A System with Two Integrable Hierarchies

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Abstract

By assuming the symmetry group and using the inverse method, we find an integrable system which is constructed from superposition of two hierarchies of integrable equations. We discuss $SU(2)$ and $SL(2, R)$ cases in detail. Furthermore, we present a procedure to construct integrable equations by taking a coefficient in the hierarchies as function of x and t .

Key Words: integrable system, hierarchy, group, dispersionless equation, symmetry

§1 Introduction and Main Results

We presented a general scheme of the inverse method starting from the Lagrangian for the coupled dispersionless system¹⁾:

$$L = \text{Tr} \left(\frac{1}{2} S_x S_t - \frac{1}{3} G[S, [S_x, S]] \right). \quad (1)$$

Here, S is defined by

$$S = \sum_a \phi_a T^a = \sum_{a,b} \eta_{ab} \phi^a T^b, \quad (2)$$

where ϕ is a vector with components $(\phi_1, \phi_2, \dots)$, T^a is a generator of the Lie algebra satisfying the commutation relation with the structure constants f^{ab}_c :

$$[T^a, T^b] = f^{ab}_c T^c, \quad (3)$$

and the tensor η_{ab} is the inverse of the metric tensor η^{ab} defined by

$$\eta^{ab} = \text{Tr}(T^a T^b). \quad (4)$$

A constant matrix G is defined in terms of a constant vector κ with components $(\kappa_1, \kappa_2, \dots)$ as

$$G = \sum_a \kappa_a T^a. \quad (5)$$

The equation of motion is given by

$$S_{xt} + [S_x, [S, G]] = 0. \quad (6)$$

This system has the conserved quantities:

$$\text{Tr}(S_x^n), \quad (7)$$

for integer n . However, all of them are not independent. Important one is the case of $n = 2$.

Inverse method of eq.(6) is written as

$$\begin{aligned} V_x &= UV, \\ V_t &= WV, \end{aligned} \quad (8)$$

and its compatibility condition as

$$U_t - W_x + [U, W] = 0. \quad (9)$$

Assume U and W with an eigenvalue λ such that

$$\begin{aligned} U &= \lambda S_x, \\ W &= W_0 + \frac{1}{\lambda} W_{-1}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} W_0 &= [S, G], \\ W_{-1} &= G. \end{aligned} \tag{11}$$

Then the compatibility condition yields eq.(6) as

$$S_{xt} + [S_x, W_0] = 0. \tag{12}$$

So given an algebra, then we can find an integrable system.

We have previously presented some examples²⁾, which we shall discuss later. For the $SU(2)$ case, introduce a vector $\mathbf{r}$ with the components $(r_1, r_2, r_3)^t$ in such a way that S is given by

$$S = i \begin{pmatrix} r_3 & r_1 - ir_2 \\ r_1 + ir_2 & -r_3 \end{pmatrix}. \tag{13}$$

The inverse method is given as

$$\begin{aligned} U &= i\lambda \begin{pmatrix} r_{3x} & r_{1x} - ir_{2x} \\ r_{1x} + ir_{2x} & -r_{3x} \end{pmatrix}, \\ W &= \begin{pmatrix} 0 & r_1 - ir_2 \\ -r_1 - ir_2 & 0 \end{pmatrix} - \frac{i}{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{14}$$

The compatibility condition gives

$$\begin{aligned} r_{1xt} - 2r_{3x}r_1 &= 0, \\ r_{2xt} - 2r_{3x}r_2 &= 0, \\ r_{3xt} + 2r_{1x}r_1 + 2r_{2x}r_2 &= 0. \end{aligned} \tag{15}$$

The conserved quantity (7) for $n = 2$ is given by

$$r_{1x}^2 + r_{2x}^2 + r_{3x}^2 = C_0. \tag{16}$$

Taking $r_2 = 0$, then we obtain the equation which is equivalent to the sine-Gordon equation^{3, 4)}. On the same way, for the case of $SL(2, R)$, if we take

$$\begin{aligned} U &= \lambda \begin{pmatrix} r_{3x} & r_{1x} + r_{2x} \\ r_{1x} - r_{2x} & -r_{3x} \end{pmatrix}, \\ W &= \begin{pmatrix} 0 & -r_1 - r_2 \\ r_1 + r_2 & 0 \end{pmatrix} + \frac{1}{2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{17}$$

we obtain

$$\begin{aligned} r_{1xt} - 2r_{3x}r_1 &= 0, \\ r_{2xt} - 2r_{3x}r_2 &= 0, \\ r_{3xt} + 2r_{1x}r_1 - 2r_{2x}r_2 &= 0. \end{aligned} \tag{18}$$

The conserved quantity for $n = 2$ is given by

$$r_{1x}^2 - r_{2x}^2 + r_{3x}^2 = C_0. \quad (19)$$

Note that C_0 takes positive, negative or null value. Taking $r_1 = 0$, then we obtain the equation which is equivalent to the sinh-Gordon equation, the cosh-Gordon equation or the Liouville equation by depending on the choice of positive, negative or null value of C_0 , respectively.

In this paper, satisfying the conserved quantity (7), which plays a key role in our discussion, we shall investigate a generalized scheme of the inverse method (8) with

$$\begin{aligned} U &= \lambda R = \sum_a \lambda R_a T^a, \\ W &= \sum_n \lambda^n W_n = \sum_{n,a} \lambda^n W_{an} T^a, \end{aligned} \quad (20)$$

where R_a and W_{an} are real functions and n takes both positive and negative integers. With components V_i , U_{ij} and W_{ij} of V , U and W , respectively, eq.(8) with eq.(20) can be written in the form of a conservation law as⁵⁾

$$\left(\sum_{i,j} U_{ij} \frac{V_j}{V_i} \right)_t = \left(\sum_{i,j} W_{ij} \frac{V_j}{V_i} \right)_x, \quad (21)$$

so, we can obtain an infinite number of the conserved quantities by expanding V_j/V_i in the power series of λ .

From eqs.(9) and (20) we obtain the following series of equations as

$$\begin{aligned} &\dots\dots\dots \\ \lambda^4 : & -W_{4,x} + [R, W_3] = 0, \end{aligned} \quad (22a)$$

$$\lambda^3 : -W_{3,x} + [R, W_2] = 0, \quad (22b)$$

$$\lambda^2 : -W_{2,x} + [R, W_1] = 0, \quad (22c)$$

$$\lambda^1 : R_t - W_{1,x} + [R, W_0] = 0, \quad (22d)$$

$$\lambda^0 : -W_{0,x} + [R, W_{-1}] = 0, \quad (22e)$$

$$\lambda^{-1} : -W_{-1,x} + [R, W_{-2}] = 0, \quad (22f)$$

$$\lambda^{-2} : -W_{-2,x} + [R, W_{-3}] = 0, \quad (22g)$$

$\dots\dots\dots$

Since T^a s are generators of the Lie algebra (3), eqs.(22) make a closed system. We should note that the equation of motion (22d) is given by two parts of hierarchy, one is W_1

determined from the upper part of a series of equations ($n > 1$) and another one is W_0 given by the lower part of that of equations ($n < 1$). W_1 is determined by a linear sum of the equations given by W_j ($j \geq 1$) with $W_i = 0$ ($i > j$). On the same way, we can obtain W_0 as a sum of the equations given by W_j ($j \leq 0$) with $W_i = 0$ ($i \leq j$). So we shall find that eq.(22d) consists of the following superposition of two hierarchies as

$$\begin{aligned} R_t - W_{11,x} - W_{12,x} - W_{13,x} - \cdots \\ + [R, W_{00}] + [R, W_{01}] + [R, W_{02}] + [R, W_{03}] + \cdots = 0, \end{aligned} \quad (23)$$

where W_{1j} is the equation obtained from W_j of the upper part and W_{0j} is the equation from W_{-j} of the lower part. Here, we use the hierarchy in the same sense as the Korteweg-de Vries hierarchy.⁶⁾

If we remove the constraint (7), we can take an arbitrary coefficient as a function of x and t in the hierarchies. Giving the function, we demonstrate a procedure to construct new integrable equations in the sense that they are not a simple linear combination of the constituent equations. We shall obtain simple or complex integrable equations depending on a choice of simple or complex forms of the function. These equations have the inverse form (8).

We shall consider $SU(2) \cong O(3)$ and $SL(2, R) \cong O(2, 1)$ as the symmetry groups. One of the interesting results is that, for the $SU(2)$ case, the equation of motion consists of the hierarchy of the localized induction equation of a thin vortex filament, which is equivalent to the Heisenberg spin equation, and the hierarchy of the dispersionless equations which is equivalent to the Lund-Regge equations^{7, 8)}. It is interesting that Fukumoto and Miyajima showed that the Lund-Regge equations is given by taking all sum of the localized induction equations in the hierarchy.⁹⁾ From this fact we see that the Lund-Regge equations and the induction equations have the same mathematical property. So it is very natural that two hierarchies is linked within the same inverse method. But we do not know how to interpret their interesting result from our point of view. Their result suggests that there must be some connection between two hierarchies.

For $SL(2, R)$, we shall obtain hierarchies of the hyperbolic type of the localized induction equations and of the dispersionless equations.

In the next section we shall discuss the case of $SU(2)$ and give the equation explicitly where we shall find the localized induction hierarchy and the hierarchy of the dispersionless system. For the case of $SL(2, R)$, we shall show their hyperbolic version in §3. In §4, we shall give a procedure to produce new integrable equations. The last section presents concluding remarks.

§2 Equation for $SU(2)$

Let us start the inverse method (8) with

$$\begin{aligned} U &= \lambda R = \sum_a \lambda R_a i\sigma_a, \\ W &= \sum_n \lambda^n W_n = \sum_{n,a} \lambda^n W_{an} i\sigma_a. \end{aligned} \quad (24)$$

Here R is defined to be the matrix with elements R_a and σ_a s denote the Pauli matrices. The conserved quantity is given by eq.(16) with $C_0 = 1$, that is, $RR = -1$.

If only $W_1 \neq 0$ and all others $W_i = 0$ ($i \geq 2$), then, from eq.(22c) we obtain

$$W_1 = A_1 R. \quad (25)$$

Here A_1 is a constant. Equation(22d) is given by

$$R_t - (A_1 R)_x = 0. \quad (26)$$

This is the first equation of the upper part of the hierarchies.

If only W_1 and W_2 are not zero, from eqs.(22c) and (22b) we get

$$\begin{aligned} W_2 &= 2A_2 R, \\ W_1 &= -A_2 R R_x = -\frac{A_2}{2} [R, R_x], \end{aligned} \quad (27)$$

because of $RR_x + R_x R = 0$. A_2 is a constant. Equation(22d) is given by

$$R_t + \frac{A_2}{2} ([R, R_x])_x = 0. \quad (28)$$

This is the second equation of the upper part of the hierarchies.

If W_1 , W_2 and W_3 are not equal zero, from eqs.(22c), (22b) and (22a) we get

$$\begin{aligned} W_3 &= 4A_3 R, \\ W_2 &= -2A_3 R R_x, \\ W_1 &= -A_3 \left(R_{xx} + \frac{3}{8} [R_x, [R, R_x]] \right), \end{aligned} \quad (29)$$

with a constant A_3 . We note that, without the second term of the right hand side of W_1 , $-A_3 R_{xx}$ satisfies eq.(22c), but the conserved quantity (16) is not satisfied, so we must included the second term. Equation(22d) is given by

$$R_t + A_3 \left(R_{xx} + \frac{3}{8} [R_x, [R, R_x]] \right)_x = 0. \quad (30)$$

This is the third equation of the upper part of the hierarchies.

Linear sum of all three equations

$$R_t - (A_1 R)_x + \frac{A_2}{2}([R, R_x])_x + A_3 \left(R_{xx} + \frac{3}{8}[R_x, [R, R_x]] \right)_x = 0. \quad (31)$$

yields the integrable equation. Higher order equations starting from W_n ($n \geq 4$) are able to be calculated on the same way. Then we obtain the localized induction hierarchy where R means the tangent vector along the filament and x denotes the arclength. The second term of eq.(31) represents contribution of the constant flow, the third term that of the swirl flow and the last term that of the axial flow. Equation(31) is also called the Heisenberg spin equation. In order to derive eq.(31), the inverse method (8) is given by

$$\begin{aligned} U &= \lambda R, \\ W &= \lambda A_1 R + 2\lambda^2 A_2 R - \lambda A_2 R R_x \\ &\quad + 4\lambda^3 A_3 R - 2\lambda^2 A_3 R R_x - \lambda A_3 \left(R_{xx} + \frac{3}{8}[R_x, [R, R_x]] \right). \end{aligned} \quad (32)$$

For the case of the lower part of eq.(22), if only $W_0 \neq 0$, we obtain from eq.(22e) as

$$W_0 = B_0. \quad (33)$$

Here B_0 is a constant matrix. Equation(22d) is given by

$$R_t - [R, B_0] = 0. \quad (34)$$

This is the first equation of the lower part of the hierarchies.

If only $W_0 \neq 0$ and $W_{-1} \neq 0$, from eqs.(22e) and (22f) we get

$$\begin{aligned} W_{-1} &= B_{-1}, \\ W_0 &= \int^x [R, B_{-1}] dx, \end{aligned} \quad (35)$$

where B_{-1} is a constant matrix. Equation(22d) is given by

$$R_t - [R, \int^x [R, B_{-1}] dx] = 0. \quad (36)$$

If we introduce

$$R = r_x, \quad (37)$$

eq.(36) becomes the dispersionless equation (6):

$$r_{xt} - [r_x, [r, B_{-1}]] = 0. \quad (38)$$

We see that eq.(36) equivalently eq.(38) gives the expression (6). If $B_{-1} = i\sigma_3$, we can reproduce eq.(15). This is the second equation of the lower part of the hierarchies.

If W_0 , W_{-1} and W_{-2} are not zero, from eqs.(22e), (22f) and (22g) we get

$$\begin{aligned} W_{-2} &= B_{-2}, \\ W_{-1} &= \int^x [R, B_{-2}] dx, \\ W_0 &= \int^x [R, \int^{x'} [R, B_{-2}] dx'] dx, \end{aligned} \quad (39)$$

with a constant matrix B_{-2} . Equation(22d) is given by

$$R_t - [R, \int^x [R, \int^{x'} [R, B_{-2}] dx'] dx] = 0. \quad (40)$$

This is the third equation of the lower part of the hierarchies.

We can also obtain a series of the equations starting from W_n ($n \leq -3$). So this set of the lower part of equations produces a hierarchy. Totally we obtain the equation

$$\begin{aligned} R_t - (A_1 R)_x + A_2 (R R_x)_x + A_3 \left(R_{xx} + \frac{3}{8} [R_x, [R, R_x]] \right)_x \\ - [R, B_0] - [R, \int^x [R, B_{-1}] dx] - [R, \int^x [R, \int^{x'} [R, B_{-2}] dx'] dx] = 0, \end{aligned} \quad (41)$$

in which the localized induction hierarchy and the hierarchy of the dispersion system are included.

Introducing vectors $\mathbf{R} = (R_1, R_2, R_3)^t$ and $\mathbf{R} = \mathbf{r}_x$ and vectors $\mathbf{B}_0$, $\mathbf{B}_{-1}$ and $\mathbf{B}_{-2}$ made by matrices B_0 , B_{-1} and B_{-2} , respectively, we can rewrite eq.(41) as

$$\begin{aligned} \mathbf{R}_t - (A_1 \mathbf{R})_x - A_2 \mathbf{R} \times \mathbf{R}_{xx} + A_3 \left[\mathbf{R}_{xx} + \frac{3}{2} \mathbf{R}_x \times (\mathbf{R} \times \mathbf{R}_x) \right]_x \\ + 2\mathbf{R} \times \mathbf{B}_0 - 4\mathbf{R} \times \left(\int^x \mathbf{R} \times \mathbf{B}_{-1} dx \right) + 8\mathbf{R} \times \left(\int^x \mathbf{R} \times \left(\int^{x'} \mathbf{R} \times \mathbf{B}_{-2} dx' \right) dx \right) = 0, \end{aligned} \quad (42)$$

and

$$\begin{aligned} \mathbf{r}_{xt} - (A_1 \mathbf{r}_x)_x + A_2 \mathbf{r}_x \times \mathbf{r}_{xxx} + A_3 \left[\mathbf{r}_{xxx} + \frac{3}{2} \mathbf{r}_{xx} \times (\mathbf{r}_x \times \mathbf{r}_{xx}) \right]_x \\ + 2\mathbf{r}_x \times \mathbf{B}_0 - 4\mathbf{r}_x \times (\mathbf{r} \times \mathbf{B}_{-1}) + 8\mathbf{r}_x \times \left(\int^x \mathbf{r}_x \times (\mathbf{r} \times \mathbf{B}_{-2}) dx \right) = 0. \end{aligned} \quad (43)$$

It should be remarked that, since, for the case of $R_2 = 0$, the conserved quantity is given by

$$R_1^2 + R_3^2 = 1, \quad (44)$$

R_1 and R_3 must rotate on a circle defined by the $R_1 R_3$ plane. However, we see that from eq.(22c), W_1 takes a value perpendicular to the plane. So we can say that there is no hierarchy produced from the upper part.

§3 Equation for $SL(2, R)$

The inverse method is given by eq.(8) where

$$\begin{aligned} U &= \lambda R = \sum_a \lambda R_a \sigma_a, \\ W &= \sum_n \lambda^n W_n = \sum_{n,a} \lambda^n W_{an} \sigma_a \end{aligned} \quad (45)$$

with the $SL(2, R)$ generators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (46)$$

The conserved quantity is given by eq.(19), that is, $RR = C_0$.

If only $W_1 \neq 0$ and all others $W_i = 0$ ($i \geq 2$), from eq.(22c) we obtain

$$W_1 = A_1 R, \quad (47)$$

where A_1 is a constant. On the same reason as the previous section, we note that other coefficients A_2 and A_3 and matrices B_0 , B_{-1} and B_{-2} in this section are also constants. Equation(22d) is given by

$$R_t - (A_1 R)_x = 0. \quad (48)$$

If only W_1 and W_2 are not equal to zero, from eqs.(22c) and (22b) we get

$$\begin{aligned} W_2 &= 2A_2 R, \\ W_1 &= -\frac{A_2}{C_0} R R_x = -\frac{A_2}{2C_0} [R, R_x]. \end{aligned} \quad (49)$$

because of $RR_x + R_x R = 0$. Equation(22d) is given by

$$R_t - \frac{A_2}{2C_0} ([R, R_x])_x = 0. \quad (50)$$

This gives the hyperbolic Heisenberg equation¹⁰⁾.

If W_1 , W_2 and W_3 are not equal to zero, from eqs.(22c), (22b) and (22a) we get

$$\begin{aligned} W_3 &= 4A_3 R, \\ W_2 &= \frac{2A_3}{C_0} R R_x, \\ W_1 &= \frac{A_3}{C_0} \left(R_{xx} - \frac{3}{8C_0} [R_x, [R, R_x]] \right). \end{aligned} \quad (51)$$

On the same reason as the $SU(2)$ case, the second term of the right hand side of W_1 must be included in order to satisfy the conserved quantity (19). So, eq.(22d) is given by

$$R_t - \frac{A_3}{C_0} \left(R_{xx} - \frac{3}{8C_0} [R_x, [R, R_x]] \right)_x = 0. \quad (52)$$

Higher order equations are calculated on the same way. Linear combination of all three equations becomes

$$R_t - (A_1 R)_x - \frac{A_2}{2C_0} ([R, R_x])_x - \frac{A_3}{C_0} \left(R_{xx} + \frac{3}{8} [R_x, [R, R_x]] \right)_x = 0. \quad (53)$$

This is a part of the hierarchy of hyperbolic type of the localized induction equations.

For the case of the lower part of eq.(22), we obtain from eq.(22e) as

$$W_0 = B_0 \quad (54)$$

if only $W_0 \neq 0$. Equation(22d) is given by

$$R_t - [R, B_0] = 0. \quad (55)$$

If only W_0 and W_{-1} are not equal zero, from eqs.(22e) and (22f) we get

$$\begin{aligned} W_{-1} &= B_{-1}, \\ W_0 &= \int^x [R, B_{-1}] dx. \end{aligned} \quad (56)$$

Equation(22d) is given by

$$R_t - [R, \int^x [R, B_{-1}] dx] = 0. \quad (57)$$

As before, if we introduce

$$R = r_x, \quad (58)$$

eq.(57) becomes the dispersionless equation (6):

$$r_{xt} - [r_x, [r, B_{-1}]] = 0, \quad (59)$$

which gives eq.(18) if $B_{-1} = \sigma_3$.

If W_0 , W_{-1} and W_{-2} are not equal to zero, from eqs.(22e), (22f) and (22g) we get

$$\begin{aligned} W_{-2} &= B_{-2}, \\ W_{-1} &= \int^x [R, B_{-2}] dx, \\ W_0 &= \int^x [R, \int^{x'} [R, B_{-2}] dx'] dx. \end{aligned} \quad (60)$$

Equations(22d) is given by

$$R_t - [R, \int^x [R, \int^{x'} [R, B_{-2}] dx'] dx] = 0. \quad (61)$$

We also get a series of equations starting from W_n ($n \leq -3$). Summing up the derived equations, we obtain the equation

$$\begin{aligned} R_t - (A_1 R)_x - \frac{A_2}{C_0} (R R_x)_x - \frac{A_3}{C_0} \left(R_{xx} - \frac{3}{8C_0} [R_x, [R, R_x]] \right)_x \\ + [R, B_0] + [R, \int^x [R, B_{-1}] dx] + [R, \int^x [R, \int^{x'} [R, B_{-2}] dx'] dx] = 0. \end{aligned} \quad (62)$$

This is the sum of the hyperbolic type of the localized induction equations and the dispersionless equations.

By defining vectors $\mathbf{R} = (R_1, R_2, R_3)^t$ and $\mathbf{r} = \mathbf{r}_x$ and vectors $\mathbf{B}_0$, $\mathbf{B}_{-1}$ and $\mathbf{B}_{-2}$ made by matrices B_0 , B_{-1} and B_{-2} , respectively, eq.(62) can be written as

$$\begin{aligned} \mathbf{R}_t - (A_1 \mathbf{R})_x - \frac{A_2}{C_0} \mathbf{R} \times \mathbf{R}_{xx} - \frac{A_3}{C_0} \left[\mathbf{R}_{xx} - \frac{3}{2C_0} \mathbf{R}_x \times (\mathbf{R} \times \mathbf{R}_x) \right]_x \\ + 2\mathbf{R} \times \mathbf{B}_0 + 4\mathbf{R} \times \left(\int^x \mathbf{R} \times \mathbf{B}_{-1} dx \right) + 8\mathbf{R} \times \left(\int^x \mathbf{R} \times \left(\int^{x'} \mathbf{R} \times \mathbf{B}_{-2} dx' \right) dx \right) = 0, \end{aligned} \quad (63)$$

and

$$\begin{aligned} \mathbf{r}_{xt} - (A_1 \mathbf{r}_x)_x - \frac{A_2}{C_0} \mathbf{r}_x \times \mathbf{r}_{xxx} - \frac{A_3}{C_0} \left[\mathbf{r}_{xxx} - \frac{3}{2C_0} \mathbf{r}_{xx} \times (\mathbf{r}_x \times \mathbf{r}_{xx}) \right]_x \\ + 2\mathbf{r}_x \times \mathbf{B}_0 + 4\mathbf{r}_x \times (\mathbf{r} \times \mathbf{B}_{-1}) + 8\mathbf{r}_x \times \left(\int^x \mathbf{r}_x \times (\mathbf{r} \times \mathbf{B}_{-2}) dx \right) = 0, \end{aligned} \quad (64)$$

where the vector product of two vectors $\mathbf{R} = (R_1, R_2, R_3)^t$ and $\mathbf{S} = (S_1, S_2, S_3)^t$ is defined by

$$\mathbf{R} \times \mathbf{S} = (-R_2 S_3 + R_3 S_2, R_3 S_1 - R_1 S_3, -R_1 S_2 + R_2 S_1)^t. \quad (65)$$

It should be remarked that, since, for the case of $R_3 = 0$, the conserved quantity is given by

$$R_1^2 - R_2^2 = C_0, \quad (66)$$

R_1 and R_2 must move on a hyperbolic curve defined by the $R_1 R_2$ plane. However, we see that from eq.(22c), W_1 takes a value perpendicular to the plane. Then we see that there is no hierarchy produced by the upper part on the same reason as the previous section.

§4 Construction of Integrable Equations

In this section, by removing the constraint (7), we give examples to construct new integrable equations in the sense that they are not given by a simple linear combination

of the constituent equations of the hierarchies. Here, we consider the $SU(2)$ case with the situation of only $W_1 \neq 0$ and $W_2 \neq 0$ of eqs.(22c) and (22b). The case of $SL(2, R)$ can be studied on the same way.

From eq.(22b), we have

$$W_2 = 2A_2R, \quad (67)$$

where A_2 is an arbitrary function of x and t . Equation(22c) becomes

$$-2A_2R_x - 2A_{2x}R + [R, W_1] = 0. \quad (68)$$

Referring eqs.(25) and (27) we assume W_1 as

$$W_1 = \alpha RR_x + \beta R. \quad (69)$$

Here α must satisfy

$$\alpha = -\frac{A_2}{C(x, t)} \quad (70)$$

where

$$C = -\frac{\text{Tr}(RR)}{2} \quad (71)$$

with

$$A_2 = \frac{A_{20}}{\sqrt{C}}. \quad (72)$$

A_{20} is a constant. β is an arbitrary functions of x and t . Giving β , we get another new equation (22d) with eq.(69)

Let us introduce the matrix I with the Pauli matrices as

$$I = \sum_n i\sigma_n, \quad (73)$$

and define N and D as

$$\begin{aligned} N &= \text{Tr}(IRR_x), \\ D &= \text{Tr}(IR). \end{aligned} \quad (74)$$

Taking β as

$$\beta = -\alpha \frac{N}{D}, \quad (75)$$

we obtain

$$W_1 = \begin{pmatrix} W_{13} & W_{11} - iW_{12} \\ W_{11} + iW_{12} & -W_{13} \end{pmatrix}. \quad (76)$$

with

$$\begin{aligned} W_{11} &= -i\alpha \frac{C(R_{3x} - R_{2x}) - \frac{C_x}{2}(R_3 - R_2)}{R_1 + R_2 + R_3}, \\ W_{12} &= -i\alpha \frac{C(R_{1x} - R_{3x}) - \frac{C_x}{2}(R_1 - R_3)}{R_1 + R_2 + R_3}, \\ W_{13} &= -i\alpha \frac{C(R_{2x} - R_{1x}) - \frac{C_x}{2}(R_2 - R_1)}{R_1 + R_2 + R_3}. \end{aligned} \quad (77)$$

This is symmetric with respect to R_i s and seems to be a new integrable equation. This symmetric property comes from the symmetric choice of β . Equation(22d) is given by

$$R_t - W_{1,x} = 0. \quad (78)$$

Note that, here, we choose β as

$$\text{Tr}(IW_1) = 0 \quad (79)$$

with eq.(69).

This is not a necessary condition, but just an example. Choosing different β s, we can obtain different equations.

§5 Concluding Remarks

We have found the system which contains the superposition of two hierarchies of integrable equations where the inverse method is given by eq.(8) with U and W of eq.(20). We have studied the $SU(2)$ and $SL(2, R)$ cases in detail. We have also found integrable equations which are not a simple linear combination of the constituent equations of the hierarchies by removing the constraint (7).

We note that the equation

$$W_{i,x} = [R, W_{i-1}] \quad (i \geq 2), \quad (80)$$

is a key to obtain a hierarchy so it plays a recurrision relation between equations in the hierarchy. As pointed out in the introduction, the relation between eq.(42) and the result by Fukumoto and Miyajima is not yet clear.

References

- [1] H. Kakuata and K. Konno: J. Phys. Soc. Japan **65** (1996) 340.
- [2] K. Konno and H. Kakuata: J. Phys. Soc. Japan **65** (1996) 713.
- [3] K. Konno and H. Oono: J. Phys. Soc. Japan **63** (1994) 3534.
- [4] R. Hirota and S. Tsujimoto: J. Phys. Soc. Japan **63** (1994) 3533.
- [5] M. Wadati, H. Sanuki and K. Konno: Prog.Theor.Phys. **53**(1975) 419.
- [6] P. G. Drazin and R. S. Johnson: *Soliton: an introduction*, (Cambridge University Press, 1989).
- [7] V.P. Kotlyarov: J. Phys. Soc. Japan **63** (1994) 3535.
- [8] K. Imai, K. Konno and H. Kakuata: J. Phys. Soc. Japan **67** 1998 4300.
- [9] Y. Fukumoto and M. Miyajima, J. Phys. A**29** (1996) 8025.
- [10] R.S. Ward and A.E. Winn, J. Phys. A**31** (1998) L261.