton sector. This method may be more convenient than the traditional method based on the comparison of the oscillation probabilities of neutrinos and antineutrinos if only a neutrino or antineutrino beam is available.

In this paper we have considered long-baseline experiments neglecting matter effects, that are rather complicated in the framework of four-neutrino mixing and will be discussed in detail in another article [18]. We think that it is important to discuss long-baseline oscillations in vacuum for at least two reasons. First, because vacuum oscillations can be treated analytically and all the effects can be understood in a clear way. The matter effects, that often must be calculated numerically, can be appreciated as corrections (sometimes large) to the oscillations in vacuum. Second, in order to measure accurately the properties of neutrinos without the uncertainty and complications caused by matter effects, it may be possible, in some indeterminate future, to perform a long-baseline experiment in vacuum. For example, it may be possible to produce a neutrino beam in a orbital station (existing space stations orbit at an altitude between 300 and 400 km) and shoot it to some detector(s) on the Earth. This experiment would allow to measure accurately the properties of neutrinos.
when the beam does not cross the Earth and, with this knowledge, to perform an accurate tomography of the Earth using the matter effects when the beam crosses the Earth. At present such an experiment is fanciful, but we know that many experiments have been performed up to now that were unthinkable or believed to be impossible in the past.
REFERENCES


FIG. 1. Allowed range for $|U_{\mu d}|^2$ as a function of $\Sigma_\mu$ obtained from Eq. (3.13) and the limit (3.5) for the Super-Kamiokande up-down asymmetry.
FIG. 2. Allowed range for $\Sigma_e$ as a function of $\Delta m_{SBL}^2$ (shadowed region). The solid line is the upper bound (3.1) obtained from the exclusion curve of the Bugey experiment. The dashed line represents the lower bound (3.19) obtained from the allowed region of the LSND experiment. The dash-dotted line shows the upper bound (3.26) obtained from the negative result of the CHOOZ experiment.
FIG. 3. Allowed range for $\Sigma_{\mu}$ as a function of $\Delta m^2_{\text{SBL}}$ (white region). The shadowed area enclosed by the solid line is excluded by the 90% CL exclusion curve of the CDHS experiment. The vertically hatched area limited by the dash-dotted line is excluded by the inequality (3.12) obtained from the Super-Kamiokande up-down asymmetry. The region above the dashed line is excluded by the upper bound (3.18) and the region below the dotted line is excluded by the lower bound (3.19).
FIG. 4. Allowed range for $1 - P_{\nu_e \rightarrow \nu_e}^{\text{LBL}}$ as a function of $\Delta m^2_{\text{SBL}}$ (shadowed region). The solid line shows the upper bound in Eq. (4.3) obtained from the exclusion curve of the Bugey experiment. The dotted line shows the lower bound in Eq. (4.3) obtained from the results of the LSND experiment. The dash-dotted line represents the upper bound obtained in the CHOOZ experiment.
FIG. 5. Allowed range (5.20) for $\langle P_{\nu \mu}^{\text{LBL}} \rangle$ as a function of $\Delta m_{\text{SBL}}^2$ (shadowed region). The dashed curve represents the unitarity constraint $\langle P_{\nu e}^{\text{LBL}} \rangle \leq 1 - \langle P_{\nu e}^{\text{LBL}} \rangle$. 