JET ACCELERATION BY TANGLED MAGNETIC FIELDS
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ABSTRACT

We explore the possibility that extragalactic radio jets might be accelerated by highly disorganized magnetic fields that are strong enough to dominate the dynamics until the terminal Lorentz factor is reached. Following the twin-exhaust model by Blandford & Rees (1974), the collimation under this scenario is provided by the stratified thermal pressure from an external medium. The acceleration efficiency then depends on the pressure gradient of that medium. In order for this mechanism to work there must be continuous tangling of the magnetic field, changing the magnetic equation of state away from pure flux freezing (otherwise conversion of Poynting flux to kinetic energy flux is suppressed). This is a complementary approach to models in which the plasma is accelerated by large scale ordered fields. We include a simple prescription for magnetic dissipation, which leads to tradeoffs among conversion of magnetic energy into bulk kinetic energy, random particle energy, and radiation. We present analytic dynamical solutions of such jets, assess the effects of radiation drag, and comment on observational issues, such as the predicted polarization and synchrotron brightness. Finally, we try to make the connection to observed radio galaxies and γ-ray bursts.

INTRODUCTION

Many extragalactic radio jets move with bulk Lorentz factors $\Gamma > 10$, as evidenced by very short variability timescales, superluminal proper motion of jet features, and diluted particle lifetimes, yet the process that actually accelerates the material is not well known, and neither is the mechanism that collimates the outflow.

One of the first serious models for the large scale dynamics of extragalactic radio jets is the 'twin exhaust' model (Blandford & Rees 74, BR74 throughout the rest of the paper). In this model, relativistic particle pressure provides the bulk acceleration via conversion of internal to kinetic energy. The collimation comes from confinement by an external medium, the pressure of which is very likely stratified in the gravitational field of the central black hole and the host galaxy and cluster (either in a hydrostatic equilibrium configuration or as a wind). Pressure balance then dictates the evolution of the jet bulk Lorentz factor $\Gamma$, and the jet diameter $R$. However, cooling processes for particles with highly relativistic random motions (necessary to produce outflows with large bulk Lorentz factors $\Gamma$) are very efficient, thus competing with and probably disabling bulk acceleration.

The problem of radiative losses is similarly limiting in the case of radiatively accelerated jets (Phinney 1982). Furthermore, these jets rely on the presence of a strong point-like radiation source (the terminal Lorentz factor is limited by the solid angle subtended by the radiation source due to relativistic aberration). The same Inverse Compton (IC) scattering effects that lead to the bulk acceleration lead to even stronger radiative losses. In fact, radiation drag can actually decelerate the jet if the external radiation field is isotropic, which leads to both random energy losses and kinetic energy losses by the same process.

Radiative losses are much less limiting if the energy is stored in the magnetic field. Organized magnetic fields have been suggested to provide both acceleration and collimation (e.g., Blandford & Payne 1982, Li, Chiueh, & Begelman 1992), however, such solutions, in which the collimation is provided by toroidal field, seem to be hampered by instabilities (Begelman 1998). Furthermore, it is difficult to achieve collimation on the basis of magnetic tension alone without contribution from an external pressure source (Begelman 1995, B95 hereafter).

Polarization measurements show that the magnetic field in jets is probably not well organized — the polarization is generally well below the maximal value of ~70%; only in knots does the polarization tend towards this value (note, however, that interpretations of polarization measurements are often ambiguous, since different field geometries can sometimes lead to the same net polarization). This argues for the presence of largely unorganized, chaotic fields, which could easily account for the high polarization measured in the knots if they are interpreted as shocks, compressing the field in the shock plane (Laing 1980, B95). This goes hand in hand with the fact that the field produced in the disk by dynamo processes is expected to be highly chaotic. Since the conditions in the jet will likely be controlled by disk physics, we should expect the same statement to be true for the magnetic field in the jet at least close to the disk. These arguments led us to investigate the dynamics of jets containing large amounts of such...
disorganized magnetic fields.

The rate of acceleration in a jet propelled by internal (isotropic) particle pressure in an external pressure gradient is limited to \( \Gamma \propto p_{\text{ext}}^{-1/4} \) (BR74), which means that the acceleration to bulk Lorentz factors of \( \Gamma \sim 10 - 100 \) would occur over length scales \( \gtrsim (1000) r_{\text{s}} \) for external pressure gradients \( p \propto z^{-2} \). One might think that an anisotropic pressure in the form of chaotic magnetic fields could increase the rate at which the jet is accelerated, if the excess momentum flux is oriented along the direction of the jet. We will show that under a given set of simple assumptions the rate of acceleration is actually the same as in the classic case considered by BR74, i.e., \( \Gamma \propto p_{\text{ext}}^{-1/4} \).

It is unlikely that the magnetic field evolves without some form of dissipation, especially if it is highly unorganized (reconnection is a diffusive process, so strong gradients in the field, as are present if the field is highly tangled on small scales, will likely lead to increased dissipation). These loss processes can compete with the efficiency of bulk acceleration by removing energy from the flow reservoir. We will investigate the effects such a tradeoff might have on the dynamics and appearance of jets.

We will lay out the simplifying assumptions going into this model in §2. Section 3 contains a brief discussion of allowed solutions and the sonic transition of such solutions. In §4 we will discuss self-similar and asymptotic solutions and present some full analytic solutions, including a simple estimate of the effects of radiation drag. Section 5 contains a discussion of the aforementioned tradeoff between dissipation and acceleration, some predicted observational consequences, and an outlook on applications of this model to radio galaxies and \( \gamma \)-ray bursts. Finally, we will summarize our results in §6.

2. THE MODEL

The model we are employing here is closely related to (and an extension of) the ‘twin-exhaust’ model put forward by BR74. We adopt a similar scenario under which the jet is launched into a stratified external medium. In the case we are considering, turbulent and highly disorganized magnetic fields dominate the internal energetics of the jet.

We have tried to illustrate the overall picture we are employing in Fig. 1: Interstellar magnetic field is advected inward by the accretion disk. Turbulent shear then amplifies the field and tangles it up (dynamo action). This effect will grow stronger with decreasing distance to the black hole. Eventually, regions of very high field strength will develop. Due to their buoyancy they will accelerate away from the black hole, forming an initial outflow. This outflow is then collimated by the pressure of the external medium. The jet channel is constrained by pressure balance, i.e., the jet will expand or contract in such a way that an equilibrium solution is set up for which the flow is stationary. As the flow expands, we assume that micro instabilities and turbulence constantly rearrange the field. As in the pure particle pressure case, the flow can go through a critical point, where the radius \( R \) has a minimum and beyond which the flow will become self-similar (if the external pressure itself behaves self-similarly with distance to the black hole) before the rest mass energy starts dominating the inertia, at which point the jet will reach a terminal Lorentz factor \( \Gamma_{\infty} \). Along the way, the field might dissipate energy via reconnection-like processes, and radiation drag might alter the dynamics.

We assume that no energy or particles are exchanged between jet and environment, except for radiative losses. However, the momentum discharge (the jet thrust) \( Q \) need not be conserved along the jet. By assuming that quantities like \( \rho \), \( U_i' \equiv (B_i'^2)/8\pi \) (where a prime denotes that the quantity is measured in the comoving frame) do not vary significantly across the jet we simplify the analysis to a quasi-1D solution. We ignore effects of shear at the jet boundaries. We assume throughout most of the paper that the advected matter is cold (i.e., enthalpy density \( h' \approx n'm_{\text{particle}}c^2 \)). We are looking for stationarity of flow along the jet (i.e., far from the terminal shock), enabling us to drop time derivatives. Finally, to make this quasi 1D treatment possible, we will need to make the assumption that the jet is narrow, which in our case implies that the opening angle is small compared to the collimation angle, i.e., \( dR/dz < \ll 1/\Gamma \). As we will later show, this also implies that the jet is in causal contact with its environment (as required by the assumption that the jet is in pressure equilibrium with the surrounding medium).

2.1. Treatment of Magnetic Field

We use cylindrical coordinates \((r, \phi, z)\) with the \( z \)-axis oriented along the jet axis. The flow velocity is not aligned with the \( z \)-axis for \( r > 0 \) (the jet expands). We assume that the sideways velocity is small compared to \( v_{\text{r}} \) but nonvanishing, i.e., the flow is well collimated. The magnetic field is expressed in a different basis, since the standard basis vectors \( \hat{e}_r \) and \( \hat{e}_z \) are not orthogonal in the comoving frame for \( r \neq 0 \). One axis of this new basis is aligned with the local velocity vector, \( \hat{e}_r \). The second unit vector \( \hat{e}_\phi \) is coincident with the \( \phi \)-unit vector of the lab coordinate system. The third unit vector \( \hat{e}_z \) is obtained from the cross product of the other two. In the comoving frame we have

\[
\mathbf{B}' = B'_r \hat{e}_r + B'_\phi \hat{e}_\phi + B'_z \hat{e}_z.
\]  

(1)

As mentioned before, the 1D approximation is only possible if the opening angle is small compared to the beam ing angle. This is because the Lorentz factor will not be nearly uniform across the jet otherwise. The assumption that \( dR/dz < \ll 1/\Gamma \) simplifies the equations of motion significantly.

Following B95, who investigated similar jets in the non-relativistic limit, we assume that the magnetic field is highly disorganized. In the comoving frame, averages over the individual components and cross terms vanish while the energy density in the individual components is not zero:

\[
\langle B'_i B'_j \rangle = 0, \quad \text{for} \ i \neq j, \quad \langle B'_i \rangle = 0,
\]

\[
\langle B'_i'^2 \rangle \equiv 8\pi U'_i \neq 0. \tag{2}
\]

Lorentz transformation of the field to the lab frame (and to the cylindrical coordinate system aligned with the jet axis) yields

\[
\mathbf{B} = \left( \frac{v_r}{v} \Gamma B'_r + \frac{v_\phi}{v} B'_\phi \right) \hat{e}_r + \Gamma B'_\phi \hat{e}_\phi
\]

\[
+ \left( \frac{v_r}{v} B'_r - \frac{v_\phi}{v} \Gamma B'_\phi \right) \hat{e}_z. \tag{3}
\]
Some of the components are now correlated. The electric field in the lab frame is

\[ E = v_k B'_\phi \delta_r - \sqrt{v_e^2 + v_s^2} \nabla B'_\phi \phi - v_r B'_\phi \delta_z. \] (4)

### 2.1.1. Magnetic Equation of State

Without the presence of turbulent rearrangement of the field, flux freezing would govern the behavior of the individual components. If we assume the presence of turbulent mixing between the different field components, we might expect the field to follow a modified evolution according to

\[ dB'_i = \sum_j \alpha_{ij} \frac{\partial B'_j}{\partial \Gamma} d(\Gamma) v + \sum_j \beta_{ij} \frac{\partial B'_j}{\partial R} dR. \] (5)

where the subscript \( ff \) denotes the value the derivative would take under flux freezing, and \( \alpha_{ij} \) and \( \beta_{ij} \) are arbitrary mixing coefficients. Based on this picture we therefore choose the following convenient ad-hoc parametrization of the field evolution with Lorentz factor \( \Gamma \) and jet radius \( R \), including rearrangement:

\[
\begin{align*}
B'_\omega &\propto B'_\phi \propto (\Gamma)^{2+\mu_1} R^{-2+\mu_2}, \\
B'_\perp &\propto (\Gamma)^{\mu_1} R^{-4+\mu_4}.
\end{align*}
\] (6)

This is the magnetic equation of state we use. In the case of pure flux freezing, \( \mu_i = 0 \) for all \( i \). In the case of a completely isotropic field we have \( \mu_1 = 2/3, \mu_2 = -2/3, \mu_3 = -4/3, \mu = 4/3 \). Note that this prescription is still fully general [until we make some limiting assumptions about the \( \mu_i(r, z) \)]. Since the re-arrangement process mixes the perpendicular and parallel components of the field, we would expect that the field behavior is changed from flux freezing in such a way that the coefficients \( \mu_i \) are bracketed by the values they take in the case of flux freezing. Since the case of a purely isotropic field must be included in our analysis, it is clear that this condition requires that \( \mu_2 < 0 < \mu_1 \) and \( \mu_3 < 0 < \mu_2 \).

We define two quantities to characterize the anisotropy of the magnetic pressure:

\[
\begin{align*}
\zeta &\equiv \frac{U'_\omega - U'_\perp}{U'_\parallel + U'_\perp}, \\
\delta &\equiv \frac{U'_\phi - U'_\omega}{U'_\parallel + U'_\perp},
\end{align*}
\] (7)

where \( U'_\perp \equiv U'_\omega + U'_\perp \). Thus, the magnetic field is purely perpendicular for \( \zeta = -1 \), and purely parallel for \( \zeta = 1 \). The perpendicular component is purely radial for \( \delta = -1 \) and purely toroidal for \( \delta = 1 \). The field is perfectly isotropic for \( \zeta = -1/3 \) and \( \delta = 0 \). It is obvious from equation (6) that \( \delta = 0 \) is constant for any combination of parameters, since \( U'_\perp \propto U'_\phi \) by assumption.

While this parametrization alone is rather unrestrictive, we can limit it to a one parameter family by assuming that the \( \mu_i \) are constants under any possible variation of \( \Gamma \) and \( R \), and that the rearrangement process does not change the total comoving energy density in the magnetic field. (Otherwise the same process would have to act as an energy sink, since we assume that the magnetic field is the dominant term in the internal energy budget. We would therefore be dealing with a dissipative process, which we will address in §2.1.2.) We can solve for \( \mu_i \) in terms of \( \zeta \) by fixing either \( \Gamma \) or \( R \) and demanding that the total energy density \( U' \equiv \sum U'_i \) behave the same as it would following flux freezing:

\[
\begin{align*}
dU' = & U'_\perp \left[(\mu_1 - 2) d(\Gamma v) + (\mu_2 - 2) dR\right] \\
+ & U'_\parallel \left[\mu_1 d(\Gamma v) + (\mu_3 - 4) dR\right] \\
= & -2U'_\perp d(\Gamma v) + 4U'_\perp dR
\end{align*}
\] (8)

for arbitrary \( d(\Gamma v) \) and \( dR \). Constancy of any of the \( \mu_i \) then implies constancy of \( \zeta \) and substitution of \( \zeta \) from equation (7) yields

\[
U_i \propto (\Gamma v)^{\zeta-1} R^{-3-\zeta}
\]

\[
\begin{align*}
\mu_1 & = 1 + \zeta, \\
\mu_2 & = -1 - \zeta, \\
\mu_3 & = -1, \\
\mu_4 & = 1 - \zeta
\end{align*}
\] (9)

which includes the isotropic case, where the magnetic field behaves like a relativistic gas, for which \( \zeta = -1/3, \delta = 0 \).

It turns out that one can find special analytic solutions with constant \( \mu_i \) that satisfy equation (8) without the requirement that \( \zeta \) be constant (see §4). For these solutions the rearrangement process conserves the comoving magnetic energy density only under the variations in \( \Gamma \) and \( R \) allowed by the Bernoulli equation [i.e., \( d(\Gamma v) \) and \( dR \) in equation (8) are not arbitrary]. The only condition on the parameters \( \mu_i \) for such a solution is that \( \mu_i/\mu_3 < \mu_3/\mu_4 \). These solutions are limited to the self-similar range, where the jet is dominated by magnetic pressure. Once they approach the terminal phase (i.e., \( \rho' \geq U' \)), the parameters \( \mu_i \) must vary with \( z \). For the rest of the paper we will assume that equation (9) holds unless indicated otherwise.

### 2.1.2. Dissipation of Magnetic Energy

It is unlikely that the tangled magnetic field evolves without any dissipative energy (e.g., via reconnection). We thus include a simple, \( ad \) hoc prescription of magnetic energy losses. We base our parametrization on the idea that the magnetic field is always in a nearly force-free equilibrium. However, it is impossible to maintain perfect force-free conditions everywhere and as the jet expands in either direction, the field responds by rearrangement between the different components [eq. (9)] and by dissipation of some of its energy. We therefore assume that the dissipation rate is roughly proportional to the divergence of the velocity in the comoving frame:

\[
\left( \frac{\partial U'_i}{\partial \tau} \right)_{\text{diss}} \approx -\Lambda U'_i |\nabla \cdot \mathbf{v}'|.
\] (10)

or, in the lab frame

\[
\left( \frac{\partial U'_i}{\partial z} \right)_{\text{diss}} \approx -\Lambda U'_i |\frac{\partial}{\partial z} (\Gamma v R^2)|.
\] (11)

This ansatz can easily be generalized to different \( \Lambda_i \) for different field components (e.g., if the Alfvén velocity factors into \( \Lambda \)). For a more realistic dissipation model see the Appendix.

We will assume that the dissipated energy goes into isotropic particle pressure, which is then either (a) radiated away immediately as isotropic radiation in the comoving frame or (b) accumulated until the jet reaches a state of equipartition between particle pressure and magnetic field.
2.2. Equations of Motion

We write the relativistic continuity equation as

$$\rho' v_\perp \Gamma R^2 = \text{const.} \quad (12)$$

The energy and momentum equations are given by $T^{\alpha\beta}v_\alpha = 0$, ($T$ is the stress-energy tensor, separable into a matter and an electromagnetic part). In the absence of gravity, this reduces to $T^{0\beta}v_0 = 0$, which will be sufficient for the analysis through most of this paper since most of the acceleration will likely take place at distance $z > r_g \equiv GM/c^2$, where $M$ is the mass of the central black hole. It turns out, however, that gravity is important in discussing the critical points of the jet, in which case we approximate the covariant derivative by a Newtonian potential ($-\Gamma_0^{00} = \Gamma_1^{33} = \Gamma_0^{40} \approx r_g/z^2$). We will comment on the accuracy of this approximation in §3.1.

Since there is no energy exchange between the jet and the environment, and using the expression for the electromagnetic field measured in the lab frame from §2.1, we can write the energy equation as $T^{0\alpha}v_\alpha = 0$ (neglecting gravity). We integrate the equation over a cross-sectional volume of the jet and convert it to a surface integral using Gauss' law. The contribution from the sidewall is zero, giving

$$\Gamma^2 (\rho' c^2 + 4p') v_\perp \pi R^2$$
$$+ \frac{1}{4\pi} v_\perp \Gamma^2 \left( B_\perp^2 + B_\perp'^{2} \right) \pi R^2$$
$$\equiv L = \text{const.} \quad (13)$$

(while the $B_\perp'^{2}$ are now averaged quantities). Dividing equation (12) into equation (13) gives the relativistic Bernoulli equation. In the more general case including radiative losses and gravity we have

$$\frac{d}{dz} \Gamma^2 v_\perp R^2 \left( \rho' c^2 + 4p' + 2B_\perp'^{2} \right)$$
$$+ 2v_\perp \Gamma^2 v_\perp R^2 \left( \rho' c^2 + 4p' + 2U'_\perp + S_{\text{rad}} \right) = 0,$$

where $S_{\text{rad}}$ is the energy lost to radiation leaving the jet. We have to make some assumption about the form of $S_{\text{rad}}$, i.e., the amount of energy radiated away [cases (a) and (b) from §2.1.2].

The $z$-momentum flux $Q$ can be calculated in much the same way (integrating $T^{33}$ across a jet cross-section). Since the jet can exchange $z$-momentum with the environment, the momentum discharge need not be conserved, however. Dropping terms of order $v_z^2$, the integration yields

$$Q \equiv \int_A dT^{33} \approx \Gamma^2 v_\perp R^2 \left( \rho' + 4p'/c^2 \right) + \pi R^2 p'$$
$$+ \pi R^2 \left[ \Gamma^2 (1 + v_\perp^2) U'_\perp - U'_\parallel \right]. \quad (15)$$

The sideways momentum equation is given by $T^{3\alpha}v_\alpha - v_\alpha T^{3\alpha}v_\alpha = 0$. The condition that the solution be stationary (i.e., $\partial R(z)/\partial t = 0$) gives the pressure balance condition between the jet and its environment. We assume that the internal structure of the field adjusts to maintain the given cross-section. Since we assume that $v_r \ll v_z$, the internal variation will be sufficiently small, $\frac{\partial}{\partial r} \sim \left( \frac{v_r}{v_z} \right) \frac{\partial}{\partial z}$, to justify the assumption of uniformity (note: this assumption is only satisfied if the jet is in causal contact). We are thus only interested in the pressure balance condition at the jet walls, $r = R$, which gives

$$p_{\text{ext}} = U'_\perp + U'_\parallel - U'_\perp + p' = \text{const.} \quad (16)$$

Note that $U'_\perp = 0$ directly at the jet boundaries, since the magnetic field is assumed not to penetrate the contact discontinuity. However, since interior pressure balance demands that $U'_\perp + U'_\parallel - U'_\perp + p'$ be constant, we can set $U'_\perp - U'_\perp = \text{const.}$ and substitute it for $U'_\perp$ at $r = R$, which gives equation (16).

![Fig. 1.— Cartoon of the general picture employed in this article.](image)

Tangled magnetic field is generated in the disk, advected inward by the disk flow, and accelerated away from the black hole to form an initial outflow. Under suitable conditions, this outflow is then collimated and accelerates away from the core, keeping pressure balance with the thermal pressure provided by the jet environment.

3. DYNAMICAL SOLUTIONS

Before we start analyzing the equations presented above, it is worth noting that in the case of a cold ($p' = 0$) jet and a magnetic field following pure flux freezing ($\mu_1 = \mu_2 = 0$) the only possible solution to equation (13) far away from the core (i.e., $z \gg r_g$) is $\Gamma(z) = \text{const.}$, i.e., the jet expands sideways to satisfy pressure balance, without accelerating.
This is because both the kinetic energy flux and the Poynting flux do not vary with \( R \), while they do vary with \( v \), so that equation (13) becomes an equation of \( \Gamma \) only. Thus, fixing the total jet power \( L \) fixes \( \Gamma \). While a scenario like this might explain the coasting phase of the jet (where no more acceleration occurs), it cannot account for the initial bulk acceleration we are looking for.

Note that this is different than the case of anisotropic, relativistic particle pressure in the absence of isotropization (i.e., simply under adiabatic behavior of the individual components). In that case, the components scale like \( p_\parallel \propto (\Gamma v)^{-2} R^{-4} \) and \( p_\perp \propto \rho_0 (\Gamma v)^{-1} R^{-3} \). We might expect a behavior like this for a relativistic turbulent pressure term. The sideways pressure is simply \( p_\parallel \). At relativistic speeds, the solution approaches the one found by BR74, \( \Gamma \propto R \propto p^{-1/4} \). Thus, unlike in the magnetic case described in this paper, it is generally possible to accelerate a jet with anisotropic particle pressure without making any arbitrary assumptions about the randomization process.

This is simply because only the perpendicular component of the field, \( U'_\perp \), contributes to the Poynting flux, while all components of the pressure enter equation (13), which introduces a dependence on \( R \), making a solution \( \Gamma \neq \text{const.} \) possible. For a magnetically dominated solution to exist, on the other hand, we need a field rearrangement process at work, such as was described in §2.1.1. But even under such favorable conditions, a proper, accelerating solution is not always guaranteed.

### 3.1. Critical Points

Since the jet will likely be injected with sub-relativistic speed, the question arises as to where the jet crosses possible critical points and at what velocity. If the jet is injected at large distances from the central black hole, we can neglect gravity; if it is injected close to the hole we will have at large distances from the central black hole, we can neglect gravity. However, as is the case in the solar wind, \( c_\ast \) reduces to the sound speed of a relativistic gas with \( \gamma_{\text{ad}} = 4/3, \ c_\ast = \sqrt{4/3} \).

Locally we can always write \( p_{\text{ext}} \propto z^{-\xi} \), thus we define

\[
\xi = -d \ln p_{\text{ext}} / d \ln z.
\]

We can then substitute the pressure balance condition (16) into equation (17) in the limit \( \rho' \ll 4p' \ll U' \) and eliminate \( R \), which yields

\[
\Gamma^2 \left[ (3 + \zeta) v^2 + (1 + 3\zeta) \right] \frac{dv}{dz} = (1 + \zeta) \xi.
\]

This equation also has a critical point with a critical speed of

\[
c_\text{t} \equiv \frac{1 + 3\zeta}{-3 - \zeta
\]

Unlike equation (17), solutions cannot cross this critical point, since there \( dv/dz \to \infty \) (but see §3.2).

We expect \( dp_{\text{ext}}/dz < 0 \), so solutions always accelerate (decelerate) for \( v > c_\text{t} (\nu < c_\text{t}) \). Since \( c_\text{t} \) only exists for \( \zeta < -1/3 \), solutions with \( \zeta > -1/3 \) always accelerate. In that case equation (17) implies that for \( v > c_\text{t} \) \( (\nu < c_\text{t}) \) the jet is expanding (contracting) in the \( r \)-direction. Since \( c_\text{t} \) only exists for \( \zeta < 0 \), solutions with \( \zeta > 0 \) always expand sideways.

If, on the other hand, \( \zeta < -1/3 \), two branches of solutions exist: (a) solutions which are injected with \( v > c_\text{t} \), which always accelerate and go through a nozzle at \( v \equiv c_\ast \geq c_\text{t} \), and (b) solutions which are injected with \( v < c_\text{t} \), which always decelerate. Thus, at sufficiently large distances from the core for gravity to be negligible (see §3.2), highly anisotropic solutions with \( \zeta \approx -1 \) have to be injected at relativistic velocities to be accelerating, since \( c_\text{t} \to 1 \) as \( \zeta \to -1 \). This corresponds to the right branch of the dashed solutions plotted in Fig. 2 (which includes the effects of gravity, see §3.2).

It is instructive to look at the case of pure anisotropic relativistic particle pressure again. We define the pressure anisotropy as \( \zeta_p \equiv (p_\parallel - p_\perp)/(p_\parallel + p_\perp) \). If we fix \( \zeta_p \) by some rearrangement process as we did for the magnetic field in §2.1.1 (which might occur, for example, if there is coupling between magnetic field and turbulent pressure as suggested by B95), the behavior is very similar in the sense that accelerating solutions for \( \zeta_p > -1/3 \) (i.e., for \( p_\perp > 2p_\parallel \)) have to be injected at super-critical velocity \( v > c_{\text{tp}} = \sqrt{(1 + 3\zeta_p)/(5 - \zeta_p)} \).

If we simply let the components of \( p \) evolve adiabatically (without rearrangement), we arrive at a different critical velocity, \( c_{\text{tp}} = \sqrt{(5 + 7\zeta_p)/(9 + 3\zeta_p)} \). Since \( c_{\text{tp}} > c_\text{t} \) for \( \zeta_p > -5/7 \), and since \( dR/dz < 0 \) for \( v < c_\ast \), solutions injected with \( v > c_{\text{tp}} \) must accelerate to satisfy pressure balance, which means that \( \zeta_p \) increases with \( z \), reducing \( c_{\text{tp}} \) and thus making the flow more super-critical (i.e., once above the critical point, the solution moves away from it).

### 3.2. The Effects of Gravity on the Sonic Transition

As seen in the previous section, a solution for \( \zeta > -1/3 \) that starts out with \( v < c_\text{t} \) will always decelerate in the absence of gravity. However, as is the case in the solar wind,
gravity can actually help a flow go through a critical point. We thus consider \( M > 0 \) in this section.

We can go through the same arguments as in §3.1. The critical speeds are still given by equations (18) and (21), but now the critical conditions are different. At \( c_\ast \), equation (14) gives

\[
[4p'_\ast (2 - 2\gamma_{ad}) - 2(1 + \zeta)U'_\perp] \frac{dR}{dz} + \left( \rho'_\ast c^2 + 4p'_\ast + 2U'_\perp \right) \frac{2g}{z_\ast} = 0
\]

(22)

instead of \( dR/dz = 0 \). Since \( \zeta \geq -1 \) and \( \gamma_{ad} > 1 \), we can infer that \( dR/dz > 0 \) at \( z_\ast \), i.e., there is no ‘geometric’ nozzle at \( z_\ast \), anymore. The solution can always adjust \( dR/dz \) to satisfy this condition, thus the first critical point \( c_\ast \) becomes irrelevant.

Inclusion of the gravity term changes equation (20) to

\[
\Gamma^2 \left[ (3 + \zeta) v^2 + (1 + 3\zeta) \right] \frac{dv}{\rho dz} = (1 + \zeta) \frac{\xi}{z} - (3 + \zeta) \frac{2g}{z^2}.
\]

(23)

Now solutions can cross the critical point \( c_\perp \), since the right hand side of equation (23) vanishes at

\[
z_\perp = \frac{3 + \zeta}{1 + \zeta} \frac{2g}{\xi}.
\]

(24)

If \( v \neq c_\perp \) at \( z_\perp \), the solution must follow \( dv/dz = 0 \) at that point. This is true for all \( \zeta \). Since that is the only zero of equation (23), we can therefore conclude that solutions accelerating at any \( z > z_\perp \) will accelerate for all \( z > z_\perp \).

Solving the equations for \( dR/dz \) instead gives

\[
[(\zeta^2 - 1) - (3 + \zeta)(v^2 + \zeta)] \frac{dR}{\rho dz} = - (v^2 + \zeta) \frac{\xi}{z} + (\zeta - 1) \frac{2g}{z^2}.
\]

(25)

which also has a critical point at \( c_\perp \). For \( v = c_\perp \), the right hand side of this equation only vanishes at \( z_\perp \). In that case, \( dR/dz \) remains finite. For all other solutions (i.e., if \( z \neq z_\perp \) when \( v = c_\perp \)), we must have singularities in both \( dv/dz \) and \( dR/dz \). The singularity in \( dv/dz \) is evident from Fig. 2 and from equation (23); pressure balance then requires that \( dR/dz \) must have a singularity of opposite sign, since \( dp_{ext}/dz \) is assumed to be finite, i.e. \( p_{ext} \) is continuous.

We have numerically integrated equation (23) for two representative cases (\( \zeta = -0.9 \) and \( \zeta = 0, \xi = 2 \)) and plotted them in Fig. 2. Solutions are qualitatively different for \( \zeta < -1/3 \) and \( \zeta > -1/3: \)

- For \( \zeta < -1/3 \), there is one accelerating transonic solution, given by the condition in equation (24), shown in the upper panel of Fig. 2 as a thick solid black curve. This is also the only solution accelerating for all \( z \). As in the case of a regular adiabatic wind (Parker 1958), there also exists a decelerating transonic solution. Regions where solutions contract in the \( r \)-direction (i.e., where \( dR/dz < 0 \)) are shown as hatched areas. There are four more branches of solutions. Two branches are double-valued (shown as dashed curves in Fig. 2). The left branch of those solutions can be rejected since those solutions only exist for \( z < z_1 \). For an accelerating solution on the right branch to exist, it must be injected with \( v > c_1 \). This corresponds to the solutions discussed in §3.1 for which gravity can be neglected.

The remaining two branches are solutions that are always sub- or supercritical (plotted as thin solid black curves in Fig. 2). The sub-critical solutions decelerate for large \( z \) and always stay sub-relativistic. They are uninteresting as possible candidates for relativistic jets. The supersonic solutions decelerate for \( z < z_1 \) and accelerate for \( z > z_1 \). These solutions correspond to the super-critical solutions mentioned in §3.1. They always expand in the sideways direction. As we let \( \zeta \to -1, z_1 \to \infty \). This is not necessarily an indication that no solution is possible for \( \zeta \approx -1 \), since for those cases the critical speed is very close to 1, thus the solution can attain large \( \Gamma \). Furthermore, as we saw above, the solution is expanding even before it goes through \( z_1 \). We can thus have a regular (though sub-critical) accelerating jet even for \( \zeta \approx -1 \).

- For \( \zeta > -1/3 \), there is only one branch of solutions, all of which start out decelerating, shown in the bottom panel of Fig. 2. As the solutions reach \( z_1 \) they begin to accelerate and behave the same way as described in §3.1. Since \( z_1 \) moves inward for increasing \( \zeta \), this is no handicap. For \( \zeta > -1/3 \) we have \( z_1 < 8r_g/\xi \) from equation (24), which, for reasonable values of \( \xi \), is well in the regime where relativistic corrections become important and inside the region where we expect the injection to occur. All of these solutions have positive sideways expansion \( dR/dz > 0 \) for all \( z \).

The transonic solution for \( \zeta < -1/3 \) has some additionally nice features: Since we know \( z_1 \), we can relate the jet cross section to the total jet power \( L_1 \) at \( z_1 \). Assuming that the jet is still magnetically dominated at \( z_1 \), the kinetic luminosity of the jet is

\[
L_1 = \pi R_1^2 \Gamma_1^2 v_0 U_\perp.
\]

(26)

The external pressure at \( z_1 \) is \( p_{ext} \) and so

\[
R_1 = \sqrt{\frac{L_1 (\delta + \frac{4\pi}{\gamma - 1}) (1 + \zeta)}{\pi p_{ext} \sqrt{-\zeta}}},
\]

(27)

While we do not know \( p_{ext} \), for most parameter choices \( L_1 \) is very nearly equal to \( L_\infty \), which we have a reasonably good handle on from an observational point of view. Furthermore, we can estimate the jet width at observable distances and scale the solution back to \( z_1 \), which gives us an estimate of \( p_1 \) and thus \( U_{\perp,1} \). This in turn will allow us to determine the original matter loading of the jet from estimates of the terminal Lorentz factor \( \Gamma_\infty \).
4. SOLUTIONS IN THE SELF-SIMILAR REGIME AND ASYMPTOTIC SOLUTIONS

For an already relativistic jet in the 'self-similar' range \( z_{\perp} \ll z \ll z_{\text{inertia}} \) (where \( z_{\text{inertia}} \) is the location where \( \rho' c^2 = 2U_{\perp} \)) a self-similar solution can be found. Equations (9) and (13) give \( \Gamma \propto R^{-\mu_2/\mu_1} \) and with equation (9) we have \( \Gamma \propto R \).

Under the conditions of equation (9), the pressure balance condition gives

\[
\Gamma \propto R \propto p_{\text{ext}}^{-1/4},
\]

the same as in the case of isotropic particle pressure considered by BR74. For future reference we define the acceleration efficiency

\[
\eta \equiv -\frac{d\ln \Gamma}{d\ln p_{\text{ext}}},
\]

thus for this simple case \( \eta = 1/4 \).

If we adopt the less limiting restriction \( \mu_1/\mu_2 = \mu_3/\mu_4 \) (see §2.1.1) instead of equation (9), we can find powerlaw solutions in three limiting cases:

(a) For \( \delta = 0 \), the solution is given by \( \Gamma \propto R^{-\mu_2/\mu_1} \propto p_{\text{ext}}^{\mu_2/\mu_1} \), which can be very efficient for \( \mu_1 < -\mu_2 \) (as pointed out in §2.1.1, one would generally expect that \( \mu_2 < 0 < \mu_1 \)).

(b) For \( \zeta \approx -1 \), the solution is given by \( \Gamma \propto R^{-\mu_2/\mu_1} \propto p_{\text{ext}}^{1/(2\mu_1/\mu_2-2)} \), which has a limiting efficiency of \( \eta \leq 1/2 \).

(c) For \( \zeta \approx 1 \) the solution is approximately the same as case (a).

If \( \delta > 0 \), the solution will in general approach solutions (b) or (c) (i.e., \( \zeta \rightarrow \pm 1 \)). For \( \delta < 0 \) it is possible that the solution approaches a finite terminal Lorentz factor and zero opening angle if radial tension cancels the pressure due to \( U_{\parallel} \) and \( U_\phi \). Fig. 3 shows the different regimes. Note once again that these solutions exist only for \( \Gamma > 1 \) and \( \rho c^2 \ll U_{\perp} \). For all other cases the coefficients \( \mu_1, \mu_2, \mu_3, \mu_4 \) are not constant. Note that we would generally expect that \( \mu_1 \sim -\mu_2 \), since otherwise the re-arrangement process would be acting preferentially for changes in geometry in one specific direction, which seems arbitrary. Thus, these results reduce to the well known \( \eta \sim 1/4 \).

We can look for solutions in the presence of dissipation of magnetic energy. We now have to consider equation (14). We assume that the energy goes completely into relativistic particles, thus energy conservation implies

\[
\frac{dp'_{\parallel}}{dz}_{\text{dissipation}} = -\frac{1}{3} \frac{dU'_{\perp}}{dz}_{\text{dissipation}}.
\]

The particle energy can subsequently be radiated away as isotropic radiation. As long as \( p' \ll U'_{\perp} \), the radiative case is no different from the non-radiative case, since the adiabatic term in the particle pressure does not contribute to the dynamics.
the ratio of the opening angle $\alpha_\text{b}$ to the beaming angle $\alpha_\text{o}$, $\sim \Gamma^{-1}$. In the absence of dissipation we can write

$$\frac{\alpha_\text{o}}{\alpha_\text{b}} = 1 \frac{\mathrm{d}R}{\mathrm{d}z} \propto z^{\xi/2-1},$$

independent of $\zeta$. Thus, for steep pressure gradients $\xi > 2$, the opening angle will grow faster than the beaming angle and will thus always become larger even if it starts out being smaller. For shallow pressure gradients $\xi < 2$, the situation is reversed, i.e., the beaming angle will eventually become larger than the opening angle. The presence of dissipation changes this behavior qualitatively: the ratio $\alpha_\text{o}/\alpha_\text{b}$ now depends on both $\Lambda$ and $\zeta$, as illustrated in Fig. 4.

This has consequences for the appearance of the jet, since the effective beaming angle is given by the larger of the two angles. Under the assumption that the jet is always collimated, the opening angle in the coasting phase will always become smaller than the beaming angle, since the jet does not accelerate anymore. This could have important consequences for the morphology of superluminal sources: if the beaming angle were smaller than the opening angle, one might expect to see larger jet misalignments, or lose the jet morphology altogether. The appearance would become sensitive to the emissivity and local Lorentz factor as a function of position across the jet cross section.

Jets that expand too fast will eventually lose causal contact with their environment. This happens when the Alfvén crossing time of the jet becomes larger than the expansion time (in the comoving frame), i.e., when

$$\tau_A' = \frac{\tau_A}{c_{\text{Alfvén}}} \approx \frac{R}{c} > \tau_{\text{exp}}' \approx \frac{R}{\Gamma v \Gamma' d z / \mathrm{d} z} \approx \frac{z}{\Gamma c^2},$$

where $\tau_{\text{exp}}'$ is the effective expansion time of the jet.
(where we approximated $v_{\text{Alfvén}} \sim c$) or
\[ R > \frac{z}{\Gamma \zeta}. \] (33)

This corresponds (up to the factor $\zeta$) to the criterion when $\alpha_0 > \alpha\gamma$. Thus, for $\zeta > 2$ (in the absence of dissipation) the jet will eventually lose causal contact with its surroundings (see Fig. 4 for values of $\zeta$ and $\xi$ where this is the case). As mentioned in §2, a quasi 1D treatment is no longer possible, since the internal pressure balance is now regulated by shocks traveling inward from the jet walls. After the jet reaches the terminal phase, it will re-gain causal contact, since the opening angle will continually decrease (assuming the jet is still collimated).

4.2. Equipartition

Constant pumping of magnetic energy into particle pressure can lead to equipartition between $U'$ and $p$. We can use the self-similar solution to estimate $\Gamma_{\text{eq}}$, where the accumulated particle pressure surpasses the magnetic energy density (including effects of adiabatic cooling on the accumulated particle pressure, where we assume that it behaves as a relativistic gas, which gives an upper limit on the resulting pressure). Thus, for $\Gamma_{\infty} \geq \Gamma_{\text{eq}}$ the solution might be altered. For some parameter values $p$ never reaches the level of $U_{\perp}$. In that case we estimate the asymptotic ratio $(p'/U_{\perp})_{\infty}$. Figure 5 shows the results of those estimates. For large $\zeta$, equipartition can be reached quickly, thus, unless the energy going into particles is subsequently radiated away, the assumption that the particle pressure be negligible compared to the magnetic field energy density will be violated beyond $\Gamma_{\text{eq}}$.

![Fig. 5.— Value of $\Gamma$ for which the pressure accumulated by non-radiative dissipation reaches equipartition with $U_{\perp}$. Neglecting $p$ is no longer justified beyond that $\Gamma$. For $\zeta \leq -0.6$ the pressure never catches up with $U_{\perp}$, in that case we plotted contours of the limiting ratio $p'/U_{\perp}$ (dashed lines). The hatched and grey regions correspond to the regions in Fig. 4.

We define the energy distribution function as
\[ f(\gamma) \equiv F_0 \gamma^{-s}, \int_{\gamma_1}^{\gamma_2} f(\gamma) d\gamma = n', \] (34)

where $\gamma$ is the Lorentz factor according to a particle’s random motion, measured in the comoving frame, and $\gamma_1 \ll \gamma_2$ are the lower and upper spectral cutoffs. Since we assume that the magnetic field is dominating the internal energy budget, synchrotron losses can be very strong, provided the particle energy spectrum is flat enough so that most of the energy is at high particle energies (i.e., $s < 2$). In that case we can expect most of the energy to be radiated away immediately and the corresponding electrons will lose most of the inertia, thus the dissipated energy will not lead to a build-up of particle pressure. If, however, synchrotron losses are weak compared to dissipation (e.g., if the spectrum is too steep, or if synchrotron self-absorption traps most of the radiation to inhibit cooling), the effects of particle pressure can become important, as demonstrated in Fig. 5. For a discussion of the observational effects of the different radiative scenarios see §5.1

4.3. Full Solutions

We can solve the full equation (14) in the regime $\Gamma \gg 1$, i.e., for relativistic jets. As mentioned before, the pressure balance condition leads to a simple algebraic equation in $\Gamma$ and $R$. In the absence of dissipation and gravity, equation (13) is in fact another algebraic equation relating $\Gamma$ and $R$, thus, the two equations can be solved for $\Gamma(p_{\text{ext}})$ using a numerical root finder. Apart from reproducing the scaling behaviors established in §3, this will enable us to determine the terminal Lorentz factors and the length scales over which the transitions between different dynamical phases occur. Furthermore, we can use the full dynamical model to investigate the evolution of such observational quantities as polarization and synchrotron brightness.

In the absence of dissipation, the terminal Lorentz factor that can be reached with such a jet is simply
\[ \Gamma_{\infty} \equiv \lim_{p_{\text{ext}} \to 0} \Gamma = \Gamma_0 \left( \frac{\beta'_0 c^2 + 2U_{\perp} + 4p_0}{\rho'_0 c^2} \right), \] (35)

where subscripts 0 denote quantities evaluated at some arbitrary upstream point.

This simple solution is no longer possible in the presence of dissipation, which introduces a sink term into equation (13). As a result, we have to use equation (14) instead. Once the energy has been converted into particle pressure, it can be radiated away as isotropic radiation, which will not affect the dynamics of the jet any further (assuming that $p$ is dynamically unimportant). If the energy is stored as particle pressure, the pressure could eventually become dynamically important (see §4.2). Until that happens, though, the two solutions are identical. The terminal Lorentz factor is always reduced (see §5.1), but the acceleration efficiency can be increased for $\zeta < -3/5$ (see §4). We have plotted the solution for the radiative case (the one case solvable analytically) in Fig. 6.
timescale. If this is not satisfied, the influence of radiation heating time scales are short compared to the adiabatic simplest possible prescription.

We assume that the IC cooling and the dissipational energy alters both the acceleration efficiency and the terminal Lorentz factor $\Gamma_\infty$.

4.4. Radiation Drag

The presence of ultra-high-energy particles in AGN jets suggests that inverse Compton (IC) enhanced radiation drag might be dynamically important. While O’Dell (1981) initially suggested that pair jets might be accelerated by the Compton rocket effect, Phinney (1982) showed that it is hard to accelerate a plasma beyond fairly modest Lorentz factors by radiation pressure without a continuous source of particle acceleration to offset the strong IC cooling of the plasma. Furthermore, if the radiation source is not point-like, the terminal Lorentz factor is limited by the solid angle $\Omega$ the radiation source subtends. On the other hand, radiation drag can hamper the bulk acceleration of plasmas containing relativistic particles in the presence of a radiation field, if those particles are continuously reheated to overcome the IC losses. The dissipation mechanism discussed above could provide such reheating. Here we will consider the effect of radiation drag in the simplest possible prescription.

We assume that the IC cooling and the dissipational heating time scales are short compared to the adiabatic timescale. If this is not satisfied, the influence of radiation drag will be reduced. We can then expect dissipational heating to nearly balance IC losses in a near equilibrium situation. Thus the amount of IC drag is controlled by how much dissipation there is. For this approximation to be valid, IC losses must dominate the loss processes of the particles, i.e., the radiation energy density $U^{\prime}_\text{rad}$ in the comoving frame must be large compared to the magnetic field energy density $U'$ (for large enough $\Gamma$ this is always going to be the case, since the external field will be Doppler boosted). Finally, we assume that $\langle \beta^2 \gamma^2 \rangle \gg 1$, where $\gamma$ is the particle Lorentz factor in the comoving frame. This sets an upper limit of

$$U^{\prime}_\text{rad} \ll 6 \times 10^3 \text{ergs cm}^{-3}$$

on the comoving radiation energy density (otherwise IC cooling would have lowered the upper spectral cutoff to $\gamma_\infty \sim 1$). These assumptions allow us to eliminate $U^{\prime}_\text{rad}$ from the equations, since the drag term and the cooling term are both proportional to $U^{\prime}_\text{rad}$. In a sense, then, we are presenting an upper limit on the importance of IC radiation drag over large length scales. It has to be kept in mind, though, that drag can be much more important in non-stationary situations (like, for example, in shocks), which are beyond the scope of this paper.

We assume the jet is moving through a radiation field that is locally isotropic in the lab frame. Following Phinney’s treatment (1982), we can calculate the loss rate and the force due to IC scattering in the comoving frame and then transform back to lab frame to find the additional term for equation (13). We find that radiation drag always decreases both the acceleration efficiency $\eta$ and the terminal Lorentz factor $\Gamma_\infty$ by moderate amounts. It does not, however, introduce qualitatively new features. To demonstrate this, we have plotted a solution including radiation drag for otherwise identical parameters in Fig. 6.

5. DISCUSSION

5.1. Tradeoff Between Dissipation and Acceleration and Synchrotron Brightness

In the following section we will investigate the observational effects of the jets we have introduced in this paper. A highly dissipative jet will radiate away a large fraction of its internal energy along the way before reaching the terminal Lorentz factor $\Gamma_\infty$, while a non-dissipative jet will convert all its internal energy into kinetic energy flux. Since the jet will ultimately terminate and reconvert its kinetic energy flux into random particle energy when it slams into the surrounding medium, the ratio of kinetic luminosity (which could be estimated based on the energy input into the lobes, based on the source size and its age) to the radiative luminosity $L_{\text{rad}}(z)$ (i.e., the integrated luminosity of the jet before reaching the terminal shock) should give us some indication of the importance of dissipation.

We have already seen in § 4.3 that the presence of dissipation can lower $\Gamma_\infty$, thus lowering the kinetic energy flux at the terminal shock, $L_\infty$ (dominated by cold kinetic energy flux), and the produced hot-spot luminosity. A given fraction $b$ of the terminal luminosity $bL_\infty$ will be radiated away, which can be estimated from the hot-spot and cocoon luminosity (calculating $b$ is, of course, a highly non-trivial matter), giving us a handle on $\Lambda$. We have plotted the ratio

$$\epsilon \equiv \frac{L_{\text{rad}}}{L_\infty}$$

in Fig. 7. To make that plot, we chose parameters such that in the absence of dissipation the jet would reach $\Gamma_\infty(\Lambda = 0) = 2U_\perp \omega_\perp (\rho'/c^2) = 100$. If we had chosen a larger (smaller) value of this parameter, the lines in the
plot would move down (up), since $\Gamma_\infty$ depends non-linearly on both $\Lambda$ and $2U_0'/(\rho_0' c^2)$, so this plot is only a representative one of a family of similar plots for different $\Gamma_\infty (\Lambda = 0)$.

Another question is what the spatial distribution of the jet emission is, since there are several competing effects: the optical depth to self absorption (and inverse Compton up-scattering), Doppler boosting and opening angle (the larger of which will determine the opening angle of the cone into which most of the radiation goes), and of course the dissipative power of the jet itself [depending on $\Lambda$ and $\Gamma(z)$, $R(z)$]. This question can be asked with respect to the frequency integrated brightness $I$ or the spectral brightness $I_\nu$. If we assume that all the dissipated energy is radiated away on the spot, we can calculate the local dissipation rate, which must then be equal to the local frequency integrated jet emissivity $j'$ in the comoving frame. This assumption depends on the injected particle energy spectrum. If the spectrum is flatter than $s = 2$, most of the energy is in the low energy particles. In that case, synchrotron cooling can be efficient enough for our assumption of on the spot radiation to be effective. If, on the other hand, $s > 2$, most of the energy is in the low energy particles, synchrotron radiation will not be efficient (unless $\gamma_1$ is very high, in which case the injected spectrum would rapidly cool to a quasi mono-energetic distribution), and the jet will accumulate particle pressure or radiate by other means (note that Compton cooling would be equally insufficient to balance heating in this case).

To investigate the first case we will set $s < 2$. Given a viewing angle we can then determine the observed total intensity $I$, given by

$$I \propto \left( \frac{dU'}{dz} \right)_{\text{diss}} D^2 \frac{R}{\sin \theta},$$

where $\theta$ is the angle between line of sight and jet axis and $D \equiv [\Gamma (1 - v \cos \theta)^{-1}$ is the Doppler factor. This expression takes relativistic beaming and the relativistic corrections to foreshortening into account. One might expect that the integrated brightness peaks at a certain distance from the core, since $D$ is strongly peaked at $\Gamma \sim 1/\theta$. However, in our prescription the dissipation drops off too fast for this effect to be important. The main difference in the brightness evolution is that a dissipative jet has a different efficiency $\eta$ and generally expands less rapidly in the sideways direction (and will reach a smaller terminal Lorentz factor $\Gamma_\infty$). This effect will become important once the jet has reached $\Gamma_\infty$ and only for $\Lambda$ large enough to significantly alter the dynamics ($\Lambda \gtrsim 0.1$). We have plotted $I$ as a function of $z$ for different values of $\Lambda$, $\zeta = 0$, $U_0' = 20 \rho_0' c^2$, and a viewing angle of $\theta = 10^\circ$ in Fig. 8, arbitrarily normalized to $I(\Lambda = 0.01)$ to increase dynamic range (the brightness decreases by many orders of magnitude along the jet). As is obvious from the plot, for small $\Lambda$, only the overall normalization of $I$ varies with $\Lambda$, whereas for large enough $\Lambda$, the brightness distribution itself changes shape due to the altered dynamics. Also shown are the Doppler factors $D$ for the different parameters, which are primarily responsible for the different shapes.

The situation can become more difficult in the opposite case, i.e., if most of the radiation is trapped (e.g., by synchrotron self-absorption, which implies a steep spectrum, $s > 3$) or if the deposited energy is simply not efficiently radiated (e.g., if $2 < s < 3$). In that case the emission of dissipated energy could be delayed, leading to a relative brightness peak downstream. To briefly investigate this possibility, we assume the latter case, i.e., $2 < s < 3$, which is not an unreasonable choice for AGNs (see §5.3, for example). We assume that the energy flux $\nu F_\nu$ peaks at high energies $\nu_\nu$: either at the spectral break of $\Delta \alpha \sim 1/2$ expected in a scenario in which high energy particles are constantly re-injected, where the spectral index $\alpha$ is given by

$$\alpha \equiv - \frac{d \ln I_\nu}{d \ln \nu},$$

or at the spectral cutoff produced by synchrotron and IC cooling (in the absence of a strong break). The position of $\nu_\nu$ depends on adiabatic effects, radiative cooling, and heating due to dissipation. The spectrum will be self-absorbed at low frequencies, which generally leads to an observed spectral index of $\alpha \sim -5/2$ (for an exact treatment of the spectral shape at the self absorption turnover see De Kool, Begelman, & Sikora 1989). If we take $2 < s < 3$ or $1/2 < \alpha < 1$, the self absorbed part contributes a negligible fraction to the total brightness and most of the energy is emitted at the high end of the spectrum.

In this case it is impossible to calculate the brightness analytically as a function only of $\Gamma$, $R$, and $z$. Rather, one can numerically integrate the evolution equation of the peak frequency under adiabatic cooling, dissipative heating [we assume a self-similar transfer of energy from magnetic field to the particles such that $(d\gamma/dz)|_{\text{diss}} \propto \gamma (dU'/dz)|_{\text{diss}}$, and synchrotron cooling on the basis of the solution $I(z)$ given above. We can estimate the run of
by scaling it with the brightness at $\nu_0$, taking account of relativistic beaming and aberration. If we choose to normalize the intensity curves as we did in the previous case, we can get around fixing the absolute normalization of $U'$, since it only enters linearly into the intensity and will thus cancel out upon normalization. We used $U'_\perp \rho = 20 \rho' \nu c^2$, along with $s = 5/2$ and plotted the frequency integrated intensity in Fig. 9 with otherwise the same parameter values as in Fig. 8, once again normalized to $I$ for $\Lambda = 0.01$. Note that for large values of $\Lambda$ our assumption that the jet be magnetically dominated and that particle pressure be negligible can break down (see Fig. 5), so curves with high $\Lambda$ are to be taken with a grain of salt. Nevertheless, it is interesting to note that the intensity drops less rapidly with $z$ for larger values of $\Lambda$, corresponding to the delayed emission mentioned above. The shapes of these curves depend only weakly on $s$. The different bends in the curves stem from the evolution of the Doppler factor (shown in Fig. 8), and the evolution of the peak frequency. The slopes of the curves are produced mostly by the evolution of the lower cutoff frequency curves are produced mostly by the evolution of the lower cutoff frequency $\gamma_1$ (see eq. [34]), which enters the expression for the synchrotron brightness through the powerlaw normalization of the particle distribution, and by the evolution of Lorentz factor and jet radius (entering through the particle density and the integration of the emissivity across the jet).

$\Gamma = \sqrt{1 + \frac{\gamma^2}{\gamma^2} \frac{\nu^2}{c^2}}$.

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$\Gamma = \sqrt{1 + \frac{\gamma^2}{\gamma^2} \frac{\nu^2}{c^2}}$.

While the degree of polarization is highest for homogeneous magnetic fields, jets with tangled or disorganized field can exhibit a net polarization if there is a net anisotropy in the field (Laing 1980). The measured polarization will depend not only on $\zeta$, but also on the viewing angle $\theta$ and the bulk Lorentz factor $\Gamma$. The polarization of radiation from a powerlaw distribution of electrons with index $s$ in a region of homogeneous field is given by

$$\Pi \equiv \frac{I_\perp - I_\parallel}{I_\perp + I_\parallel} = \frac{s + 1}{s + 7/3} \frac{\sin^2 \theta}{s + 7/3} \frac{\sin^2 \theta}{\sin^2 \theta + 7}$$

where $I$ is the intensity at a given optically thin wavelength. To calculate the integrated polarization, we average across the jet. To do this we decompose the radiation into polarization along the jet axis and perpendicular to it. Furthermore, we assume that field directions are distributed among all solid angles and introduce a weighting function that distributes the field orientations to the required anisotropy,

$$w(\theta) = \sin \theta^\kappa$$

where $\theta$ is the angle between the field and the $z$-axis and $\kappa$ is determined by the anisotropy. We can solve for $\kappa$ under the condition that $\langle B_\parallel^2 \rangle = (1 + \zeta)/(1 - \zeta)$:

$$\kappa = \frac{1 + 3\zeta}{1 + \zeta}.$$
seen head on. Furthermore, the angle between line of sight and magnetic field is important in determining the relative brightness of a region. Figure 10 shows the predicted polarization for the cases shown in Fig. 6, a spectral index of $\alpha = 0.5$, and a viewing angle of $\theta = 10^\circ$. Since the anisotropy of the jet is fixed, the variation in the polarization $\Pi(\theta)$ is solely caused by changes in the viewing angle due to relativistic aberration. Generally, the polarization will be perpendicular to the jet axis if $\zeta < -1/3$ and parallel if $\zeta > -1/3$. As long as equation (9) holds, an extremum in $\Pi(\theta)$ will be present and it should indicate the position where $\Gamma = 1/\sin \theta$, i.e., where the viewing angle corrected for aberration is $\theta' = 90^\circ$. Note that this polarization is averaged across the jet. In order to compare these predictions to actual measurements a relatively small correction for the emission weighted averaging across the jet at different angles must be made. The qualitative predictions of this section should be unaffected by that lack of better knowledge we will use the larger values obtained from the optical data. The average polarization at kpc distances (where the jet has likely reached terminal velocity) is roughly $\Pi \sim 10\%$ parallel to the jet axis, which corresponds to $\zeta \sim 0.3$ with the numbers given above.

Reynolds et al. (1996) showed that the jet probably consists of pairs rather than ionized gas, so the jet might actually still be accelerating (if indeed a large fraction of the particles has $\gamma > 10$). Note, however, that the presence of shocks, clearly visible as knots in all wavelengths, calls for a more sophisticated model which takes time-dependence and MHD instability effects into account. For a discussion of the nature of the shocks observed in M87 see Bicknell & Begelman (1996). Furthermore, there is evidence that the jet is not magnetically dominated on large scales ($z > 100$ pc) and that the magnetic field is actually somewhat below equipartition (Heinz & Begelman 1997), which means in this context that the magnetic field must have been dissipated non-radiatively, accounting for the particle pressure at larger distances. The observed spectral index at optically thin wavelengths is $\alpha \sim 0.6$, which is at least in the correct regime for synchrotron radiation to be inefficient at radiating away the dissipated energy (see §5.1).

At a viewing angle of $20^\circ$ the jet radius at knot A ($z \sim 3$ kpc) is $R \sim 35$ pc with an approximate opening angle of $\alpha_0 \sim 0^\circ.7$, much smaller than the beaming angle of $\alpha_0 \gtrsim 6^\circ$, thus the jet is narrow at least at VLA resolution. Note that entrainment of ambient material may be important before the jet reaches knot A (Bicknell & Begelman 1996). Using knot D as a reference point does not change this analysis significantly, however. There are no good estimates of the pressure gradient in the innermost regions of the M87 X-ray atmosphere (this situation should change, though, with the launch of Chandra). The pressure at large distances is approximately $p_{\text{ISM}} \sim 10^{-10}$ dyn cm$^{-2}$, but Bicknell and Begelman (1996) argue that the pressure in the radio lobes is significantly higher than this, $p_{\text{ext}} \sim 1.5 \times 10^{-9}$ dyn cm$^{-2}$. For lack of better information we will assume the latter value and a smooth pressure gradient with $\xi = \text{const}$. The radiative luminosity of the jet itself is $L_{\text{rad}} \sim 3 \times 10^{42}$ ergs sec$^{-1}$ (Reynolds et al. 1996), which argues for $\Lambda > 4 \times 10^{-3}$. If (as we speculated above) much of the particle pressure at large distances was indeed produced by dissipation, $\Lambda$ could be much larger.

The jet probably reaches terminal $\Gamma_{\infty} \sim 10$ somewhere between VLBI scales and VLA scales, $2 \times 10^3 r_g \sim 0.2$ pc $< z_0 < 200$ pc $\sim 2 \times 10^6 r_g$. For $\Lambda$ small enough to be dynamically unimportant we can assume that $\eta \sim 1/4$ in the self similar regime. Arbitrarily setting $z_0 \sim 10 r_g$ gives $0.8 \lesssim \xi \lesssim 1.74$. The central pressure $p_0$ is then $2 \times 10^{-4}$ dyn cm$^{-2} < p_0 < 200$ dyn cm$^{-2}$, which requires an rms magnetic field of $0.09 G < B'_{\text{0}} < 0.9 G$ for pressure balance (using $\zeta \sim 0.3$). The initial jet width strongly depends on $\xi$: $8 r_g < R_0 < 9000 r_g$ for an assumed $z_0 = 10 r_g$. The latter value is unrealistic (and inconsistent with the limits put on the jet width by VLBI observations, Junor & Biretta 1995), since most of the energy output of the disk into the jet will be provided close to the black hole. It is therefore most reasonable to assume that $z_\infty \sim 2 \times 10^4 r_g$ and $\zeta \sim 1.7$. The total jet power implied by the num-

![Fig. 10.—Optically thin polarization $\Pi$ as a function of $z$ for the same parameter values as in Fig. 6 and a viewing angle of $\theta = 10^\circ$. We chose a spectral index of $\alpha = 0.5$. The variation in $\Pi$ is solely due to changes in the aberrated viewing angle along the jet.](attachment:image)
bers given is of order $L \sim 2 \times 10^{44}$ ergs sec$^{-1}$, consistent with the estimate by Bicknell & Begelman (1996). Overall it seems that this model is consistent with the observed properties of M87 to first order if we adopt a pressure gradient following $\xi \sim 1.7$.

The total isotropic energy output of $E \gtrsim 10^{54}$ ergs of GRB 990123 (Bloom et al. 1999, Bloom et al. 1999) argues strongly in favor of non-isotropic gamma-ray burst (GRB) scenarios, so jet models explaining the apparently beamed nature of these sources (e.g., Mészáros & Rees 1997, Sari, Piran, & Halpern 1999) enjoy newly enhanced popularity. One of the standard scenarios for the energy sources of GRBs is a massive accretion event (either a neutron star - neutron star merger or the accretion of a neutron star by a black hole: Narayan, Paczyński, & Piran 1992), which leads to the formation of a disk after tidal disruption of one of the objects. Since neutron stars already display large magnetic fields, one might expect strong shear amplification of this field in the disk, leading to a scenario similar to what we described in §2. Similarly, the hypernova approach (Paczyński 1998, Woosley 1993) can produce a precollimated outflow, which might then evolve into a jet. Application of our model to GRBs is, however, not as straightforward as in the case of mature radio galaxies. This is because GRBs are highly time dependent (corresponding to the adolescent stages of radio galaxies): the lifetime of the jet (roughly the order of the light travel time of the material) is shorter than the sound crossing time of the bubble the jet blows into the environment, so a pressure balanced solution as described above (which will be set up after the jet and the ambient material have equilibrated, i.e., after the jet has existed for a few sound crossing times) might not be a good approximation. The investigation of jet dynamics in the context of GRBs and in the framework of tangled fields as introduced above will therefore be the subject of another paper. Here we simply wish to point out the benefits of jet models in general and an approach based on our model in particular:

- Acceleration of GRB outflows by Poynting flux has the advantage that collimation can be provided not only by the external medium, but also by the field geometry itself (note that for $\delta = 0$ the perpendicular component of the field does not contribute to the sideways pressure, so for $\zeta \sim -1$ the jet can have a large Poynting flux yet orders of magnitude smaller sideways pressure than a particle dominated jet would have for the same energy flux).

- One would not need to invoke shocks to produce emission in this context, since the internal dissipation of magnetic energy would provide a natural source of high energy particles and photons to produce gamma rays (see Thompson 1994 for an example of how internal dissipation can power GRBs). The short term variability seen in GRBs could then be explained by inhomogeneities (e.g., variations in the inhomogeneity of the field) imprinted in the outflow by variability in the central engine itself, which is expected to have time scales of the same order as the ones observed.

6. CONCLUSIONS

We have presented simple analytic models of jet acceleration. The jets are accelerated by tangled magnetic fields, with collimation being provided by pressure from an external medium. This is a new approach, based on Begelman (1995), as previous models of MHD jet acceleration concentrated mostly on large scale organized fields. Our analytical quasi-1D approach is limited to narrow jets, for which the opening angle is smaller than the beaming angle, coincident with the condition that the jet be in causal contact with its environment. We introduce an ad-hoc process that redistributes energy between perpendicular and parallel field to facilitate efficient conversion of internal energy to kinetic energy. Without such a process, stationary jet acceleration by disorganized fields is impossible. In the absence of dissipation, the rate of acceleration achieved under this scenario is the same as in the case of relativistic particle pressure, $\Gamma \propto p_{\text{jet}}^{-1/4}$. We also find analytic solutions beyond the self-similar region, which enable us to calculate the terminal bulk Lorentz factors of such jets. In order for these jets to reach the observed $\Gamma > 10$, they initially must be magnetically dominated.

We estimate the impact of dissipation of magnetic energy on the dynamics of the flow by considering a simple, phenomenological prescription of the loss process. The presence of dissipation lowers the terminal Lorentz factor $\Gamma_{\infty}$ and generally changes the rate at which the jet is accelerated (the latter effect is noticeable only if the dissipation rate is comparable to the adiabatic expansion rate). We also include the effects of radiation drag in the simplest scenario, which always lowers the efficiency $\eta$ and $\Gamma_{\infty}$. The amount of radiation drag in our model is controlled by the amount of dissipation replenishing the high energy particle pool but seems to be dynamically unimportant.

We calculate the frequency integrated surface brightness for the extreme cases where all or very little of the dissipated energy is radiated away on the spot and find that, while the brightness drops off very rapidly in all considered cases due to the expansion of the jet, values of the dissipation efficiency $\Lambda \gtrsim 0.05$ can have a significant impact on the intensity as a function of $z$. In the marginally non-radiative case the buildup of particle energy leads to a slower decline in intensity with $z$ for larger $\Lambda$. Finally, we apply this model to the prototypical radio galaxy M87 and find that it is consistent with the observed properties.

This research was supported in part by NSF grants AST95–29170 and AST98–76887. MCB also acknowledges support from the Institute for Theoretical Physics under NSF grant PHY94–07194, and from a Guggenheim Fellowship. We thank Chris Reynolds, Jim Chiang, and Chris Thompson for helpful discussions.
assumption that \(\zeta\) (and possibly a shock) at a fixed distance. This process would produce an observable hot-spot [BR74]. Eventually, the massless approximations will break down, in which case the Alfvén velocity will drop, lowering the dissipation rate, which leads to a run-away process. In this limit, the magnetic field dissipates away too quickly to satisfy pressure balance and the jet must contract and deaccelerate to increase its internal pressure, thereby increasing its dissipation rate, which leads to a run-away process.

It is possible to solve the set of equations in the magnetically dominated case (i.e., setting \(\rho' + 4\rho = 0\)) under the assumption that \(\zeta = \text{const}\). and in the relativistic limit, \(\Gamma \gg 1\). In this limit \(v_{\text{Alfvén}} = 1\), which simplifies the treatment significantly. We assume that the dissipated energy is radiated away immediately, which leads to a modified equation (14)

\[
\frac{\Lambda}{\Gamma R} \left( \frac{2 + 6\zeta}{3 - 3\zeta} \right) + (2 + 2\zeta) \frac{d\Gamma}{dz} - (2 + 2\zeta) \frac{dR}{dz} = 0.
\]

The pressure balance equation (16) is also modified:

\[
(\zeta - 1) \frac{d\Gamma}{dz} - (3 + \zeta) \frac{dR}{dz} - \frac{\Lambda}{\Gamma R} = -\xi/z
\]

where we used \(U' \propto \rho_{\text{ext}}\), and equation (19). This set of equations can be solved to give

\[
\Gamma(z) = \Gamma_0 \left(\frac{z}{z_0}\right)^{1/4} \left\{ 1 - A \left[ \left(\frac{z}{z_0}\right)^{1-\xi/2} - 1 \right] \right\}^{(3+\zeta)/(4+4\zeta)},
\]

with

\[
A \equiv \frac{1}{\Gamma_0 R_0} \frac{1}{(1-\xi/2)(1+3\zeta)}.
\]

Subscripts 0 denote quantities evaluated at some arbitrary upstream point \(z_0\). In the limit of \(\Lambda \to 0\) this solution appropriately reduces to the result without dissipation, i.e., \(\Gamma \propto z^{3/4}\). The solution can essentially take on three different behaviors: If \(\xi > 2\), the solution will asymptotically approach \(\Gamma \propto z^{3/4}\), i.e., \(\eta = 1/4\). If \(\xi < 2\), on the other hand, two different scenarios can occur: the flow can either approach a self-similar behavior with \(\eta \neq 1/4\) (but still constant) in the limit of \(z \gg z_0\), or, if \(A > 0\) (i.e., \(\xi > -1/3\)), the flow can actually stall, i.e., \(\Gamma \to 0\) for \(z \to z_0 \left(1 - \xi/2\right) (1 + 3\xi) / A - 1 \right)^{1/(1-\xi/2)}\). In this limit, the magnetic field dissipates away too quickly to satisfy pressure balance and the jet must contract and decelerate to increase its internal pressure, thereby increasing its dissipation rate, which leads to a run-away process. Eventually, the massless approximation will break down, in which case the Alfvén velocity will drop, lowering the dissipation rate (furthermore, the particle pressure will gain in importance, eventually stabilizing the jet against external pressure, in which case the jet would behave as described by BR74). This process would produce an observable hot-spot (and possibly a shock) at a fixed distance.

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**APPENDIX**

### A MORE REALISTIC DISSIPATION LAW

The dissipation law we assumed in §2.1.2 was only one of many plausible ad hoc models. Since (for finite conductivity, i.e., beyond the limit of perfect MHD) reconnection will occur whenever there is field reversal on sufficiently small scales (which would certainly be the case in a highly tangled geometry), we would expect dissipation to occur even if there were no change in field geometry due to expansion or acceleration. In that case we might expect that the dissipation timescale is proportional to the time it takes a disturbance to travel a given characteristic length (e.g., the jet width) in the comoving frame, i.e.,

\[
\frac{dU'}{dz} \bigg|_{\text{diss}} \sim A U_{\text{Alfvén}} \frac{\Gamma}{R},
\]

where the parameter \(\Lambda\) absorbs the effects of resistivity and all the unknown physics of the reconnection process. Once again, it is straightforward to generalize to the case of different \(\Lambda\) for different components of the field.

It is possible to solve the set of equations in the magnetically dominated case (i.e., setting \(\rho' + 4\rho = 0\)) under the assumption that \(\zeta = \text{const}\). and in the relativistic limit, \(\Gamma \gg 1\). In this limit \(v_{\text{Alfvén}} = 1\), which simplifies the treatment significantly. We assume that the dissipated energy is radiated away immediately, which leads to a modified equation (14)

\[
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\]

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\[
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\]

where we used \(U' \propto \rho_{\text{ext}}\) and equation (19). This set of equations can be solved to give

\[
\Gamma(z) = \Gamma_0 \left(\frac{z}{z_0}\right)^{1/4} \left\{ 1 - A \left[ \left(\frac{z}{z_0}\right)^{1-\xi/2} - 1 \right] \right\}^{(3+\zeta)/(4+4\zeta)},
\]

with

\[
A \equiv \frac{1}{\Gamma_0 R_0} \left(\frac{z}{z_0}\right)^{1-\xi/2}(1+3\zeta).
\]