Irreducible Hamiltonian approach to the Freedman-Townsend model

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Abstract

The irreducible BRST symmetry for the Freedman-Townsend model is derived. The comparison with the standard reducible approach is also addressed.

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1 Introduction

It is well-known that the Hamiltonian BRST formalism [1]–[5] stands for one of the strongest and most popular quantization methods for theories with first-class constraints. In the irreducible context the ghosts can be interpreted like one-forms dual to the vector fields corresponding to the first-class constraints. This geometrical interpretation fails within the reducible framework due to the fact that the vector fields are no longer independent, hence they cannot form a basis. The redundant behaviour generates the appearance of ghosts with ghost number greater than one, traditionally called ghosts for ghosts, of their canonical conjugated momenta, named antighosts, and, in the meantime, of a pyramidal non-minimal sector. The ghosts for ghosts
ensure a straightforward incorporation of the reducibility relations within the cohomology of the exterior derivative along the gauge orbits, while their antighosts are required in order to kill the higher resolution degree nontrivial co-cycles from the homology of the Koszul-Tate differential. Among the reducible systems, the Freedman-Townsend model [6] plays a special role due on the one hand to its link with Witten’s string theory [7], and, on the other hand, to its equivalence to the non-linear σ-model [6]. This model was approached within the antifield-BRST framework [8]–[12] and only partially at the Hamiltonian BRST level [8], but in both settings was studied along an on-shell reducible context.

The purpose of this paper consists in proving that it is possible to quantize the Freedman-Townsend model, which is an example of on-shell first-stage reducible Hamiltonian theory, in the framework of an irreducible Hamiltonian BRST procedure. The idea of replacing reducible systems by some irreducible ones is not new. At the Hamiltonian level, this idea appears in [5] and [13], but it hasn’t been either consistently developed, or applied so far to the quantization of reducible Hamiltonian systems. Our treatment mainly relies on the replacement of the on-shell first-stage reducible Hamiltonian model by an irreducible one, and on the subsequent quantization of the resulting theory in the Hamiltonian BRST context. The irreducible Hamiltonian system is completely derived from the requirement that under a suitable redefinition of the antighost number one antighosts all the antighost number one co-cycles of the reducible Koszul-Tate differential should identically vanish. This further prevents the appearance of antighosts with antighost number two. Moreover, the reducible and irreducible systems possess the same physical observables, hence the zeroth order cohomological groups of the corresponding BRST operators coincide. This enables us to replace the BRST symmetry for the starting reducible theory by that for the irreducible system. As a consequence of our irreducible approach to the Freedman-Townsend model the ghosts for ghosts are absent. Thus, the three-ghost coupling term is discarded from the gauge-fixed action, and the corresponding gauge-fixed Lagrangian BRST symmetry becomes off-shell nilpotent in our procedure.

The comparison between the reducible analysis and our irreducible treatment in the case of this model is instructive at emphasizing some interesting aspects revealed by our formalism. For instance, within the reducible approach to this model the ghosts for ghosts are bosonic and display massless scalar field propagators, hence they are fields with correct spin-statistics re-
lations. On the other hand, it is well-known that in quantum field theory the ghosts do not describe physical particles, such that it appears necessary to recover ‘wrong’ spin-statistics relations in connection with these fields. Our method presents the nice feature that it restores this required type of ‘wrong’ relations. Another interesting feature is that the emerging gauge-fixed action is manifestly Lorentz covariant and displays a simpler form than the one inferred within the reducible procedure, allowing thus a more straightforward perturbative approach. In the meantime, our irreducible analysis can be applied to investigating the possible consistent couplings that can be introduced among a system of free two-form gauge fields. The last problem was studied in [12] accordingly the reducible background, being shown that in four dimensions the Freedman-Townsend interaction vertex defines the only consistent interaction that deforms nontrivially the gauge transformations of free two-forms. It is possible that an irreducible approach to this matter will reveal new aspects or enlighten other features of the already known results. All these considerations motivate the necessity of an irreducible analysis for the Freedman-Townsend model.

Our paper is structured in four sections. In Section 2 we realize the irreducible quantization of the Freedman-Townsend model. Initially, we derive an irreducible theory associated with the starting reducible model by means of some homological ideas. Subsequently we prove that we can substitute the Hamiltonian BRST quantization of the original redundant model by that of the irreducible theory. In the final part of this section we deduce the path integral for the irreducible system in the context of the Hamiltonian BRST quantization, which is found manifestly Lorentz covariant. In Section 3 we realize the comparison between the gauge-fixed action derived in Section 2 and the usual gauge-fixed action of the Freedman-Townsend model obtained in the reducible antifield context. Section 4 ends the paper with some conclusions.

2 Irreducible treatment of the Freedman-Townsend model

In this section we construct the path integral for the Freedman-Townsend model by using an irreducible Hamiltonian BRST procedure. We start with
the canonical analysis and derive some irreducible first-class constraints associated with the reducible ones following homological arguments. Next, we obtain the irreducible BRST symmetry associated with the above mentioned irreducible constraints and reveal its relationship with the standard reducible BRST symmetry via proving that the physical observables corresponding to the irreducible and reducible theories coincide. This makes permissible the replacement of the Hamiltonian BRST quantization for the original reducible model by that of the irreducible system. Finally, we deduce the path integral of the irreducible theory, which is found manifestly Lorentz covariant.

2.1 Canonical analysis of the model

We begin with the Lagrangian action of the Freedman-Townsend model [6]

\[ S_0^L \left[ A_\mu^a, B_{\mu}^{\alpha\beta} \right] = \frac{1}{2} \int d^4 x \left( -B_{\mu}^{\alpha\beta} F_{\mu}^{\alpha^i} + A_\mu^a A_\mu^a \right), \]  

(1)

where \( B_{\mu}^{\alpha\beta} \) stands for a set of antisymmetric tensor fields, and the field strength of \( A_\mu^a \) reads as

\[ F_{\mu}^{\alpha^i} = \partial_\mu A^{\alpha^i}_\mu - \delta^i_\alpha A_{\mu}^a - f_{\alpha\beta}^{c} A^{c}_{\mu} A_{\mu}^c. \]

The canonical analysis of this theory outputs the constraints

\[ \Phi^{(1)a}_{i} \equiv \epsilon_{0ijk} \pi^{jka} \approx 0, \quad \Phi^{(2)a}_{i} \equiv \frac{1}{2} \epsilon_{0ijk} \left( F_{ij}^{ka} - \left( D_{[i} \right)^{a}_{b} \pi_{j0b} \right) \approx 0, \]  

(2)

\[ \chi^{(1)a}_{i} \equiv \pi_{0i}^a \approx 0, \quad \chi^{(2)a}_{i} \equiv \pi_{i}^a + B_{0i}^a \approx 0, \quad \chi^{(3)a}_{i} \equiv \pi_{0}^a \approx 0, \]  

(3)

\[ \chi^{(2)}_a \equiv A^0_a + f_{ab}^c B_{ci}^{0i} + (D_i)^b_a \pi_{bi} \approx 0, \]  

(4)

and the first-class Hamiltonian

\[ H = \int d^3 x \left( \frac{1}{2} B_{\mu}^{ij} \left( F_{ij}^{a} - \left( D_{[i} \right)^{a}_{b} \pi_{j0b} \right) \right) - \frac{1}{2} A^{a}_\mu A_{\mu}^a - A^{a}_0 \left( \left( D_i \right)^b_a \pi_{bi}^a + f_{ab}^c B_{ci}^{0i} \right) - A^i_0 \left( \pi_{0i}^a - \partial_\mu \pi_{0}^a \right) \right). \]  

(5)

The symbol \([ij]\) appearing in (5) signifies the antisymmetry with respect to the indices between brackets. In the above the notations \( \pi_{0}^a \) and \( \pi_{\mu}^a \) denote the momenta respectively conjugated in the Poisson bracket to the fields \( A_\mu^a \) and \( B_{\mu}^{\alpha\beta} \), while the covariant derivatives are defined by \( (D_i)^a_b = \delta^a_b \partial_i + f_{ac}^b A^c_i \) and \( (D_i)^a_b = \delta^a_b \partial_i - f_{ac}^b A^c_i \). By computing the Poisson brackets between the constraint functions (2–4) we find that (2) are first-class and
(3–4) second-class. In addition, the functions $\Phi_i^{(2)a}$ from (2) are on-shell first-stage reducible
\[
\left( (D_i)^a_b + \sum_{bc} \pi^{0ic} \right) \Phi_i^{(2)b} = -\epsilon^{0ijk} \sum_{cd} \chi^{(1)c}_d \chi^{(1)d}_k \approx 0. \tag{6}
\]
In order to deal with the Hamiltonian BRST formalism, it is useful to eliminate the second-class constraints with the help of the Dirac bracket \[14\] built with respect to themselves. By passing to the Dirac bracket, the constraints (3–4) can be regarded as strong equalities with the help of which we can express $A^a_i$, $\pi^0_a$, $\pi^i_a$, and $\pi^a_0$ in terms of the remaining fields and momenta, such that the independent ‘co-ordinates’ of the reduced phase-space are $A^a_i$, $B_{bi}^a$, $B_{aj}^i$ and $\pi_{ij}^a$. The non-vanishing Dirac brackets among the independent components are expressed by
\[
\left[ B_{ai}^b (x), A_{bj}^i (y) \right]^* = \delta^b_a \delta^i_j \delta^3 (x - y), \tag{7}
\]
\[
\left[ B_{ai}^j (x), \pi_{kl}^b (y) \right]^* = \frac{1}{2} \delta^b_a \delta_{ij}^l \delta_{kl}^j \delta^3 (x - y). \tag{8}
\]
In terms of the independent fields, the first-class constraints and first-class Hamiltonian respectively take the form
\[
\gamma_i^{(1)a} \equiv \epsilon_{0ijk} \pi^{jka} \approx 0, \quad G_i^{(2)a} \equiv \frac{1}{2} \epsilon_{0ijk} F^{jka} \approx 0, \tag{9}
\]
\[
H' = \frac{1}{2} \int d^3 x \left( B_{ai}^j F^{ja} - A_{ai}^j A_{aj}^i + \left( (D_i)^a_b B_{bi}^a \right) (D_j)^c_d B_{cj}^d \right) \equiv \int d^3 x h', \tag{10}
\]
while the reducibility relations
\[
(D_i)^a_b G_i^{(2)b} \equiv Z_{ai}^b G_i^{(2)b} = 0, \tag{11}
\]
hold off-shell in this case. Moreover, the first-class constraints (9) remain abelian in terms of the Dirac bracket. In the sequel we work with the theory based on the reducible first-class constraints (9), the first-class Hamiltonian (10) in the context of the Dirac bracket defined by (7–8).

2.2 Irreducible constraints. Irreducible Hamiltonian BRST symmetry

The Hamiltonian BRST symmetry for our reducible first-class model $s_R = \delta_R + \sigma_R + \cdots$ contains two crucial differentials. The Koszul-Tate differential
$\delta_R$ realizes an homological resolution of smooth functions defined on the surface (9), while the model of longitudinal derivative $\sigma_R$ takes into account the gauge invariances. The main property of $\delta_R$, the acyclicity, is gained via introducing some new fields, called antighosts, and denoted by $\mathcal{P}^a_{(1)i}$, $\mathcal{P}^a_{(2)i}$ and $\mathcal{P}^a$. The first two sets of antighosts are fermionic and possess the antighost number one, while the last set is bosonic and displays the antighost number two. The standard definitions of $\delta_R$ on the Koszul-Tate generators are given by

$$
\delta_R z^A = 0, \quad \delta_R \mathcal{P}^a_{(1)i} = -\gamma^{(1)a}_i, \quad (12)
$$

$$
\delta_R \mathcal{P}^a_{(2)i} = -G^{(2)a}_i, \quad (13)
$$

$$
\delta_R \mathcal{P}^a = -\left(D^i\right)^a_b \mathcal{P}^b_{(2)i}, \quad (14)
$$

where $z^A$ can be any of the reduced phase-space ‘co-ordinates’. The antighosts $\mathcal{P}^a$ are required by the acyclicity of $\delta_R$ at antighost number one. Indeed, from (11) and (13) we find that

$$
\mu^a \equiv \left(D^i\right)^a_b \mathcal{P}^b_{(2)i}, \quad (15)
$$

are nontrivial co-cycles, which are restored trivial with the help of (14).

The basic idea of our irreducible approach consists in redefining the antighosts $\mathcal{P}^a_{(2)i}$ in such a way that the new co-cycles of the type (15) vanish identically. If we solve this problem, the antighosts $\mathcal{P}^a$ will be discarded from the theory, hence we get an irreducible Hamiltonian system. In view of this aim, we perform the transformation

$$
\mathcal{P}^a_{(2)i} \rightarrow \tilde{\mathcal{P}}^a_{(2)i} = N^a_{bij} \mathcal{P}^b_{(2)j}, \quad (16)
$$

where $N^a_{bij}$ are some functions that may involve the $z^A$’s taken such that

$$
\left(D^i\right)^a_b N^bjc = 0, \quad N^a_{bij}G^{(2)b}_j = G^{(2)a}_i. \quad (17)
$$

Multiplying (13) by $N^b_{aij}$ and taking into account (16–17) we find

$$
\delta \tilde{\mathcal{P}}^a_{(2)i} = -G^{(2)a}_i, \quad (18)
$$

which further yield the co-cycles $\nu^a \equiv \left(D^i\right)^a_b \tilde{\mathcal{P}}^b_{(2)i}$ that vanish identically due to the former relations in (17), hence (18) describe an irreducible theory.
The vanishing of these co-cycles induces the removal of the antighost number two antighosts \( P^a \) from the antighost spectrum, so it is natural to replace the notation \( \delta_R \) by \( \delta \) in (18) in order to emphasize that these relations correspond to an irreducible system. If we take

\[
N^{a^j}_{b^i} = \delta^a_b \delta^j_i - (D_i)^{a^c} \tilde{M}_d^c \left( D^d_j \right)^d_b,
\]

(19)

with \( \tilde{M}_d^c \) the inverse of \( M_d^c = (D_i)^{c^b} \left( D^b_j \right)_d \), the relations (17) are automatically fulfilled. It is clear that \( M_d^c \) is invertible because our model is first-stage reducible, i.e., the equations \( (D_i)^{a^b} u^b \approx 0 \) possess only trivial solutions. The concrete form of \( \tilde{M}_d^c \) is not important in the context of our analysis, but only its existence in principle. Substituting (19) in (18) we arrive at the relations

\[
\delta \left( P^{a^i}_{(2)i} - (D_i)^{a^c} \tilde{M}_d^c \left( D^d_j \right)^d_b \; P^{b^j}_{(2)j} \right) = -G^{(2)a}_i.
\]

(20)

The last formulas describe an irreducible theory underlying some irreducible first-class constraints to be further determined. In this light we add the supplementary bosonic canonical pairs \( (\varphi_a, \pi^a) \) equal in number with the number of the reducibility relations (11), with \( \pi^a \) the non-vanishing solutions to the equations

\[
M^{a^b}_b \pi^b = \delta \left( \left( D^j \right)^{a^c} \; P^{b^j}_{(2)j} \right).
\]

(21)

Due to the invertibility of \( M^{a^b}_b \), the system (21) possesses non-vanishing solutions if and only if \( \delta \left( \left( D^j \right)^{a^c} \; P^{b^j}_{(2)j} \right) \neq 0 \). Thus, the non-vanishing solutions enforce the irreducibility as \( \mu^a \) given by (15) are not co-cycles in this case. Obviously, the opposite situation (i.e., the trivial solutions \( \pi^b = 0 \)) leads to the equations \( \delta \left( \left( D^j \right)^{a^c} \; P^{b^j}_{(2)j} \right) = 0 \), that reveal the appearance of the co-cycles (15), and thus of the reducibility. On behalf of (21) the relations (20) can be written under the form

\[
\delta P^{a^i}_{(2)i} = -G^{(2)a}_i + (D_i)^{a^b} \pi^b.
\]

(22)

Relations (22) together with \( \delta z^A = 0, \; \delta P^{a^i}_{(1)i} = -\gamma^{(1)a}_i \) completely define an irreducible Koszul-Tate complex based on the irreducible first-class constraints

\[
\gamma^{(1)a}_i \equiv \epsilon_{0ijk} \pi^{jka} \approx 0, \quad \gamma^{(2)a}_i \equiv G^{(2)a}_i - (D_i)^{a^b} \pi^b \approx 0.
\]

(23)
In this manner we associated an irreducible theory based on the irreducible first-class constraints (23) with the initial redundant model. It is simple to see that the new constraints remain abelian. It is well-known that the Lagrangian action (1) of the original reducible model is invariant under some Lorentz covariant gauge transformations [8]–[12]. At the Hamiltonian level, the covariant behaviour of the original gauge transformations is basically ensured by the presence, besides the gauge parameters corresponding to the first-class constraints (2), of the supplementary gauge parameters associated with the first-stage reducibility functions appearing in the left-hand side of (6). On the contrary, the gauge parameters available within the Hamiltonian context of our irreducible system are fewer than the original ones due to the lack of reducibility. Thus, we find ourselves in the position to infer some non-covariant transformations at the level of the Lagrangian action underlying the irreducible theory. In order to surpass this inconvenient and, moreover, to derive a manifestly covariant path integral for the irreducible theory it is necessary to enhance the field and constraint spectrum in such a way to preserve the irreducibility. A natural and also simple manner of realizing this scope is to enlarge the phase-space with the canonical bosonic pairs \((\varphi^{(1)}_a, \pi^{(1)a})\), \((\varphi^{(2)}_a, \pi^{(2)a})\), which we set to be constrained by

\[
\gamma^{(1)a} \equiv -\pi^{(1)a} \approx 0, \quad (24)
\]

\[
\gamma^{(2)a} \equiv -\pi^{(2)a} \approx 0. \quad (25)
\]

The constraints (23) and (24–25) are first-class and irreducible. It is well known that one can always add a combination of first-class constraints to a first-class constraint without afflicting the theory. In this respect, we remark that (11) and (23) allow us to express \(\pi^a\) under the form

\[
\pi^a = -M^a_b \left(D^i\right)_c \gamma^{(2)c}_i, \quad (26)
\]

such that we can replace (24) (and still maintain the irreducibility) by

\[
\gamma^{(1)a} \equiv \pi^a - \pi^{(1)a} \approx 0. \quad (27)
\]

So far, we derived the irreducible first-class constraints

\[
\gamma^{(1)a}_i \equiv \varepsilon_{0jk} \pi^{jk}a \approx 0, \quad \gamma^{(2)a}_i \equiv \frac{1}{2} \varepsilon_{0jk} F^{jk}a - (D_i)^a_b \pi^b \approx 0, \quad (28)
\]

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\( \gamma^{(1)a} \equiv \pi^a - \pi^{(1)a} \approx 0, \quad \gamma^{(2)a} \equiv -\pi^{(2)a} \approx 0. \) \hfill (29)

The first-class Hamiltonian in the Dirac bracket defined by (7–8) with respect to the above constraints can be chosen of the type

\[
\tilde{H} = \int d^3x \left( \frac{1}{2} B_{ij}^{a} \left( F_{ij}^{a} + \varepsilon_{0ijk} (D^{k})_{b}^{a} \pi^{b} \right) - \frac{1}{2} A_{i}^{a} A_{a}^{i} + \varphi_{a} \pi^{(2)a} + \frac{1}{2} \left( (D_{i})_{a}^{b} B_{b}^{0i} - f_{ab}^{c} \left( \varphi_{c} \pi^{b} - \varphi_{c}^{(1)} \pi^{b} + \varphi_{c}^{(2)} \pi^{(2)b} \right) \right)^2 - \varphi_{a}^{(2)} (D_{i})_{a}^{b} \left( D^{i} \right)_{c}^{b} \pi^{c} \right) \equiv \int d^3 x \tilde{h},
\hfill (30)
\]

where we employed the notation

\[
\left( (D_{i})_{a}^{b} B_{b}^{0i} - f_{ab}^{c} \left( \varphi_{c} \pi^{b} - \varphi_{c}^{(1)} \pi^{b} + \varphi_{c}^{(2)} \pi^{(2)b} \right) \right)^2 \equiv \left( (D^{j})_{a}^{b} B_{b}^{0j} - f_{ab}^{c} \left( \varphi_{c} \pi^{b} - \varphi_{c}^{(1)} \pi^{b} + \varphi_{c}^{(2)} \pi^{(2)b} \right) \right) \times \left( (D^{j})_{a}^{b} B_{b}^{0j} - f_{ab}^{c} \left( \varphi_{c} \pi^{b} - \varphi_{c}^{(1)} \pi^{b} + \varphi_{c}^{(2)} \pi^{(2)b} \right) \right) \gamma_{b}^{(2)c} \gamma_{a}^{(2)c}.
\hfill (31)
\]

The irreducible first-class constraints are abelian in the Dirac bracket, while the remaining gauge algebra relations read as

\[
\left[ \gamma_{i}^{(1)a}, \tilde{H} \right]^{*} = -\gamma_{i}^{(2)a}, \quad \left[ \gamma_{i}^{(2)a}, \tilde{H} \right]^{*} = - (D_{i})_{a}^{b} \gamma_{b}^{(2)c} + f_{bc}^{a} \left( (D^{j})_{a}^{d} B_{d}^{0j} - f_{de}^{b} \left( \pi^{d} \varphi^{c} - \pi^{d} \varphi^{(1)} c + \pi^{d} \varphi^{(2)c} \right) \right) \gamma_{i}^{(2)c},
\hfill (32)
\]

\[
\left[ \gamma_{i}^{(1)a}, \tilde{H} \right]^{*} = \gamma_{i}^{(2)a}, \quad \left[ \gamma_{i}^{(2)a}, \tilde{H} \right]^{*} = (D_{i})_{a}^{b} \gamma_{b}^{(2)c} + f_{bc}^{a} \left( (D^{j})_{a}^{d} B_{d}^{0j} - f_{de}^{b} \left( \pi^{d} \varphi^{c} - \pi^{d} \varphi^{(1)} c + \pi^{d} \varphi^{(2)c} \right) \right) \gamma_{i}^{(2)c}.
\hfill (33)
\]

As it will be further evidenced, the gauge algebra (32–33) ensures the Lorentz covariance of the irreducible approach.

At this point we are in the position to construct the irreducible BRST symmetry corresponding to the irreducible theory derived so far. The minimal antighost spectrum of the irreducible Koszul-Tate differential is organized as

\[ \mathcal{P}_{\Gamma} \equiv \left( \mathcal{P}_{i}^{(1)a}, \mathcal{P}_{i}^{(2)a}, \mathcal{P}_{i}^{(1)a}, \mathcal{P}_{i}^{(2)a} \right), \hfill (34) \]
where all the variables are fermionic of antighost number one, being associated with (28–29). Using the standard definitions

\[ \delta z^A = 0, \quad \delta P_i^{(\Delta) a} = -\gamma_i^{(\Delta) a}, \quad \delta P^{(\Delta) a} = -\gamma^{(\Delta) a}, \quad \Delta = 1, 2, \]

the Koszul-Tate operator is nilpotent and acyclic. The generators of the longitudinal derivative along the gauge orbits are given by

\[ \eta^\Gamma \equiv \left( \eta^{(1)i}_a, \eta^{(2)i}_a, \eta^{(1)}_a, \eta^{(2)}_a \right), \]

and are fermionic with pure ghost number one. The definitions of the longitudinal derivative along the gauge orbits read as

\[ \sigma z^A = \sum_{\Delta=1}^{2} \left( \left[ z^A, \gamma^{(\Delta) a}_i \right]^* \eta^{(\Delta) i}_a + \left[ z^A, \gamma^{(\Delta) a} \right]^* \eta^a_\Delta \right), \]

\[ \sigma \eta^\Gamma = 0. \]

With these definitions at hand, \( \sigma \) is strongly nilpotent. Extending \( \delta \) to the ghosts (37) and \( \sigma \) to the antighosts (34) through

\[ \delta \eta^\Gamma = 0, \quad \sigma P_\Gamma = 0, \]

the homological perturbation theory [15]–[18] guarantees that the irreducible BRST symmetry \( s_I = \delta + \sigma \) exists and is nilpotent. To conclude with, at this moment we managed to derive an irreducible BRST symmetry associated with the original reducible one.

### 2.3 Relation with the reducible BRST symmetry

In the sequel we establish the relationship between the irreducible and reducible BRST symmetries discussed in the above. In this respect we prove that the physical observables of the irreducible theory coincide with those of the reducible model. Let \( F \) be an observable of the irreducible system. Consequently, it satisfies the equations

\[ \left[ F, \gamma^{(1)a}_i \right]^* \approx 0, \quad \left[ F, \gamma^{(2)a}_i \right]^* \approx 0, \]

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\[
\begin{align*}
[F, \gamma^{(1)a}]^* &\approx 0, \quad [F, \gamma^{(2)a}]^* \approx 0,
\end{align*}
\]  
(42)

where the weak equality holds on the surface defined by (28–29). Using (26) the equations \([F, \gamma^{(2)a}_i]^* \approx 0\) lead to
\[
[F, G^{(2)a}_i]^* \approx 0,
\]  
(43)
on the surface (28–29). Employing (26), by direct computation we find that
\[
[F, \pi^a]^* \approx 0,
\]  
(44)
on the same surface. On behalf of (44) it follows that the equations (42) are equivalent with
\[
[F, \pi^{(1)a}]^* \approx 0, \quad [F, \pi^{(2)a}]^* \approx 0.
\]  
(45)
Thus, any observable of the irreducible theory should verify the former equations in (41) and (43–45) on the surface (28–29).

Next we show that the first-class constraints (28–29) are equivalent with
\[
\gamma^{(1)a}_i \approx 0, \quad G^{(2)a}_i \approx 0, \quad \pi^a \approx 0, \quad \pi^{(1)a} \approx 0, \quad \pi^{(2)a} \approx 0.
\]  
(46)
Indeed, the constraints \(\gamma^{(1)a}_i \approx 0\), and \(\pi^{(2)a} \approx 0\) pertain to both sets, so it is enough to emphasize that the constraints
\[
G^{(2)a}_i \approx 0, \quad \pi^a \approx 0, \quad \pi^{(1)a} \approx 0,
\]  
(47)
are equivalent to
\[
\gamma^{(2)a}_i \approx 0, \quad \gamma^{(1)a} \approx 0.
\]  
(48)
It is obvious that when (47) hold (48) hold, too. The converse results as follows. Substituting (26) in the concrete expression of \(\gamma^{(2)a}_i\) we arrive at
\[
G^{(2)a}_i = \left(\delta^a_d \delta^j_i - (D_i)^a_b \bar{M}^b_{bc} (D^j)^c_d\right) \gamma^{(2)d}_j \approx 0.
\]  
(49)
From (26) and (49) we directly get that if (48) hold, then (47) also hold. This proves the equivalence between the first-class constraints (28–29) and (46).

By virtue of the above discussion, we have that any observable of the irreducible theory, which we found that should verify the former equations in
(41) and (43–45) on the surface (28–29), will check these equations also on the surface (46). As a consequence, the observables of the irreducible theory and those of the theory based on the constraints (46) coincide. Now, we show that the observables of the system possessing the constraints (46) and the observables corresponding to the original reducible theory also coincide. We observe that the surface (46) can be obtained in a trivial manner from \(\gamma_i^{(1)a} \approx 0\), \(G_i^{(2)a} \approx 0\) by adding the canonical pairs \((\varphi_a, \pi^a)\), \((\varphi_a^{(1)}, \pi^{(1)a})\), \((\varphi_a^{(2)}, \pi^{(2)a})\) and requiring that their momenta vanish. Thus, the observables of the original model are unaffected by the introduction of the new canonical pairs. More precisely, the difference between an observables \(F\) of the theory based on the constraints (46) and one of the original theory \(\bar{F}\) is of the type

\[
F - \bar{F} = \lambda_a \pi^a + \lambda_a^{(1)} \pi^{(1)a} + \lambda_a^{(2)} \pi^{(2)a}.
\]

As any observables that differ through a combination of first-class constraints can be identified, it follows that the observables of the original system and of the theory based on the constraints (46) coincide. In consequence, we proved that the observables of the theory with the constraints (46) coincide with those of the irreducible system, as well as with those of the original model. This leads to the conclusion that the observables of the irreducible and original reducible models coincide. In turn, this matter has strong implications at the BRST level.

As we noticed earlier, there exists a consistent Hamiltonian BRST symmetry satisfying the general grounds of homological perturbation theory [15]–[18] associated with the irreducible system. Comparing the standard reducible Hamiltonian BRST symmetry corresponding to the Freedman-Townsend model \(s_R\) with that for the irreducible theory \(s_I\) from the point of view of the basic equations underlying the BRST formalism, we have that

\[
s_R^2 = 0, \quad s_I^2 = 0, \quad H^0 (s_R) = H^0 (s_I) = \{\text{physical observables}\}.
\]

The above relations enable us to substitute the Hamiltonian BRST quantization of the Freedman-Townsend model by that of the irreducible system.

### 2.4 Irreducible path integral

Based on the last conclusion, we pass to the Hamiltonian BRST quantization of the irreducible first-class theory, which is described by the first-class constraints (28–29) and the first-class Hamiltonian (30). The ghost and
antighost spectra are respectively given by (37) and (34). At the same time, we add the non-minimal sector

\[
\eta^a_i (\partial_i^a b^a i) , \quad (P_i^{(b) a}, b_i^{(1)b}) , \quad (P_i^{(\bar{\eta}) a}, \bar{\eta}_i^a) , \quad (P_i^{(\bar{\eta}) a}, \bar{\eta}_i^{(1)a}) ,
\]

where the first two sets of non-minimal fields are bosonic with ghost number zero, while the last two sets are fermionic, with the \(\bar{\eta}\)'s of ghost number minus one and the \(P_{(\bar{\eta})}\)'s of ghost number one. The former variables in every set are taken as fields, while the latter represent their momenta. Under these considerations, the non-minimal BRST charge reads as

\[
\Omega = \int d^3 x \left( \sum_{\Delta=1}^{2} (\eta^{(\Delta)} a_i \gamma^{(\Delta)} a_i + \eta^{(\Delta)} a_i \gamma^{(\Delta)} a_i) + P_i^{(\bar{\eta}) a_i} + P_i^{(\bar{\eta}) a_i} b_i^{(1)a}) \right),
\]

while the BRST-invariant extension of the first-class Hamiltonian \(\tilde{H}\) is expressed by

\[
\tilde{H}_B = \tilde{H} + \int d^3 x \left( \eta^{(1) a} (\partial^a_i b_i + \bar{\eta}_i^a) + \bar{\eta}_i^{(1) a} \gamma^{(1) a} + \eta_i^{(2) a} \gamma^{(2) a} \right) + P_i^{(\bar{\eta}) a_i} + P_i^{(\bar{\eta}) a_i} b_i^{(1)a}) \right),
\]

Choosing the gauge-fixing fermion

\[
K = \int d^3 x \left( \partial^a_i (\bar{\eta}_i^a + \eta_i^{(1) a}) + \bar{\eta}_i^{(1) a} \gamma^{(1) a} + \eta_i^{(2) a} \gamma^{(2) a} \right),
\]

and computing the gauge-fixed action, we find after eliminating some auxiliary variables the path integral

\[
Z_K = \int \mathcal{D} A_i^a \mathcal{D} B_a^{\mu a} \mathcal{D} \varphi_a^{(1)} \mathcal{D} b_a^a \mathcal{D} \bar{\eta}_a \mathcal{D} \bar{\eta}_a^a \exp \left( i \tilde{S}_K \right),
\]

where

\[
\tilde{S}_K = \int d^4 x \left( -A_i^a B^a_i + \frac{1}{2} A_i^a A_i^a - \frac{1}{2} B^i_a F^a_i - \right.
\]
\[ \frac{1}{2} \left( (D_i)_a^b B_{0i}^a + f_{ab} \left( \left( \bar{\eta}_i^a - \partial_i \bar{\eta}_0^a \right) \eta_c^i - \left( \partial^\mu \bar{\eta}_0^a \right) \eta_c^0 \right) \right)^2 - \\
\frac{1}{2} \partial_i \bar{\eta}_0^a (D_i)_a^b \eta_0^b - \left( \partial^\mu \bar{\eta}_0^a \right) \left( \eta_0^0 + (D_i)_a^b \eta_i^b \right) + \\
\frac{1}{2} \partial_i \left( \eta_i^a \right) (D_i)_a^b \bar{\eta}_0^b - \left( \partial^\mu \eta_i^a \right) \left( \eta_i^0 + (D_0)_a^b \eta_i^b \right) \right). \]  

\[ (57) \]

In deriving the above expressions we performed the identifications

\[ b_\mu^a = \left( \pi^{(1)a}, \tilde{b}_a^a \right), \quad \bar{\eta}_\mu^a = \left( \mathcal{P}^{(1)a}, -\bar{\eta}_0^a \right), \quad \eta_\mu^a = \left( \eta^{(2)a}, \eta_0^{(2)a} \right). \]  

\[ (58) \]

We remark that we can introduce an auxiliary field \( H_0^a \) in the path integral by means of the relation

\[ \exp \left( -\frac{i}{2} \int d^4x \left( (D_i)_a^b B_{0i}^b + f_{ab} \left( \left( \bar{\eta}_i^b - \partial_i \bar{\eta}_0^b \right) \eta_c^i - \left( \partial^\mu \bar{\eta}_0^b \right) \eta_c^0 \right) \right)^2 \right) = \]

\[ \int DH_0^a \exp \left[ i \int d^4x \left( \frac{1}{2} H_0^a H_0^a - \\
H_0^a (D_i)_a^b B_{0i}^b + f_{ab} \left( \left( \bar{\eta}_i^b - \partial_i \bar{\eta}_0^b \right) \eta_c^i - \left( \partial^\mu \bar{\eta}_0^b \right) \eta_c^0 \right) \right) \right). \]  

\[ (59) \]

With the help of the above Gaussian average and by realizing the identification

\[ A_\mu^a = \left( H_0^a, A_\mu^a \right), \]  

\[ (60) \]

the path integral (56) takes the manifestly covariant form

\[ Z_K = \int \mathcal{D}A_\mu^a \mathcal{D}B_{\mu\nu}^a \mathcal{D}\varphi_a^{(1)} \mathcal{D}H_0^a \mathcal{D}H_0^a \exp \left( iS_K \right), \]  

\[ (61) \]

where the gauge-fixed action \( S_K \) reads as

\[ S_K = S_0^L \left[ A_\mu^a, B_{\mu\nu}^a \right] + \int d^4x \left( b_\mu^a \left( \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \partial_\nu B_{\lambda\rho} + \partial^\mu \varphi_a^{(1)} \right) - \\
\frac{1}{2} \partial_\mu \bar{\eta}_\nu^a \left( D_\nu \right)_a^b \eta_\mu^b - \left( \partial^\mu \bar{\eta}_0^a \right) \left( D_\nu \right)_a^b \eta_\mu^b \right). \]  

\[ (62) \]

In the above \( S_0^L \left[ A_\mu^a, B_{\mu\nu}^a \right] \) is nothing but the original action (1), and, in addition, we adopted the notations

\[ (D_0)_a^b = \delta_a^b \partial_0 + f_{ac}^a H_0^c, \quad (D_0)_a^b = \delta_a^b \partial_0 - f_{ac}^b H_0^c. \]  

\[ (63) \]
The BRST symmetries of (62) are expressed by
\[ sB_{a}^{\lambda} = \varepsilon^{\mu\nu\lambda\rho} (D_{\lambda})_{b}^{c} \eta^{db}, \quad sA_{a}^{\mu} = 0, \quad s\varphi^{(1)a} = (D_{\mu})_{b}^{c} \eta^{cb}, \] (64)
\[ s\eta^{a}_{\mu} = 0, \quad s\bar{\eta}^{a}_{\mu} = b_{\mu}^{a}, \quad sb_{a}^{\mu} = 0. \] (65)
This completes the irreducible Hamiltonian BRST treatment of the Freedman-Townsend model.

3 Comparison with the standard reducible approach

In the following we make the comparison between the results obtained in our irreducible context and those deriving in the standard reducible BRST approach. In the reducible approach the gauge-fixed action can be brought to the form
\[ S_{\psi} = \int d^{4}x \left( -\frac{1}{2} \partial_{\mu} \bar{C}_{\nu} + \frac{1}{8} \varepsilon^{\mu\nu\lambda\rho} f_{a}^{bc} \partial_{\mu} \bar{C}_{c} \partial_{\nu} \bar{C}_{b} + \frac{1}{2} \varepsilon^{\mu\nu\lambda\rho} \partial_{\nu} B_{\lambda\rho a} - (D_{\mu})_{b}^{c} \bar{C}_{b} - (D^{\nu})_{b}^{c} \bar{C}_{b} \right) \] (66)
where \( C_{\nu}^{a} \) represent the ghost number one ghosts, and \( C^{a} \) signify the ghosts for ghosts. The gauge-fixed BRST symmetries of (66) read as
\[ s_{\psi} B_{a}^{\mu\nu} = \varepsilon^{\mu\nu\lambda\rho} (D_{\lambda})_{b}^{c} \bar{C}_{b} - (D_{\mu})_{b}^{c} \bar{C}_{b}, \quad s_{\psi} C_{a}^{\mu} = 0, \] (67)
\[ s_{\psi} A_{a}^{\mu} = 0, \quad s_{\psi} \bar{C}_{a}^{\mu} = B_{a}^{\mu}, \quad s_{\psi} \bar{C}_{a}^{\mu} = 0. \] (68)
The above symmetries are on-shell nilpotent with respect to some of the fields
\[ s_{2}^{2} B_{a}^{\mu\nu} = -\varepsilon^{\mu\nu\lambda\rho} f_{bc}^{a} \frac{\delta S_{\psi}}{\delta B_{\lambda\rho c}} \bar{C}_{b}, \quad s_{2}^{2} \bar{C}_{a}^{\mu} = \frac{\delta S_{\psi}}{\delta C_{a}^{\mu}}, \quad s_{2}^{2} \bar{C}_{a}^{\mu} = \frac{\delta S_{\psi}}{\delta C_{a}^{\mu}}, \] (69)
and off-shell nilpotent otherwise. It is obvious that the ghosts for ghosts \( \left( \bar{C}_{a}, C^{a} \right) \) in the gauge-fixed action (66) possess massless scalar field propagators, hence they output correct spin-statistics relations, in disagreement
with the requirement of not describing physical particles. In the meantime, we cannot surpass this inconvenient by integrating over the ghosts for ghosts. This is because the integration over \((\bar{C}_a, C^a)\) is cumbersome as the matrix of their quadratic parts depends on the fields and the structure constants, such that one cannot eliminate this term from the path integral. By contrast, the ghosts for ghosts are absent within the gauge-fixed action (62), so the above disagreement is surpassed within the irreducible analysis. By performing the identifications
\[
C^a \mu \leftrightarrow \eta^a \mu, \quad \bar{C}^a \leftrightarrow \bar{\eta}^a, \quad \bar{C}'^{(1)} a \leftrightarrow \varphi^{(1)} a, \quad B^a \leftrightarrow b^a,
\]
(70) among the variables involved with the gauge-fixed actions derived within the irreducible and reducible approaches, (62), respectively, (66), the difference between the two gauge-fixed actions becomes
\[
S_\psi - S_K = \int d^4 x \left( - (\partial^\mu \bar{C}_a) (D_\mu)^a \right)_b C^b + \frac{1}{8} \epsilon^{\mu \nu \lambda \rho} f^{a} \partial \mu \partial \nu \partial \lambda \partial \rho \bar{C}_a \bar{C}_c C^c C^b \right). \quad (71)
\]
We remark that \(S_\psi - S_K\) is proportional with the ghosts for ghosts, \(C^b\), which are some essential compounds of the reducible BRST quantization. Although identified at the level of the gauge-fixed actions, the fields \(C^a\) and \(\varphi^{(1)} a\) play different roles within the two formalisms. More precisely, the presence of \(\varphi^{(1)} a\) within the irreducible model prevents the appearance of the reducibility, while the \(\bar{C}'^{(1)} a\)'s represent an effect of the reducibility. In fact, the Lagrangian effect of introducing the fields \(\varphi^{(1)} a\) resides in forbidding the existence of the zero modes which are present within the original reducible theory. In consequence, all the ingredients connected with the zero modes, like the ghosts for ghosts or the non-minimal pyramid, are no longer involved. In this light, we suggestively call the fields \(\varphi^{(1)} a\) ‘antimodes’. The antimodes also contribute to the difference between the corresponding gauge-fixed BRST symmetries. Indeed, the absence of the ghosts for ghosts in our irreducible context makes the gauge-fixed BRST symmetries (64–65) off-shell nilpotent, by contrast with the reducible situation (see (69)), and, moreover, removes the three-ghost coupling term from our irreducible procedure. Finally, we mention that the number of antimodes is equal with the number of the pairs \((\bar{C}_a, C^a)\) generated by the zero modes. Obviously, neither the ghost for ghost pairs nor the antimodes describe physical particles and are however governed by correct spin-statistics relations, but, while the ghosts for ghosts cannot
be eliminated from the path integral by direct integration, the antimodes do not produce this difficulty due to the possibility of a safely integration over them by making a Gaussian average in (62).

4 Conclusion

In this paper we show that the Freedman-Townsend model, which is a typical example of on-shell first-stage reducible Hamiltonian system, can be consistently approached along the irreducible Hamiltonian BRST line by replacing the original reducible Hamiltonian theory with an irreducible one. The construction of the irreducible system is based on “killing” the initial redundancy at the level of the Koszul-Tate complex. As the physical observables associated with the irreducible and reducible versions coincide, the main equations underlying the Hamiltonian BRST formalism make legitimate the substitution of the reducible BRST symmetry by the irreducible one. The gauge-fixed action of the irreducible model is derived within the Hamiltonian BRST context by using an appropriate gauge-fixing fermion. Finally, we emphasize the comparison between our irreducible procedure and the standard reducible BRST treatment of the Freedman-Townsend model.

References


