Instantons on Noncommutative $R^4$ and Projection Operators

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Abstract

I study noncommutative version of ADHM construction of instantons, which was proposed by Nekrasov and Schwarz. Noncommutative $R^4$ is described as algebra of operators acting in Fock space. In ADHM construction of instantons, one looks for zero-modes of Dirac-like operator. The feature peculiar to noncommutative case is that these zero-modes project out some states in Fock space. I clarify the mechanism of these projections when the gauge group is $U(1)$. I also construct some zero-modes when the gauge group is $U(N)$ and demonstrate that the projections also occur, and the mechanism is similar to the $U(1)$ case.

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Introduction

One of the most important reasons to consider physics on noncommutative spacetime is that the behavior of the theory at short distance is expected to become manageable due to the noncommutativity of the spacetime coordinates. It was shown [1] that noncommutative geometry appear in a definite limit of string theory, BFSS matrix theory [2] and IIB matrix theory [3]. In these cases noncommutativity should be relevant to the short scale physics of D-branes.

Among D-brane systems, Dp-brane-D(p+4)-brane bound states are of interest because this system has two different descriptions: in terms of worldvolume theory of Dp-branes and in terms of worldvolume theory of D(p + 4)-branes. D-flat condition of the worldvolume theory of Dp-brane coincides with ADHM equations [4][5], and Dp-branes are described as instantons in D(p + 4)-brane worldvolume theory. These descriptions should be equivalent because both describe the same system, and indeed the moduli space of the worldvolume theory of Dp-branes is identical to the instanton moduli space. Turning constant NS-NS B-field in the worldvolume of D(p + 4)-branes cause noncommutativity in the worldvolume theory of D(p + 4)-branes, and adds Feyet-Iliopoulos D-term in the worldvolume theory of Dp-branes [15][10]. The equivalence of both descriptions follows from the pioneering work of Nekrasov and Schwarz [11]. In order to construct instantons on noncommutative $\mathbb{R}^4$, one adds a constant (corresponding to the Feyet-Iliopoulos term) to the ADHM equations.\footnote{The case of equivariant instanton is studied in [13].} The modified ADHM equations describe the resolutions of singularities in the moduli space of instantons on $\mathbb{R}^4$ [7]. This moduli space has been an important clue to the nonpertabative aspects of string theory [16][17][18][19][20][21]and matrix theory [10][22][23][24][25]. Further study from both string theory and noncommutative geometry points of view was recently given by [14]. More recently, there appears a paper which gives interesting viewpoints to the short scale spacetime structure [12].

In [11] Nekrasov and Schwarz explicitly constructed some instanton solutions and showed that they are non-singular. The interesting point is that they are non-singular even if their commutative counterparts are singular, so-called small instantons. In these cases noncommutativity of the coordinates actually eliminates the singular behavior of the field configurations. What is special to the noncommutative case is the appearance of projection operators which project out potentially dangerous states in Fock space, where Fock space is introduced to describe the noncommutative $\mathbb{R}^4$ [11]. The purpose of this paper is to investigate this mechanism. It is shown that this mechanism has rich structures, and gives insight in the short scale structures near the core of instantons on noncommutative space.
1 Preliminaries

In this section we briefly review the theory of gauge field on noncommutative $\mathbb{R}^4$ and ADHM construction on commutative $\mathbb{R}^4$, as preliminaries to ADHM construction on noncommutative $\mathbb{R}^4$.

1.1 Gauge Field on Noncommutative $\mathbb{R}^4$

Noncommutative $\mathbb{R}^4$ is described by algebra generated by $x^\mu$ ($\mu = 1, \cdots, 4$) obeying the commutation relations:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (1.1)$$

where $\theta^{\mu\nu}$ is real and constant. In this paper we restrict ourselves to the case where $\theta^{\mu\nu}$ is self-dual and set:

$$\theta^{12} = \theta^{34} = \zeta. \quad (1.2)$$

Then the algebra depends only one constant parameter $\zeta$.

Introduce the generators of noncommutative $\mathbb{C}^2 \cong \mathbb{R}^4$ by

$$z_1 = x_2 + ix_1, \quad z_2 = x_4 + ix_3. \quad (1.3)$$

Their commutation relations are:

$$[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -\frac{\zeta}{2}, \quad \text{(others: zero.)} \quad (1.4)$$

We choose $\zeta > 0$. The commutation relations (1.1) has an automorphisms of the form $x^\mu \mapsto x^\mu + c^\mu$, where $c^\mu$ is a commuting real number. We denote the Lie algebra of this group by $\mathfrak{g}$. Following [11], we start with the algebra $\text{End} \mathcal{H}$ of operators acting in the Fock space $\mathcal{H} = \sum_{(n_1, n_2) \in \mathbb{Z}^2_{\geq 0}} C|n_1, n_2\rangle$, where $z, \bar{z}$ are represented as creation and annihilation operators:

$$\sqrt{\frac{2}{\zeta}} z_1 |n_1, n_2\rangle = \sqrt{n_1 + 1} |n_1 + 1, n_2\rangle, \quad \sqrt{\frac{2}{\zeta}} \bar{z}_1 |n_1, n_2\rangle = \sqrt{n_1} |n_1 - 1, n_2\rangle,$$
$$\sqrt{\frac{2}{\zeta}} z_2 |n_1, n_2\rangle = \sqrt{n_2 + 1} |n_1, n_2 + 1\rangle, \quad \sqrt{\frac{2}{\zeta}} \bar{z}_2 |n_1, n_2\rangle = \sqrt{n_2} |n_1, n_2 - 1\rangle. \quad (1.5)$$

We consider an algebra $\mathcal{A}_\zeta$ consists of smooth operators $a$, i.e. $a\partial_\mu a$ is smooth ($\partial_\mu$ is understood as the action of $\mathfrak{g} = \mathbb{R}^4$ on $\mathcal{A}_\zeta$ by translation). The $U(N)$ gauge field on

\[\text{See [14] for the meaning of this choice of parameter in string theory.}\]

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noncommutative $\mathbb{R}^4$ is defined as follows. First we consider $N$-dimensional vector space $\mathcal{E} := (\mathcal{A}_\zeta)^{\otimes N}$ which carries a right representation of $\mathcal{A}_\zeta$: $^{3}$

$$\mathcal{E} \times \mathcal{A}_\zeta \ni (e, a) \mapsto ea \in \mathcal{E}, \quad e(ab) = (ea)b,$$
$$e(a + b) = ea + eb,$$
$$(e + e')a = ea + e'a,$$  \hspace{1cm} (1.6)

for any $e, e' \in \mathcal{E}$ and $a, b \in \mathcal{A}_\zeta$. The elements of $\mathcal{E}$ can be thought of as an $N$-dimensional vector with entries in $\mathcal{A}_\zeta$. Let us consider the unitary action of $U$ to the element of $\mathcal{E}$:

$$e \rightarrow Ue,$$  \hspace{1cm} (1.7)

where $U$ is a $N \times N$ matrix with its components in $\mathcal{A}_\zeta$, satisfying $UU^\dagger = U^\dagger U = \text{Id}_{\mathcal{A}_\zeta} \otimes \text{Id}_N$. $\text{Id}_N$ is a $N \times N$ identity matrix. Under the action of this unitary transformation, $De$, the covariant derivative of $e \in \mathcal{E}$, is required to transform covariantly:

$$De \rightarrow UDe.$$  \hspace{1cm} (1.8)

The covariant derivative $D$ is written as

$$D = d + A.$$  \hspace{1cm} (1.9)

Here the $U(N)$ gauge field $A$ is introduced to ensure the covariance, as explained below. $A$ is a matrix valued one-form: $A = A_\mu dx^\mu$ with $A_\mu$ being anti-hermitian $N \times N$ matrix. The action of exterior derivative $d$ is defined as:

$$da := (\partial_\mu a) dx^\mu, \quad a \in \mathcal{A}_\zeta.$$  \hspace{1cm} (1.10)

d$x^\mu$'s commute with $x^\mu$ and anti-commute among themselves, and hence $d^2 a = 0$ for $a \in \mathcal{A}_\zeta$. From (1.7) and (1.8), the covariant derivative transforms as

$$D \rightarrow UDU^\dagger.$$  \hspace{1cm} (1.11)

Hence the gauge field $A$ transforms as

$$A \rightarrow UdU^\dagger + UAU^\dagger.$$  \hspace{1cm} (1.12)

The field strength is defined by

$$F := D^2 = dA + A^2.$$  \hspace{1cm} (1.13)

For later purpose, let us consider projection operator $P \in \mathcal{A}_\zeta$, $P^\dagger = P, P^2 = P$. For every projection operator $P$, we can consider $N$-dimensional vector space $P\mathcal{E},^{4}$ where the

\footnotesize
$^{3}\mathcal{E}$ is a right module over $\mathcal{A}_\zeta$. See, for example, [26][27].
$^{4}P\mathcal{E}$ is a right projective module over $\mathcal{A}_\zeta$. 
element of $PE$ can be thought of as $N$-dimensional vector with entries in the form $Pa$, $a \in A$. We can construct covariant derivative $D_r$ for $PE$ by

$$D_r = Pd + A, \quad A = PAP.$$  \hfill (1.14)

Notice that $D_r = PD_r$. The unitary transformation on $PE$ is given by

$$e \rightarrow U_r e, \quad U_r \equiv PUP,$$  \hfill (1.15)

where $U$ again satisfies $UU^\dagger = U^\dagger U = \text{Id}_{A} \otimes \text{Id}_N$. We require $D_r e$ to transform as

$$D_r e \rightarrow U_r D_r e.$$  \hfill (1.16)

Then the covariant derivative $D_r$ must transform as

$$D_r \rightarrow U_r D_r U_r^\dagger.$$  \hfill (1.17)

For any $e \in PE$,

$$U_r D_r U_r^\dagger e = U_r (Pd + A) U_r^\dagger e = U_r dP(U_r^\dagger e) + U_r AU_r^\dagger e$$

$$= U_r PdU_r^\dagger e + PU_r (U_r^\dagger de) + U_r AU_r^\dagger e \quad (U_r P = PU_r)$$

$$= Pde + (U_r dU_r^\dagger + U_r AU_r^\dagger )e.$$  \hfill (1.18)

Hence the gauge field $A$ transforms as

$$A \rightarrow U_r dU_r^\dagger + U_r AU_r^\dagger.$$  \hfill (1.19)

The field strength becomes

$$F := D_r^2$$

$$= PdA + A^2 + PdPdP.$$  \hfill (1.20)

Indeed, for $e \in PE$,

$$Fe = (Pd + A)(Pd + A)e$$

$$= Pd(Pde) + Pd(Ae) + APde + A^2 e$$

$$= Pd(Pde) + PdAe + A^2 e,$$  \hfill (1.21)

and since $e = Pe$ and $P^2 = P$,

$$Pd(Pde) = Pd(Pd(Pe))$$

$$= Pd(PdPe + Pde)$$

$$= PdPdPe + PdPde$$

$$= PdPdPe.$$  \hfill (1.22)

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Gauge field $A$ is called anti-self-dual, or instanton, if its field strength satisfies the conditions:

$$F^+ := \frac{1}{2} (F + *F) = 0,$$

(1.23)

where $*$ is the Hodge star.\(^5\)

### 1.2 Review of ADHM construction on commutative $\mathbb{R}^4$

ADHM construction of instantons [6] is the way to obtain anti-self-dual gauge field on $\mathbb{R}^4$ from solutions of some quadratic matrix equations. More specifically, in order to construct anti-self-dual $U(N)$ gauge field with instanton number $k$, one starts from the following data:

1. A pair of complex hermitian vector spaces $V = \mathbb{C}^k$ and $W = \mathbb{C}^N$.
2. The operators $B_1, B_2 \in \text{Hom}(V,V), I \in \text{Hom}(W,V), J = \text{Hom}(V,W)$ satisfying the equations

$$
\begin{align*}
\mu_R &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - JJ^\dagger = 0, \\
\mu_C &= [B_1, B_2] + IJ = 0.
\end{align*}
$$

(1.24)

Next define Dirac-like operator $D_z : V \oplus V \oplus W \rightarrow V \oplus V$ by

$$(\tau_z, \sigma_z^\dagger)$$

$$
\begin{align*}
\tau_z &= (B_2 - z_2, B_1 - z_1, I), \\
\sigma_z^\dagger &= (-B_1^\dagger - \bar{z}_1, B_2^\dagger - \bar{z}_2, J^\dagger).
\end{align*}
$$

(1.25)

(1.24) is equivalent to the set of equations

$$
\begin{align*}
\tau_z \tau_z^\dagger &= \sigma_z^\dagger \sigma_z, \\
\tau_z \sigma_z &= 0,
\end{align*}
$$

(1.26)

which are important conditions in ADHM construction. There are $N$ zero-modes of $D_z$:

$$D_z \psi^{(a)} = 0, \quad a = 1, \ldots, N.$$

(1.27)

We can choose orthonormalized basis of the space of zero-modes:

$$\psi^{(a)} \psi^{(b)} = \delta^{ab}.$$

(1.28)

\(^5\)In this paper we only consider the case where the metric on $\mathbb{R}^4$ is flat: $g_{\mu\nu} = \delta_{\mu\nu}$. 
The change of basis in the space of orthonormalized zero-modes \( \psi^{(a)} \) become \( U(N) \) gauge symmetry. Anti-self-dual \( U(N) \) gauge field is constructed by the formula

\[
A^{ab} = \psi^{(a)} \dagger d\psi^{(b)}.
\] (1.29)

There is an action of \( U(k) \) that does not change (1.29):

\[
(B_1, B_2, I, J) \mapsto (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}), \quad g \in U(k).
\] (1.30)

The moduli space of anti-self-dual \( U(N) \) gauge field with instanton number \( k \) is given by

\[
\mathcal{M}(k, N) = \mu^{-1}_R(0) \cap \mu^{-1}_C(0)/U(k).
\] (1.31)

where the action of \( U(k) \) is the one given in (1.30). When \((B_1, B_2, I, J)\) is a fixed point of \( U(k) \) action, \( \mathcal{M}(k, N) \) is singular. Such singularity corresponds to the instanton shrinking to zero size.

## 2 ADHM Construction on Noncommutative \( R^4 \) and the Appearance of Projection Operator

The singularities in (1.31) has a natural resolution [7]. Modify (1.24) to

\[
\begin{align*}
\mu_R &= [B_1, B_1^*] + [B_2, B_2^*] + II^* - J^*J = \zeta \text{Id}_V, \\
\mu_C &= [B_1, B_2] + IJ = 0,
\end{align*}
\] (2.1)

and consider the space

\[
\mathcal{M}_{\zeta}(k, N) = \mu^{-1}_R(\zeta \text{Id}_V) \cap \mu^{-1}_C(0)/U(k).
\] (2.2)

Then \( \mathcal{M}_{\zeta}(k, N) \) is a smooth \( 4kN \) dimensional hyper-Kähler manifold. Although the absence of singularities is interesting from the physical point of view, construction of instantons from (2.1) does not work straightforwardly. The main obstruction is that the key equation (1.26) is not satisfied on the usual commutative \( R^4 \). However, Nekrasov and Schwarz noticed that \( \tau_z, \sigma_z \) do satisfy (1.26) if the coordinates are noncommutative as in (1.4) [11]. Once (1.26) is satisfied, we can expect that the construction of instantons is similar to the usual commutative case. But there are some features peculiar to the noncommutative case. Especially, since the ADHM construction on noncommutative \( R^4 \) starts from (2.2) where small instanton singularities have been resolved, one expects that crucial difference will appear when the size of the instanton is small. It is interesting to study such situations and see how the effects of noncommutativity appear.
ADHM construction on noncommutative $\mathbb{R}^4$ is as follows [11]. We define operator $D_z : (V \oplus V \oplus W) \otimes \mathcal{A}_\zeta \to (V \oplus V) \otimes \mathcal{A}_\zeta$ by the same formula (1.25):

$$D_z = \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix},$$

$$\tau_z = (B_2 - z_2, B_1 - z_1, I),$$

$$\sigma_z^\dagger = (-(B_1^\dagger - \bar{z}_1), B_2^\dagger - \bar{z}_2, J^\dagger).$$  \hspace{1cm} (2.3)

The operator $D_z D_z^\dagger : (V \oplus V) \otimes \mathcal{A}_\zeta \to (V \oplus V) \otimes \mathcal{A}_\zeta$ has a block diagonal form

$$D_z D_z^\dagger = \begin{pmatrix} 2z_0 & 0 \\ 0 & 2z_0 \end{pmatrix},$$

$$2z_0 \equiv \tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z.$$ \hspace{1cm} (2.4)

which is a consequence of (1.26) and important for ADHM construction. Next we look for solutions of the equation

$$D_z \Psi^{(a)} = 0 \quad (a = 1, \ldots, N),$$ 

(2.5)

where $\Psi^{(a)}$'s are operators: $\Psi^{(a)} : \mathcal{A}_\zeta \to (V \oplus V \oplus W) \otimes \mathcal{A}_\zeta$. If we can normalize $\Psi^{(a)}$'s as

$$\Psi^{(a)} \Psi^{(b)} \equiv \delta^{ab} \text{Id}_{\mathcal{A}_\zeta},$$ \hspace{1cm} (2.6)

we can construct anti-self-dual $U(N)$ gauge field by the same formula (1.29):

$$A_{ab} = \Psi^{(a)} \Psi^{(b)} ,$$ \hspace{1cm} (2.7)

where $a,b$ are $U(N)$ indices. Then field strength becomes

$$F = \tilde{F}^- \equiv \Psi^\dagger \begin{pmatrix} \tau_z^\dagger & \sigma_z^\dagger \\ 1 & 1 \end{pmatrix} \begin{pmatrix} d\tau_z & d\sigma_z \\ 1 & 1 \end{pmatrix} \Psi$$

$$= \begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger & \xi^\dagger \\ \psi_1 & \psi_2 & \xi \end{pmatrix} \begin{pmatrix} dz_1 \frac{1}{dz_z} d\bar{z}_1 + d\bar{z}_2 \frac{1}{dz_z} d\bar{z}_2 & -dz_1 \frac{1}{dz_z} d\bar{z}_1 + d\bar{z}_2 \frac{1}{dz_z} d\bar{z}_1 & 0 \\ -dz_2 \frac{1}{dz_z} d\bar{z}_1 + d\bar{z}_2 \frac{1}{dz_z} d\bar{z}_1 & dz_2 \frac{1}{dz_z} d\bar{z}_2 + d\bar{z}_1 \frac{1}{dz_z} d\bar{z}_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix}$$ \hspace{1cm} (2.8)

where we have written

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} \equiv \begin{pmatrix} \Psi^{(1)} \cdots \Psi^{(N)} \end{pmatrix}, \quad \psi_1 : \mathbb{C}^N \otimes \mathcal{A}_\zeta \to V \otimes \mathcal{A}_\zeta,$$

$$\psi_2 : \mathbb{C}^N \otimes \mathcal{A}_\zeta \to V \otimes \mathcal{A}_\zeta,$$

$$\xi : \mathbb{C}^N \otimes \mathcal{A}_\zeta \to W \otimes \mathcal{A}_\zeta.$$

The derivation is similar to the commutative case. (2.8) is anti-self-dual.

However, sometimes $\Psi^{(a)}$ annihilates some states in $\mathcal{H}$ for some $a$. By “$\Psi^{(a)}$ annihilates state” we mean all the components of $\Psi^{(a)}$ annihilate that state. This is not a special
phenomenon, and the study of this kinds of zero-modes is the purpose of this paper. Let us consider the case where there is one such zero-mode $\Psi^{(1)}$. In that case we cannot normalize $\Psi^{(1)}$ as in (2.6). We may normalize $\Psi^{(1)}$ as

$$\Psi^{(1)^\dagger} \Psi^{(1)} = P,$$

where $P$ is the projection operator that projects out the states annihilated by $\Psi^{(1)}$. However, the existence of the projection operator in (2.9) gives additional contribution to the field strength, because the projection operator depends on $z$ and $\bar{z}$. The derivative of the projection operator gives additional contribution to the field strength, which is not anti-self-dual.

The appearance of projection operator $P$ indicates that we should consider restricted vector space $PE$ rather than $E$. Indeed, as we will see shortly, ADHM construction perfectly works in this setting.

From hereafter we will concentrate on the simplest $U(1)$ instanton solutions (until section 4). Let us consider covariant derivative (1.14):

$$D_P = Pd + A,$$

with $A = PAP$. The field strength is given by (1.20):

$$F = PdA + A^2 + PdPdP.$$

We can construct anti-self-dual gauge field by putting

$$A = \Psi^\dagger d\Psi P,$$

where $\Psi$ is a zero-mode of $D_z$ and normalized as $\Psi^\dagger \Psi = P$ and $\Psi^\dagger = P\Psi^\dagger$. Let us check that (2.12) is really anti-self-dual. The first term in (2.11) becomes

$$PdA = Pd\Psi^\dagger d\Psi P - P\Psi^\dagger d\Psi dP.$$

The last term above can be rewritten as

$$P\Psi^\dagger d\Psi dP = P(d(\Psi^\dagger\Psi) - d\Psi^\dagger\Psi) dP = PdPdP - Pd\Psi^\dagger \Psi dP.$$

The first term in (2.14) cancels $PdPdP$ in (2.11). The last term in (2.14) vanishes when acting on $e = Pe \in PE$, since $\Psi dPP = -\Psi dP(1 - P)P = 0$. The second term in (2.11) becomes

$$A^2 = P\Psi^\dagger d\Psi^\dagger d\Psi P = P(d(\Psi^\dagger\Psi)) - d\Psi^\dagger\Psi \Psi^\dagger d\Psi P = PdP\Psi^\dagger d\Psi - Pd\Psi^\dagger \Psi^\dagger d\Psi P.$$

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6For example, $|0,0\rangle \langle 0,0| = : e^{-\frac{1}{2}x_1^2\bar{x}_1 + x_2\bar{x}_2} :$ , where $:$ $:$ means normal ordering.
The first term in (2.15) vanishes because \( PdP\Psi^\dagger = -P(1 - P)\Psi^\dagger = 0 \). Then the field strength becomes

\[
F = Pd\Psi^\dagger (1 - \Psi\Psi^\dagger)d\Psi P
= P\tilde{F}^- P = \tilde{F}^-,
\]

(2.16)

where \( \tilde{F}^- \) is defined in (2.8) and is anti-self-dual.

The absence of the singular behavior in the field configuration follows rather straightforwardly from above constructions. In the commutative case, source of divergence is the normalization factor of the \( \psi^{(a)} \) in (1.29). At small instanton singularity, the solution of (1.27) is trivial, i.e. \( \psi^{(a)} = 0 \) and cannot define norm of the zero-mode. However, in the noncommutative case, we only normalize \( \Psi \) in the subspace where \( \Psi \) has norm. Moreover, as shown in appendix A, \( \Box_z \) has no zero-mode and hence its inverse does not cause divergence. Therefore from the explicit formula (2.8), we can see that there is no source of divergence in (2.8) or (2.16).

3 \( U(1) \) Instantons and Projection Operators

3.1 Projection Operators in \( U(1) \) Instanton Solutions and Relation to the Ideal

In the previous section I have explained how to construct anti-self-dual gauge field when the zero-mode annihilates some states. Then the natural question is: how the states annihilated by zero-mode are determined? In this section we will answer this question when the gauge group is \( U(1) \).

Let us consider the solution of the equation

\[
\mathcal{D}_z \langle U \rangle = 0,
\]

(3.1)

where \( \langle U \rangle \in \mathcal{H} \oplus \mathcal{H} \), i.e. the components of \( \langle U \rangle \) are vectors in the Fock space \( \mathcal{H} \). We call \( \langle U \rangle \) “vector zero-mode” and call \( \Psi \) in (2.5) “operator zero-mode”. We can construct operator zero-mode if we know all the vector zero-modes. The advantage of considering vector zero-modes is that we can relate them to the ideal discussed in [8][9]. The point is that we can regard vector zero-modes as holomorphic bundle described in purely commutative terms. Noncommutativity appear when we construct operator zero-mode treating all the vector zero-modes as a whole.

Let us write

\[
\langle U \rangle = \begin{pmatrix} |u_1\rangle & |u_2\rangle \\ |f\rangle \end{pmatrix}, \quad |u_1\rangle \equiv u_1(z_1, z_2) |0, 0\rangle, \quad |u_2\rangle \equiv u_2(z_1, z_2) |0, 0\rangle, \quad |f\rangle \equiv f(z_1, z_2) |0, 0\rangle
\]

(3.2)
where \(|u_1\rangle, |u_2\rangle \in \mathcal{H}^\otimes k\) i.e. they are vectors in \(V = \mathbb{C}^k\) and vectors in \(\mathcal{H}\), and \(|f\rangle \in \mathcal{H}\). The space of the solutions of (3.1), i.e. \(\ker \mathcal{D}_z = \ker \tau_z \cap \ker \sigma_z^\dagger \simeq \ker \tau_z/\text{Im} \sigma_z\) is isomorphic to the ideal \(\mathcal{I}\) defined by

\[
\mathcal{I} = \left\{ f(z_1, z_2) \mid f(B_1, B_2) = 0 \right\}. \tag{3.3}
\]

In \(U(1)\) case, one can show \(J = 0\), and the isomorphism is given by the inclusion of the third factor in (3.2) [8][9].

\[
\ker \tau_z/\text{Im} \sigma_z \hookrightarrow \mathcal{O}_{\mathbb{C}^2} : \quad |U\rangle = \begin{pmatrix} |u_1\rangle \\ |u_2\rangle \\ |f\rangle \end{pmatrix} \hookrightarrow f(z_1, z_2). \tag{3.4}
\]

Let us define “ideal state” by

\[
|\varphi\rangle \in \text{ideal states of } \mathcal{I} \iff \exists f(z_1, z_2) \in \mathcal{I}, \quad |\varphi\rangle = f(z_1, z_2) |0, 0\rangle, \tag{3.5}
\]

and denote the space of all the ideal states by \(\mathcal{H}_\mathcal{I}\). We define \(\mathcal{H}/\mathcal{I}\) as a subspace in \(\mathcal{H}\) orthogonal to \(\mathcal{H}_\mathcal{I}\):

\[
|g\rangle \in \mathcal{H}/\mathcal{I} \iff \forall f(z_1, z_2) \in \mathcal{I}, \quad f^\dagger(\bar{z}_1, \bar{z}_2) |g\rangle = 0. \tag{3.6}
\]

\(\mathcal{H}/\mathcal{I}\) is a \(k\) dimensional space. Let us denote the complete basis of \(\mathcal{H}/\mathcal{I}\) by \(|g_\alpha\rangle\), \(\alpha = 1, 2, \cdots, k\), and the complete basis of \(\mathcal{H}_\mathcal{I}\) by \(|f_i\rangle\), \(i = k+1, k+2, \cdots\). They together span complete basis of \(\mathcal{H}\). We can label them by positive integer \(n\):

\[
\{ |h_n\rangle, \quad n \in \mathbb{Z}_+ \} = \{ |g_\alpha\rangle, |f_i\rangle, \quad \alpha = 1, 2, \cdots, k, \quad i = k+1, k+2, \cdots \}. \tag{3.7}
\]

As we can see from (3.4), zero-modes (3.1) are completely determined by the ideal \(f_i(z_1, z_2)\) [9]:

\[
|\mathcal{U}(f_i)\rangle = \begin{pmatrix} |u_1(f_i)\rangle \\ |u_2(f_i)\rangle \\ |f_i\rangle \end{pmatrix}. \tag{3.8}
\]

We can construct operator zero-mode (2.5) by the following formula:

\[
\Psi = \sum_i \sum_n (\Psi)^{in} |\mathcal{U}(f_i)\rangle \langle h_n|, \tag{3.9}
\]

\(^7\)Notice that since \(\tau_z \sigma_z = 0 \, ((1.26))\), \(\ker \tau_z/\text{Im} \sigma_z\) is well defined. \(\ker \tau_z \cap \ker \sigma_z^\dagger \simeq \ker \tau_z/\text{Im} \sigma_z\) is understood as follows: The condition \(\ker \sigma_z^\dagger |\mathcal{U}\rangle = 0\) fixes the “gauge freedom” mod \(\text{Im} \sigma_z\) in \(\ker \tau_z/\text{Im} \sigma_z\).

\(^8\)The meaning of this notation is as follows: \(\mathcal{H}/\mathcal{I}\) corresponds to \(\mathbb{C}[z_1, z_2]/\mathcal{I}\), where \(\mathbb{C}[z_1, z_2]\) is the ring of polynomials of \(z_1, z_2\).
where \((\Psi)_{in}\) is a commuting number. We call \(\Psi_0\) “minimal zero-mode” if it has the form:

\[
\Psi_0 = \sum_{ij} (\Psi_0)_{ij} |U(f_i)\rangle \langle f_j|, \quad \forall j, \exists i \quad \text{such that} \quad (\Psi_0)_{ij} \neq 0,
\]

i.e. \((\Psi_0)_{in} = 0\) for \(n = \alpha = 1, 2, \cdots, k\). We call it “normalized” minimal zero-mode if it is normalized in \(\mathcal{H}_I\):

\[
\Psi_0^\dagger \Psi_0 = P, \tag{3.11}
\]

where \(P\) is a projection operator that project out the states in \(\mathcal{H}_I\). The uniqueness of the minimal zero-mode (up to gauge transformation) is shown in appendix B. From above definition, \(\Psi_0 |\varphi\rangle = 0\) for \(|\varphi\rangle \in \mathcal{H}_I\). Note that if we write

\[
\Psi_0 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix}, \tag{3.12}
\]

then \(\xi |\varphi\rangle = 0 \Rightarrow \psi_1 |\varphi\rangle = \psi_2 |\varphi\rangle = 0\). Hence the states annihilated by minimal zero-mode \(\Psi_0\) are completely determined by \(\xi\).

The interesting point is that the noncommutative operator \(\sim \) infinit runk matrix appear from ideal described in purely commutative terms, by treating infinit number of the elements of ideal at one time.

As an illustration, let us construct \(U(1)\) one-instanton solution from the ideal. First, let us recall \(U(1)\) one-instanton solution constructed in [11]. The solution to the modified ADHM equation (2.1) is given by

\[
B_1 = B_2 = 0, \quad I = \sqrt{\zeta}, \quad J = 0. \tag{3.13}
\]

There is a solution \(\tilde{\Psi}_0\) to the equation \(\mathcal{D}_2 \tilde{\Psi}_0 = 0\):

\[
\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} \sqrt{\zeta} \bar{z}_2 \\ \sqrt{\zeta} \bar{z}_1 \\ (z_1 \bar{z}_1 + z_2 \bar{z}_2) \end{pmatrix}. \tag{3.14}
\]

Notice that all the components of \(\tilde{\Psi}_0\) annihilates \(|0, 0\rangle\). As a consequence \(\tilde{\Psi}_0^\dagger \tilde{\Psi}_0 = (z_1 \bar{z}_1 + z_2 \bar{z}_2)(z_1 \bar{z}_1 + z_2 \bar{z}_2 + \zeta)\) annihilates \(|0, 0\rangle\). The inverse of \((z_1 \bar{z}_1 + z_2 \bar{z}_2)\) is only defined in the subspace of Fock space where \(|0, 0\rangle\) is projected out:

\[
(z_1 \bar{z}_1 + z_2 \bar{z}_2)^{-1} = P(z_1 \bar{z}_1 + z_2 \bar{z}_2)^{-1}P, \tag{3.15}
\]

where \(P\) is a projection operator that project out \(|0, 0\rangle\). Hence

\[
\Psi_0 = \tilde{\Psi}_0 (\tilde{\Psi}_0^\dagger \tilde{\Psi}_0)^{-1/2} \tag{3.16}
\]
is normalized as $\Psi_0^\dagger \Psi_0 = P$.

Let us reconstruct this zero-mode from ideal. The ideal which corresponds to (3.13) is $I = (z_1, z_2)$. The basis vector of the $\mathcal{H}_I$ is $|0, 0\rangle$ which is orthogonal to all the ideal states. We can use $|n_1, n_2\rangle, (n_1, n_2) \neq (0, 0)$ as basis vectors of $\mathcal{H}_I$, the space of ideal states. The solutions of $\mathcal{D}_z |U\rangle = 0$ are given by

$$\mathcal{D}_z |U\rangle = 0 :$$

$$\Psi = \sum_{(m_1, m_2) \neq (0, 0)} \sum_{(n_1, n_2)} (\Psi)(m_1, m_2)(n_1, n_2) |U_{m_1 m_2}\rangle \langle n_1, n_2|,$$  \hspace{1cm} (3.18)

The normalized minimal zero mode $\Psi_0$ is required to satisfy

$$\Psi_0 = \sum_{(m_1, m_2) \neq (0, 0)} \sum_{(n_1, n_2) \neq (0, 0)} (\Psi)(m_1, m_2)(n_1, n_2) |U_{m_1 m_2}\rangle \langle n_1, n_2|,$$

$$\Psi_0^\dagger \Psi_0 = \text{Id}_{\mathcal{A}_z} - |0, 0\rangle \langle 0, 0|.$$  \hspace{1cm} (3.19)

From the normalization condition in (3.19), it follows

$$\sum_{(m_1, m_2) \neq (0, 0)} 1/2 (m_1 + m_2)(m_1 + m_2 + 2) (\Psi^\dagger)(l_1, l_2)(m_1, m_2)(\Psi)(m_1, m_2)(n_1, n_2) = \delta(l_1, l_2)(n_1, n_2).$$  \hspace{1cm} (3.20)

The solution of (3.20) is

$$(\Psi_0)(m_1, m_2)(n_1, n_2) = \sqrt{2/(n_1 + n_2)(n_1 + n_2 + 2)} \delta(m_1, m_2)(n_1, n_2).$$  \hspace{1cm} (3.21)

(3.19), (3.21) are equivalent to (3.14),(3.16).

### 3.2 Some $U(1)$ Instanton Solutions

Construction of operator zero-mode from vector zero-modes is useful for understanding the notion of minimal zero-mode. But for actual calculation it is more easier to directly look for the operator zero-mode. It is interesting to see that the obtained operator zero-modes really annihilate states in $\mathcal{H}_I$. 

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Let us study the two-instanton solutions degenerating at the origin. The corresponding solution of the matrix equations (2.1) is given by

$$B_1 = \begin{pmatrix} 0 & \sqrt{\zeta} \lambda_1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \sqrt{\zeta} \lambda_2 \\ 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 \\ \sqrt{2 \zeta} \end{pmatrix}, \quad J = 0. \quad (3.22)$$

where $\lambda_1$ and $\lambda_2$ are complex numbers satisfying $|\lambda_1|^2 + |\lambda_2|^2 = 1$. Notice that $B_1$ and $B_2$ are upper half triangle matrices. $\lambda_1, \lambda_2$ (partially) remember the direction before two instantons collide [9][19]. The corresponding ideal is $I = (z_1^2, -\lambda_2 z_1 + \lambda_1 z_2)$. Hence the state orthogonal to all the ideal states is annihilated by $\bar{z}_1^2$, $-\lambda_2^* \bar{z}_1 + \lambda_1^* \bar{z}_2$. The states annihilated by $z_1^2$ is $|0, n_2\rangle$, $|1, n_2\rangle$ for all non-negative integer $n_2$. The states annihilated by $-\lambda_2^* \bar{z}_1 + \lambda_1^* \bar{z}_2$ are better be described by the Fock space constructed by rotated creation and annihilation operators $\tilde{z}'$:

$$z_1' \equiv \lambda_1^* z_1 + \lambda_2^* z_2, \quad z_2' \equiv -\lambda_2 z_1 + \lambda_1 z_2, \quad |0, 0\rangle = |0, 0\rangle_\lambda,$$

$$\sqrt{\frac{2}{\zeta}} z_1' |n_1', n_2'^\lambda\rangle = \sqrt{n_1' + 1} |n_1' + 1, n_2'^\lambda\rangle, \quad \sqrt{\frac{2}{\zeta}} \bar{z}_1' |n_1', n_2'^\lambda\rangle = \sqrt{n_1' + 1} |n_1' - 1, n_2'^\lambda\rangle,$$

$$\sqrt{\frac{2}{\zeta}} z_2' |n_1', n_2'^\lambda\rangle = \sqrt{n_2' + 1} |n_1', n_2' + 1\rangle, \quad \sqrt{\frac{2}{\zeta}} \bar{z}_2' |n_1', n_2'^\lambda\rangle = \sqrt{n_2' + 1} |n_1', n_2' - 1\rangle. \quad (3.23)$$

Then, the states annihilated by $z_2' = -\lambda_2^* \bar{z}_1 + \lambda_1^* \bar{z}_2$ are $|n_1', 0\rangle_\lambda$ for all non-negative $n_1'$. Therefore the basis vectors of the states orthogonal to all the ideal states is $|0, 0\rangle, |1, 0\rangle_\lambda$. Now let us study operator zero-mode. The (unnormalized) minimal zero mode can be directly obtained from (3.22):

$$\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1' \\ \tilde{\psi}_2' \\ \tilde{\xi} \end{pmatrix}, \quad \tilde{\psi}_1' = \begin{pmatrix} \sqrt{\zeta} \bar{z}_2' \\ \bar{z}_2' \bar{z}_1' (\hat{N} - 1) + \zeta \lambda_1 z_2' \end{pmatrix},$$

$$\tilde{\psi}_2' = \begin{pmatrix} \sqrt{\zeta} \bar{z}_1' \\ \bar{z}_1' (\hat{N} - 1) - \zeta \lambda_2 \bar{z}_2' \end{pmatrix}, \quad \tilde{\xi} = \frac{1}{\sqrt{2\zeta}} \left( \frac{\zeta}{2} \right)^2 (\hat{N} - 1 + 2n_2'), \quad (3.24)$$

where $\frac{\zeta}{2} \hat{N} \equiv z_1 \bar{z}_1 + z_2 \bar{z}_2$. $\frac{\zeta}{2} \hat{n}_2' \equiv z_2' \bar{z}_2'$. (3.24) is really minimal: $|0, 0\rangle$ and $|n_1', 0\rangle_\lambda$ are annihilated by all the components of $\tilde{\Psi}_0$. 

\footnote{Although we only consider solutions of matrix equation (2.1) and do not construct gauge field explicitly, we call the solutions “instanton solutions” because in principle we can construct instanton from the matrix data. We regard $k$ as a number of instantons.}

\textbf{U(1) two-instanton solution\footnote{Although we only consider solutions of matrix equation (2.1) and do not construct gauge field explicitly, we call the solutions “instanton solutions” because in principle we can construct instanton from the matrix data. We regard $k$ as a number of instantons.}}
**U(1) three-instanton solutions**

Let us consider the \( k = 3 \) solutions corresponding to the following simple ideal:  

\[
\mathcal{I} = \left\{ f(z_1, z_2) = \sum_{n_1, n_2} a_{n_1 n_2} z_1^{n_1} z_2^{n_2} \mid a_{n_1 n_2} = 0 \text{ when } (n_1, n_2) \text{ belongs to the Young tableau (Y1).} \right\}
\]  

\( (1, 0) \quad (0, 0) \quad (0, 1) \)  

(3.26)

The solutions to (2.1) is  

\[
B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\zeta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & \sqrt{\zeta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3\zeta} \end{pmatrix}, \quad J = 0.
\]  

(3.27)

We can find the (unnormalized) minimal zero-mode:  

\[
\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\xi} \end{pmatrix}, \quad \tilde{\psi}_1 = \begin{pmatrix} \sqrt{\zeta} z_1^2 \\ \sqrt{\zeta} \bar{z}_1 \bar{z}_2 \\ \xi z_1 \bar{z}_2 \end{pmatrix}, \quad \tilde{\psi}_2 = \begin{pmatrix} \sqrt{\zeta} z_1 \bar{z}_2 \\ \sqrt{\zeta} \bar{z}_1^2 \xi \\ \bar{z}_1 \bar{z}_2 \end{pmatrix}, \quad \tilde{\xi} = \frac{1}{\sqrt{3\zeta}} \left( \frac{\zeta}{2} \right)^2 \bar{N}(\bar{N} - 1).
\]  

(3.28)

(3.28) really annihilates \(|0, 0\rangle, |1, 0\rangle, |0, 1\rangle\) and hence minimal.

Next consider the ideal corresponding to the following Young tableau:  

\[
(2, 0) \quad (1, 0) \quad (0, 0) \]  

(3.29)

The solution of (2.1) is given by  

\[
B_1 = \begin{pmatrix} 0 & \sqrt{\zeta} & 0 \\ 0 & 0 & \sqrt{2\zeta} \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = 0, \quad I = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3\zeta} \end{pmatrix}, \quad J = 0.
\]  

(3.30)

The (unnormalized) minimal zero-mode is given as  

\[
\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\xi} \end{pmatrix}, \quad \tilde{\psi}_1 = \begin{pmatrix} 2\zeta z_1^2 \bar{z}_2 \\ \sqrt{2\zeta} \xi \bar{N} \bar{z}_1 \bar{z}_2 \\ \left( \xi^2 \right)^2 \left\{ (\bar{N} + 1)(\bar{N} + 4) - 2(\bar{n}_1 - 1) \right\} \bar{z}_2 \end{pmatrix},
\]  

10This kind of ideal corresponds to a fixed points of \( T^2 \) action in \([8][9]\).
\[ \tilde{\psi}_2 = \begin{pmatrix} 2\zeta \hat{z}_1^3 \\ \sqrt{2\zeta} \frac{\hat{N}}{2} \hat{z}_1^2 \\ (\frac{\zeta}{2})^2 \left\{ (\hat{N} + 1)\hat{N} - 2\hat{n}_1 \right\} \hat{z}_1 \end{pmatrix}, \]

\[ \tilde{\xi} = \frac{1}{\sqrt{3\zeta}} \left( \frac{\zeta}{2} \right)^3 \hat{N} \left\{ \hat{N}(\hat{N} + 3) - 2(3\hat{n}_1 - 1) \right\}. \] (3.31)

We can check (3.31) annihilates \(|0,0\rangle, |1,0\rangle, |2,0\rangle\).

4 \( U(N) \) Instantons and Projection Operators

In the previous section we have clarified the notion of minimal zero-mode for \( U(1) \) case. In this section we will study the \( U(2) \) instanton solutions and see that projection of states by zero-modes also occur. Since the \( U(N) \) instanton solutions are essentially embedding of \( U(2) \) instanton solutions to \( U(N) \), this means projection of states is general phenomenon in instantons on noncommutative \( \mathbb{R}^4 \). In the following, we will observe two things.

1. Minimal zero-mode appear in the \( U(1) \) subgroup of \( U(2) \) gauge group. It annihilates some states even when the size of instanton is not small.

2. When the size of instanton becomes small, only the contribution from \( U(1) \) subgroup described by the minimal zero-mode remains.

Although we have not defined minimal zero-mode for \( U(N) \) case, the zero-modes similar to minimal zero-mode in \( U(1) \) case appear in the explicit solutions. Hence in the above we have also called them minimal zero-modes. The second observation can be understood as follows. We may define “small instanton” on noncommutative \( \mathbb{R}^4 \) as \( J = 0 \) solution of modified ADHM equations (2.1). Then, the solution is essentially the embedding of \( U(1) \) instantons to \( U(N) \).

\( U(2) \) one-instanton solution

The solution to the modified ADHM equation (2.1) is given by\(^{11}\)

\[ B_1 = B_2 = 0, \quad I = \begin{pmatrix} \sqrt{p^2 + \zeta} & 0 \\ 0 & \rho \end{pmatrix}, \quad J^\dagger = \begin{pmatrix} 0 & \rho \end{pmatrix}, \] (4.1)

\(^{11}\)There are of course family of solutions with different orientation in gauge group \( U(2) \). The resulting conclusions are same.
where $\rho$ is a real non-negative number and parameterizes the size of the instanton. The two ortho"normalized" zero-modes of $D_z$ are given by

$$\Psi^{(1)} = \begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \\ \xi^{(1)} \end{pmatrix} = \begin{pmatrix} \sqrt{\rho^2 + \zeta \bar{z}_2} \\ \sqrt{\rho^2 + \zeta \bar{z}_1} \\ (z_1 \bar{z}_1 + z_2 \bar{z}_2) \end{pmatrix} \left(z_1 \bar{z}_1 + z_2 \bar{z}_2 + \zeta + \rho^2\right)^{-1/2},$$

(4.2)

$$\Psi^{(2)} = \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \xi^{(2)} \end{pmatrix} = \begin{pmatrix} -\rho \bar{z}_1 \\ \rho \bar{z}_2 \\ 0 \end{pmatrix} \left(\left(z_1 \bar{z}_1 + z_2 \bar{z}_2 + \frac{\zeta}{2}\right) \left(z_1 \bar{z}_1 + z_2 \bar{z}_2 + \frac{\zeta}{2} + \rho^2\right)\right)^{-1/2}.$$  

(4.3)

The zero-mode $\Psi^{(1)}$ is a straightforward modification of (3.16). $\Psi^{(1)}$ annihilates $|0, 0\rangle$ for any values of $\rho$, and normalized in the subspace where $|0, 0\rangle$ is projected out. The zero-mode $\Psi^{(2)}$ annihilates no states in $\mathcal{H}$ and manifestly nonsingular even if $\rho = 0$. When $\rho = 0$, $\psi_1^{(2)} = \psi_2^{(2)} = 0$, and from (2.8) $\Psi^{(2)}$ does not contribute to the field strength. Therefore the structure of the instanton at $\rho = 0$ is completely determined by the $U(1)$ subgroup described by minimal zero-mode $\Psi^{(1)}$.

**$U(2)$ two-instanton solution**

We can also construct two-instanton solution and check the statements in the beginning of section. Here we only construct one simple solution. The solution of modified ADHM equation (2.1) is given by

$$B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = 0, \quad I = \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}, \quad J^\dagger = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

(4.4)

We can obtain two (unnormalized) zero-modes orthogonal to each other:

$$\begin{align*}
\Psi^{(1)} &= \begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \\ \xi^{(1)} \end{pmatrix}, \\
\psi_1^{(1)} &= \begin{pmatrix} \sqrt{\rho^2 + \zeta \bar{z}_1 \bar{z}_2} \\ \sqrt{\rho^2 + \zeta \bar{z}_2} (z_1 \bar{z}_1 + z_2 \bar{z}_2 + \frac{\zeta}{2}) \end{pmatrix}, \\
\psi_2^{(1)} &= \begin{pmatrix} \sqrt{\rho^2 + \zeta \bar{z}_1^2} \\ \sqrt{\rho^2 + \zeta \bar{z}_2} (z_1 \bar{z}_1 + z_2 \bar{z}_2 - \frac{\zeta}{2}) \end{pmatrix}, \\
\xi^{(1)} &= \begin{pmatrix} \frac{1}{\sqrt{2}} (z_1 \bar{z}_1 + z_2 \bar{z}_2)(z_1 \bar{z}_1 + z_2 \bar{z}_2 - \frac{\zeta}{2}) + \zeta \bar{z}_2 \bar{z}_2 \\ 0 \end{pmatrix}.
\end{align*}$$

(4.5)
\( \Psi^{(2)} = \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \xi^{(2)} \end{pmatrix}, \quad \psi_1^{(2)} = \begin{pmatrix} \rho \sqrt{\zeta} \bar{z}_2 z_2 \\ \rho z_1 (z_1 \bar{z}_1 + z_2 \bar{z}_2 + \frac{\zeta}{2}) \end{pmatrix}, \quad \psi_2^{(2)} = \begin{pmatrix} \rho \sqrt{\zeta} \bar{z}_1 z_2 \\ \rho z_2 (z_1 \bar{z}_1 + z_2 \bar{z}_2 + \frac{\zeta}{2}) \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} (z_1 \bar{z}_1 + z_2 \bar{z}_2) (z_1 \bar{z}_1 + z_2 \bar{z}_2 + \frac{\zeta}{2} + \zeta (z_2 \bar{z}_2 + \frac{\zeta}{2})) \end{pmatrix}. \quad \text{(4.6)} \)

\( \Psi^{(1)} \) is a slight modification of (3.24) with \((\lambda_1, \lambda_2) = (1,0)\). It annihilates \(|0,0\rangle, |1,0\rangle\).

\( \Psi^{(2)} \) is apparently non-singular. Hence when the size of the instanton is small, only the \(U(1)\) subgroup described by \(\Psi^{(1)}\) is relevant.

**Conclusions**

In this paper we have learned that the appearance of projection operator is a general phenomenon in ADHM construction on noncommutative \( \mathbb{R}^4 \). The interesting point is that the instanton solutions are always smooth, since projection operators project out potentially dangerous states from the beginning. Although the smoothness of the solutions follows straightforwardly from the noncommutative setting, such mechanism is peculiar to noncommutative space and similar phenomenon cannot be expected in commutative setting.

Since the small instantons have been a good clue to the nonperturbative aspects of string theory, we expect our results also have applications in string theory and matrix theory.

**Acknowledgments**

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A The Absence of Zero-mode of $\Box_z$

In this section we show that $\Box_z$ in (2.4) has no zero-mode. Suppose

$$\Box_z |v\rangle = 0 \quad (A.1)$$

for some $|v\rangle$, where $|v\rangle \in \mathcal{H}^\oplus k$, i.e. $|v\rangle$ is vector in $V = \mathbb{C}^k$ and vector in $\mathcal{H}$. Then,

$$\langle v| \Box_z |v\rangle = 0 \Rightarrow \langle v| \tau_z \tau_z^\dagger |v\rangle.$$  

From (A.2),

$$\langle v| \zeta |v\rangle = 0 \Rightarrow \langle v| \tau_z \tau_z^\dagger |v\rangle.$$  

hence $|v\rangle = 0$.

B The Uniqueness of the Normalized Minimal Zero-mode

In this appendix we show the uniqueness of the normalized minimal zero-mode (up to gauge transformation) when the gauge group is $U(1)$. Let us consider operator zero-mode which has the following form:

$$\Psi_0 = \sum_{i,j} (\Psi_0)_{ij} |U(f_i)\rangle \langle f_j|.$$  

Then its norm is:

$$\Psi_0^\dagger \Psi_0 = \sum_{i,k} (\Psi_0^\dagger)_{ik} (\Psi_0)_{kj} |U(f_k)\rangle |U(f_i)\rangle \langle f_j|.$$  

where

$$|U(f_k)\rangle |U(f_i)\rangle = |u_1(f_k)|u_1(f_i)\rangle + |u_2(f_k)|u_2(f_i)\rangle + |f_k|f_i\rangle.$$  

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Let us rewrite the equation \( D_z |U(f_i)\rangle = 0 \) as
\[
D u(f_i) = -f_i, \tag{B.4}
\]
where
\[
D = \begin{pmatrix}
B_2 - z_2 & B_1 - z_1 \\
-(B_1^* - \bar{z}_1) & B_2^* - \bar{z}_2
\end{pmatrix}, \quad u(f_i) = \begin{pmatrix}
|u_1(f_i)\rangle \\
|u_2(f_i)\rangle
\end{pmatrix}, \quad f_i = \begin{pmatrix}
|f_i\rangle I \\
0
\end{pmatrix}. \tag{B.5}
\]

Since the correspondence between the elements of ideal and vector zero-modes is one-to-one, we can consider inverse operator of \( D \):
\[
u(f_i) = -\frac{1}{D} f_i. \tag{B.6}
\]

Then, (B.3) can be written as
\[
\langle U(f_k)|U(f_l)\rangle = u^\dagger(f_k) u(f_l) + f_k^\dagger f_l = f_k^\dagger \left( \frac{1}{DD^\dagger} + 1 \right) f_l = \langle f_k | \left( f^\dagger \left( \frac{1}{DD^\dagger} \right)_{11} I + 1 \right) |f_l\rangle, \tag{B.7}
\]

where we denote the components of \((DD^\dagger)^{-1}\) as
\[
(DD^\dagger)^{-1} = \begin{pmatrix}
(DD^\dagger)^{-1}_{11} & (DD^\dagger)^{-1}_{12} \\
(DD^\dagger)^{-1}_{21} & (DD^\dagger)^{-1}_{22}
\end{pmatrix}. \tag{B.8}
\]

From (B.7), the matrix \( C_{kl} = \langle U(f_k)|U(f_l)\rangle \) has no zero-eigenvalue-vector and we can consider \((C^{-1})_{kl}\). The normalized minimal zero-mode is uniquely determined (up to phase factor):
\[
(\Psi_0)_{ij} = (C^{-1/2})_{ij}. \tag{B.9}
\]
References


