On the antipode of Kreimer’s Hopf algebra

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We give a new formula for the antipode of the algebra of rooted trees, directly in terms of the bialgebra structure. This provides a short proof of the equivalence of the Bogoliubov-Parasiuk-Hepp and the Zimmermann renormalization procedures.

Keywords: Antipode, rooted tree, renormalization.

1. Introduction

More than two years ago, Kreimer [9] discovered that there is a Hopf algebra structure encoding Zimmermann’s forest formula [12] in perturbative renormalization theory. Shortly afterwards, an essential coincidence was found between Kreimer’s algebra and the Hopf algebras introduced by Connes and Moscovici in connection with the index problem for K-cycles on foliations [5].

A unified treatment in terms of the algebra of rooted trees $H_R$ was developed in [2]: there to each (superficially divergent) Feynman diagram a sum of rooted trees is assigned; the assignment is straightforward when the diagram contains only disjoint or nested subdivergences (as then it leads to a single tree), but it also works for overlapping divergences.

The central role in the application of Kreimer-Connes-Moscovici algebras is played by the antipode. In [2] two equivalent definitions of the antipode in $H_R$ were given, representing respectively the recursive Bogoliubov-Parasiuk-Hepp procedure for renormalizing Feynman integrals with subdivergences, and Zimmermann’s forest formula that solves that recursion; that indeed they correspond to the antipode of the Hopf algebra of rooted trees is implied rather than proven.

Here we construct the antipode for $H_R$, giving a new formula for computing it in terms of the coproduct; and then we show its equivalence to each of the formulae by Connes and Kreimer in turn.

2. The antipode of the Hopf algebra of rooted trees

To establish the notation, we briefly recall some basic facts concerning the antipode of a Hopf algebra (consult [1,6,8,11] for proofs), and then the algebra of rooted trees.

Given a unital algebra $(A, m, u)$ and a counital coalgebra $(C, \Delta, \varepsilon)$ over a field $\mathbb{F}$, the convolution of two elements $f, g$ of the vector space of $\mathbb{F}$-linear maps $\text{Hom}(C, A)$ is defined as the map $f \ast g \in \text{Hom}(C, A)$ given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A.$$
This product turns Hom(C,A) into a unital algebra, where the unit is the map \( u \circ \varepsilon \). In this paper \( \mathbb{F} \) is the field of real numbers \( \mathbb{R} \).

A Hopf algebra is a bialgebra \( H = (A, m, u, \Delta, \varepsilon) \) together with a (necessarily unique) convolution inverse \( S \) for the identity map \( \text{id}_H \). The map \( S \) is usually called the antipode of \( H \). The property \( \text{id}_H \ast S = S \ast \text{id}_H = u \circ \varepsilon \) boils down to the commutativity of the diagram:

\[
\begin{array}{c}
H \otimes H \xrightarrow{\Delta} H \xrightarrow{\Delta} H \otimes H \\
\downarrow \text{id} \otimes S \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
The set of nontrivial simple cuts of a tree $T$ will be denoted by $C(T)$; we consider also the “empty cut” $c = \emptyset$, for which $R_\emptyset(T) = T$ and $P_\emptyset(T) = \emptyset$.

The algebra of rooted trees $H_R$ is the commutative algebra generated by symbols $T$, one for each isomorphism class of rooted trees, plus a unit 1 corresponding to the empty tree; the product of trees is written as the juxtaposition of their symbols. The counit $\epsilon : H_R \to \mathbb{R}$ is the linear map defined by $\epsilon(1) := 1_\mathbb{R}$ and $\epsilon(T_1T_2 \ldots T_n) = 0$ if $T_1, \ldots, T_n$ are trees. Kreimer defined a map $\Delta : H_R \to H_R \otimes H_R$ on the generators, extending it as an algebra homomorphism, as follows:

$$\Delta 1 := 1 \otimes 1; \quad \Delta T := T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T). \quad (2.3)$$

Notice that $P_c(T)$ is the product of the several subtrees pruned by the cut $c$. For instance,

$$\begin{align*}
\Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1, \\
\Delta(t_2) &= t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1, \\
\Delta(t_{31}) &= t_{31} \otimes 1 + 1 \otimes t_{31} + t_2 \otimes t_1 + t_1 \otimes t_2, \\
\Delta(t_{32}) &= t_{32} \otimes 1 + 1 \otimes t_{32} + 2t_1 \otimes t_2 + t_1^2 \otimes t_1, \\
\Delta(t_{41}) &= t_{41} \otimes 1 + 1 \otimes t_{41} + t_{31} \otimes t_1 + t_2 \otimes t_2 + t_1 \otimes t_{31}, \\
\Delta(t_{42}) &= t_{42} \otimes 1 + 1 \otimes t_{42} + t_1 \otimes t_{32} + t_2 \otimes t_2 + t_1 \otimes t_{31} + t_2t_1 \otimes t_1 + t_1^2 \otimes t_2, \\
\Delta(t_{43}) &= t_{43} \otimes 1 + 1 \otimes t_{43} + 3t_1 \otimes t_{32} + 3t_1^2 \otimes t_2 + t_1^3 \otimes t_1, \\
\Delta(t_{44}) &= t_{44} \otimes 1 + 1 \otimes t_{44} + t_{32} \otimes t_1 + 2t_1 \otimes t_{31} + t_1^2 \otimes t_2. \quad (2.4)
\end{align*}$$

A most useful tool is the sprouting of a new root; namely the morphism $L : H_R \to H_R$ given by the linear map defined by

$$L(T_1 \ldots T_k) := T,$$

where $T$ is the rooted tree obtained by conjuring up a new vertex as its root and extending lines from this vertex to each root of $T_1, \ldots, T_k$. For instance,

$$L\left(\begin{array}{cc}
\circ & \circ \\
\end{array}\right) = \begin{array}{cc}
\circ \\
\circ \\
\end{array} \quad \text{and} \quad L\left(\begin{array}{cc}
\circ & \circ \\
\circ & \circ \\
\end{array}\right) = \begin{array}{cc}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \quad (2.5)$$

The proof that $\Delta$ is indeed a coproduct is based on the formula

$$\Delta \circ L = L \otimes 1 + (\text{id} \otimes L) \circ \Delta. \quad (2.6)$$

For details see [2] or [6].

When dealing with particular Hopf algebras, the antipode is often determined by specific properties of the algebras in question, and the defining property of the antipode is scarcely used. The latter turns out to be extremely useful in our context, however. We compute the antipode $S : H_R \to H_R$ by exploiting its very definition as the convolution inverse of the identity in $H_R$, via a geometric series:

$$S := (\text{id})^{-1} = (u \circ \epsilon - (u \circ \epsilon - \text{id}))^{-1} = u \circ \epsilon + (u \circ \epsilon - \text{id}) + (u \circ \epsilon - \text{id})^2 + \cdots$$
Lemma 2.1. If $T$ is a rooted tree with $n$ vertices, the geometric series expansion of $S(T)$ has at most $n + 1$ terms.

Proof. The claim is certainly true for $t_1$. Assume that it holds for all trees with $n$ vertices. Let $T$ be a rooted tree with $n + 1$ vertices; then

$$(u \circ \varepsilon - \operatorname{id})^{*(n+2)}(T) = (u \circ \varepsilon - \operatorname{id}) \ast (u \circ \varepsilon - \operatorname{id})^{*(n+1)}(T)$$

$$= m \circ [(u \circ \varepsilon - \operatorname{id}) \otimes (u \circ \varepsilon - \operatorname{id})^{*(n+1)}] \circ \Delta(T)$$

$$\quad = m \circ [(u \circ \varepsilon - \operatorname{id}) \otimes (u \circ \varepsilon - \operatorname{id})^{*(n+1)}]$$

$$\quad \quad \quad \quad = (T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T)).$$

The first and second term vanish because $(u \circ \varepsilon - \operatorname{id})1 = 0$. By the induction hypothesis the third term is zero. \qed

As an immediate corollary we obtain that $S$ so defined is indeed the antipode.

One of the advantages of this formulation is that we obtain a fully explicit formula for $S$ from the coproduct table. If $a \in H^n$, $\Delta(a) = \sum a'_i \otimes a''_i$, $\Delta(a''_i) = \sum a'_{i_2} \otimes a''_{i_1}$ and in general $\Delta(a'_{i_1, \ldots, i_k}) = \sum a'_{i_1, \ldots, i_{k+1}} \otimes a''_{i_2, \ldots, i_{k+1}}$, then

$$(u \circ \varepsilon - \operatorname{id})^{*k+1}(a) = (-1)^{k+1} \sum_{i_1, \ldots, i_k} b'_1 b'_{i_1, i_2} \cdots b'_{i_k} b''_{i_1, \ldots, i_k},$$

where

$$b'_{i_1, \ldots, i_j} := \begin{cases} 0 & \text{if } a'_{i_1, \ldots, i_j} = 1 \text{ or } a''_{i_1, \ldots, i_j} = 1, \\ a'_{i_1, \ldots, i_j} & \text{otherwise}, \end{cases}$$

and

$$b''_{i_1, \ldots, i_j} := \begin{cases} 0 & \text{if } a''_{i_1, \ldots, i_j} = 1, \\ a''_{i_1, \ldots, i_j} & \text{otherwise}. \end{cases}$$

For instance, using (2.4),

$$S(t_{42}) = -t_{42} + (t_{11} t_{32} + t_{2}^2 + t_{1} t_{31} + 2t_{1}^2 t_{2}) - (5t_{1}^2 t_{2} + 2t_{1}^4) + 3t_{1}^4$$

$$\quad = -t_{42} + t_{11} t_{32} + t_{2}^2 + t_{1} t_{31} - 3t_{1}^2 t_{2} + t_{1}^4. \quad (2.7)$$

Similarly, if we denote by $t'$ the rooted tree in (2.5) with 5 vertices, then

$$S(t') = -t' + (2t_{1} t_{42} + 2t_{2} t_{31} + t_{2}^2 t_{32} + 3t_{1} t_{2}^2)$$

$$\quad - (2t_{1} t_{32} + 6t_{1} t_{2}^2 + 2t_{2}^2 t_{31} + 8t_{1}^3 t_{2}^2 + t_{2}^5) + (12t_{1}^3 t_{2} + 6t_{1}^5) - 6t_{1}^5$$

$$\quad = -t' + 2t_{1} t_{42} + 2t_{2} t_{31} - t_{2}^2 t_{32} - 3t_{1} t_{2}^2 - 2t_{1}^2 t_{31} + 4t_{1} t_{2}^2 - t_{1}^5. \quad (2.7)$$

In correspondence with Bogoliubov’s recursive formula for renormalization, equations $m \circ (S \otimes \operatorname{id}) \circ \Delta(T) = 0$ and (2.4) suggest to define the antipode recursively, as indeed done by Connes and Kreimer:

$$S_B(T) := -T - \sum_{c \in C(T)} S_B(P_c(T)) R_c(T).$$
For instance,

\[ S_B(t_{42}) = -t_{42} - S_B(t_1)t_{32} - S_B(t_2)t_2 - S_B(t_1)t_{31} - S_B(t_2)t_1 - S_B(t_1)t_1, \]

which gives again (2.7). We next check that \( S_B \) is indeed the antipode.

**Proposition 2.2.** If \( T \) is any rooted tree, then \( S(T) = S_B(T) \).

**Proof.** For convenience, we abbreviate \( \eta := u \circ \varepsilon - \text{id} \). The statement holds, by a direct check, if \( T \) has 1, 2 or 3 vertices. If it holds for all rooted trees with at most \( n \) vertices and if \( T \) is a rooted tree with \( n + 1 \) vertices, then

\[
S(T) = \eta(T) + \sum_{j=1}^{n} \eta^{*j} \otimes \eta(T) = -T + m \circ \left( \sum_{j=1}^{n} \eta^{*j} \otimes \eta \right) \circ \Delta(T)
\]

\[
= -T + m \circ \sum_{j=1}^{n} \eta^{*j} \otimes \eta \left( T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T) \right)
\]

\[
= -T - \sum_{c \in C(T)} \sum_{j=1}^{n} \eta^{*j}(P_c(T)) R_c(T)
\]

\[
= -T - \sum_{c \in C(T)} S_B(P_c(T)) R_c(T) = S_B(T),
\]

where the penultimate equality uses the inductive hypothesis. \( \square \)

Zimmermann’s forest formula corresponds to the following nonrecursive formula for the antipode:

\[
S_Z(1) := 1, \quad S_Z(T) := -\sum_{d \in D(T)} (-1)^{\#d} P_d(T) R_d(T),
\]

where \( D(T) \) is the set of all cuts, not necessarily simple, including the empty cut, and \( \#d \) is the cardinality of \( d \).

**Proposition 2.3.** If \( T \) is any rooted tree, then \( S(T) = S_Z(T) \).

**Proof.** First we prove that, for an arbitrary rooted tree \( T \),

\[
S(L(T)) = -L(T) - S(T) t_1 - \sum_{c \in C(T)} S(P_c(T)) L(R_c(T)). \tag{2.8}
\]

Indeed, if \( T \) has \( n \) vertices, then, by Lemma 2.1 and (2.6),

\[
S(L(T)) = -L(T) + m \circ \left( \sum_{j=1}^{n} \eta^{*j} \otimes \eta \right) \circ \Delta(L(T))
\]

\[
= -L(T) + m \circ \sum_{j=1}^{n} \eta^{*j} \otimes \eta(L(T) \otimes 1 + (\text{id} \otimes L) \circ \Delta(T))
\]
\[ S_Z(L(T)) = -\left( \sum_{d \in A} + \sum_{d \in B} \right) (-1)^{#d} P_d(L(T)) R_d(L(T)). \] (2.9)

If \( d \in A \), then \( e = d \setminus \{ \ell_0 \} \) is a cut of \( T \); moreover, \( R_d(L(T)) = t_1, P_d(L(T)) = P_e(T) R_e(T) \) and \( #d = #e + 1 \), so that the first sum of (2.9) equals \( -S(T) t_1 \).

For a given \( d \in B \setminus \{ \emptyset \} \) and each \( j \in K = \{ 1, \ldots, k \} \), let \( \ell_j \) be the line in \( d \) closer to the root that is linked to \( v_j \) (if any). Then \( c' := \{ \ell_j: j \in K \} \) is a simple cut of \( T \). If \( #c' \) is odd, we set \( c := c' \), whereas if \( #c' \) is even, we set \( c := c' \setminus \{ \ell_s \} \), where \( s \) is the smallest integer in \( K \) for which there is a line with the required property. In either case, we take \( e := d \setminus c \). Clearly \( R_d(L(T)) = L(R_e(T)) \), \( (-1)^{#d+1} = (-1)^{#e} \), and \( P_d(L(T)) = P_e(Tc) R_e(Tc) \), where we use the temporary notation \( T_c := P_e(T) \). It follows that the second sum of (2.9) equals

\[ \sum_{c \in C(T)} \sum_{e \in D(T_c)} (-1)^{#e} P_e(Tc) R_e(Tc) L(R_e(T)) = - \sum_{c \in C(T)} S_Z(P_c(T)) L(R_c(T)). \]

Finally, since the summand for the empty cut is \( -L(T) \), the proposition is proved. \( \square \)

In summary, modulo the distinction between the antipode and the “twisted” or “renormalized” antipode \([3,4,10]\), Kreimer’s algebraic approach allows a new, indirect but rather quick, proof of the equivalence of the Bogoliubov-Parasiuk-Hepp and the Zimmermann procedure for renormalizing Feynman integrals with subdivergences.

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